# Remarks on Tanimura's reformulation of general relativistic version of Feynman's proof about Maxwell equations

Hitoshi MIYAZAKI and Kanji FUJII

*Department of Physics, Faculty of Sciences, Hokkaido University, Sapporo 060, Japan*

**Abstract**

We reformulate the genaral relativistic version given by S.Tanimura of Feynman's consideration. Velocity and momentum of a point particle are assumed to be transformed as (contravariant and covariant) vectors under the point transformation in quantum mechanical sense. Such an assumption restricts strongly the equation of motion allowed by the basic commutation relations. This equation is equivalent to Hamilton-type one including scalar, vector and gravitational fields covariantly.

1. *Purpose* Since F.J.Dyson [1] reconstructed the Feynman's proof of Lorentz force and the homogeneous Maxwell equations, some papers have been published on quantum mechanical equations of motion of a point particle, in which some generalizations have been done by taking into consideration the Lorentz covariance, the spin or the non-abelian internal degree of freedom [2,4]. In the Tanimura's paper [3], he tried to extend the Feynman's proof to the general relativistic case. In his formulation, however, the point transformation properties of relevant quantities are not clear; in particular, the force $F^{\mu}$ defined in Ref.[3] is not transformed as a vector.

The aim of the present paper is to reformulate the general relativistic version given by Tanimura [3], and to give the 'covariant' equation of motion (in quantum mechanical sense) for a point particle with its mass m under the point transformation of curvilinear coordinates

$$q^{\mu}(\tau)\rightarrow q'^{\mu}(\tau)=q'^{\mu}(q(\tau)). \tag{1}$$

Here, a particle is moving in N-dimensional space-time with coordinates $q^{\mu}(\tau), \mu = 1,2,\ldots,N$; $\tau$ is a parameter, the meaning of which is discussed in Ref. [3]. It should be noted that the transformation property of velocity (or momentum) under the point transformation (1) has to be modified so as to behave as a "quantum" vector, and that in such a sense one can give the equation of motion for a point particle with the "quantum" covariance, consistently with the basic commutation relations. We point out that the covariance requirement in an extended sense restricts strongly a form of the equation of motion, which is shown to be equivalent to Hamilton-type equation of motion including scalar, vector and gravitational fields acting on a point particle.

2. *Definition of relevant quantities and the basic assumptions* First we assume the commutation relations

$$[q^{\alpha}(\tau),q^{\beta}(\tau)]=0, \qquad [q^{\alpha}(\tau),m\dot{q}^{\beta}(\tau)]=i\hbar g^{\alpha\beta}(q), \qquad \dot{q}(\tau):=\frac{dq^{\beta}(\tau)}{d\tau}, \tag{2.1}$$

where $g^{\alpha\beta}(q)$ is the "metric" of the space-time. The transformation property of $\dot{q}^{\alpha}$ under (1) is taken to be written as

$$\dot{q}'^{\beta}(\tau)=<\partial_{\alpha}q'^{\beta}(\tau),\dot{q}^{\alpha}(\tau)>=<\xi_{\alpha}{}^{\beta},\dot{q}^{\alpha}>. \tag{2.2}$$

Here, $< A,B >$ means $(AB+BA)/2$. We call the quantity with the above trnsformation property the quantum (contravariant) vector. Next, we define the momentum $p_\mu$ as

$$p_\mu := m < g_{\mu\alpha}, \dot{q}^\alpha >; \tag{2.3a}$$

$g_{\mu\alpha}$ is the inverse to $g^{\alpha\beta}$. We obtain

$$[q^\alpha, p_\beta] = m < g_{\beta\gamma}, [q^\alpha, \dot{q}^\gamma] >= i\hbar g_{\beta\gamma} g^{\alpha\gamma} = i\hbar \delta^\alpha{}_\beta. \tag{2.3b}$$

As easily confirmed, $p_\mu$ behaves as a "quantum" (covariant) vector; due to $g'_{\mu\alpha} = e^\gamma{}_\mu e^\delta{}_\alpha g_{\gamma\delta}$ with $e^\beta{}_\mu := \frac{\partial q^\beta}{\partial q'^\mu}$, we obtain

$$p'_\mu =< g'_{\mu\alpha}, \dot{q}'^\alpha >=< e^\beta{}_\mu, p_\beta > . \tag{2.4}$$

For a tensor $T^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_m}$, we have

$$[p_\mu, T^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_m}(q)] = -i\hbar \partial_\mu T^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_m}(q), \tag{2.5}$$

which is form-invariant (or covariant) under the point transformation (1);

$$[p'_\mu, T'^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_m}(q')] = -i\hbar \partial'_\mu T'^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_m}(q').$$

3. *Properties of* $[\dot{q}^\alpha, \dot{q}^\beta]$ *and* $[p_\mu, p_\nu]$ For the quantity $W^{\alpha\beta}$, defined by

$$m^2 [\dot{q}^\alpha, \dot{q}^\beta] := i\hbar W^{\alpha\beta}, \tag{3.1a}$$

we obtain

$$[q^\gamma, W^{\alpha\beta} + m < (-g^{\alpha\mu} g^{\beta\nu} + g^{\beta\mu} g^{\alpha\nu}) \partial_\mu g_{\nu\rho}, \dot{q}^\rho >] = 0; \tag{3.1b}$$

therefore, $[q^\gamma, F^{\alpha\beta}(q)] = 0$ with

$$F^{\alpha\beta} := W^{\alpha\beta} + m < (-g^{\alpha\mu} g^{\beta\nu} + g^{\beta\mu} g^{\alpha\nu}) \partial_\mu g_{\nu\rho}, \dot{q}^\rho > . \tag{3.1c}$$

$F^{\alpha\beta}$ is a function only of $q^\alpha$'s. By employing the transformation property of $\dot{q}^\beta$ given by (2.2), we can confirm the tensor propertyof $F^{\alpha\beta}$;

$$F'^{\alpha\beta}(q') = \xi_\mu{}^\alpha \xi_\nu{}^\beta F^{\mu\nu}(q). \tag{3.2}$$

Similarly, from Jacobi identity and the transformation property of $p_\mu$ (2.4), one obtains

$$[q^\gamma,[p_\mu,p_\nu]] = -[p_\mu,[p_\nu,q^\gamma]] - [p_\nu,[q^\gamma,p_\mu]] = 0, \tag{3.3a}$$

$$[p'_\mu,p'_\nu] = e^\alpha{}_\mu e^\beta{}_\nu [p_\alpha,p_\beta], \tag{3.3b}$$

which mean that $[p_\mu,p_\nu]$ is a function only of $q^\alpha$'s and $[p_\alpha,p_\beta]$ is transformed as a tensor. Further, by using (2.3a) , one obtains

$$[< e^\alpha_\mu, p_\alpha >, < e^\beta{}_\nu, p_\beta >] = i\hbar < e^\alpha{}_\nu \partial_\alpha e^\beta{}_\mu - e^\alpha{}_\mu \partial_\alpha e^\beta{}_\nu, p_\beta > + i\hbar e^\alpha{}_\mu e^\beta{}_\nu F_{\alpha\beta}.$$

$$[p_\mu,p_\nu] = i\hbar g_{\rho\alpha} g_{\lambda\beta} F^{\alpha\beta} := i\hbar F_{\rho\lambda}. \tag{3.4}$$

From Jacobi identity for three $p_\rho$'s, one obtains Bianchi identity for $F_{\rho\lambda}$;

$$\partial_\mu F_{\rho\lambda} + \partial_\rho F_{\lambda\mu} + \partial_\lambda F_{\mu\rho} = 0. \tag{3.5}$$

Thus we see that there exists the quantity $A_\mu(q)$, satisfying $F_{\alpha\beta} = \partial_\alpha A_\beta(q) - \partial_\beta A_\alpha(q)$. Using this $A_\mu(q)$, we define $\pi_\beta$ as follows;

$$\pi_\beta := p_\beta + A_\beta(q). \tag{3.6a}$$

Then we can find the following properties of $\pi_\beta$'s;

$$[\pi_\beta,\pi_\gamma] = i\hbar F_{\beta\gamma} - i\hbar(\partial_\beta A_\gamma(q) - \partial_\gamma A_\beta(q)) = 0, \tag{3.6b}$$

$$[q^\alpha,\pi_\beta] = i\hbar\delta^\alpha_\beta. \tag{3.6c}$$

$\pi_\beta$ is assumed to behave as a "quantum" vector; then the two equations (3.6b) and (3.6c) are form-invariant, since

$$\begin{aligned} [\pi'_\beta,\pi'_\gamma] &= [< e^\mu{}_\beta, \pi_\mu >, < e^\nu{}_\gamma, \pi_\nu >] \\ &= i\hbar < e^\nu{}_\gamma \partial_\nu e^\mu{}_\beta - e^\nu{}_\beta \partial_\nu e^\mu{}_\gamma, \pi_\mu > = 0, \end{aligned} \tag{3.7a}$$

$$[q'^\alpha,\pi'^\beta] = i\hbar e^\nu{}_\beta \partial_\nu q'^\alpha = i\hbar\delta^\alpha{}_\beta, \tag{3.7b}$$

where integrability condition

$$\frac{\partial}{\partial q'^\mu} e^\nu{}_\rho = \frac{\partial}{\partial q'^\rho} e^\nu{}_\mu, \tag{3.7c}$$

is used in (3.7a).

With the use of the covariant derivative $\nabla_\beta$ expressed in terms of Riemann-Christoffel symbol $\Gamma^\alpha_{\beta\gamma} := \frac{1}{2}g^{\alpha\rho}(\partial_\beta g_{\rho\gamma} + \partial_\gamma g_{\rho\beta} - \partial_\rho g_{\beta\gamma})$, one obtains

$$F_{\beta\gamma} = \nabla_\beta A_\gamma - \nabla_\gamma A_\beta \tag{3.8a}$$

with $\nabla_\beta A_\gamma := \partial_\beta A_\gamma - \Gamma^\alpha_{\beta\gamma}A_\alpha$, and also

$$F^{\beta\gamma} = g^{\beta\rho}\nabla_\rho A^\gamma - g^{\gamma\rho}\nabla_\rho A^\beta. \tag{3.8b}$$

It should be noted that, according to our quantization procedure, the vector field $A_\mu(q)$ has no sources such as the magnetic monopole in the sense of (3.5).

4. *Form of the force* From

$$0 = \frac{d}{d\tau}[q^\alpha, p_\mu] = \frac{1}{m}[< g^{\alpha\rho}, p_\rho >, p_\mu] + [q^\alpha, \frac{dp_\mu}{d\tau}],$$

one obtains

$$[q^\alpha, \frac{dp_\mu}{d\tau}] = -\frac{i\hbar}{m}g^{\alpha\rho}F_{\rho\mu} - \frac{i\hbar}{m} < \partial_\mu g^{\alpha\rho}, p_\rho > . \tag{4.1}$$

The classical form of the absolute derivative of $p_\mu$ which behaves as a vector is

$$\frac{dp_\mu}{d\tau} - \frac{1}{m}\Gamma^\lambda_{\mu\nu}g^{\nu\rho}p_\lambda p_\rho. \tag{4.2}$$

The last term of (4.1) comes from the quantum-mechanical term corresponding to the second term of (4.2). If we can find such a quantity that reduces (4.2) in the classical limit and that behaves as a quantum vector, we may call it the "quantum" absolute derivative, $(\frac{\delta p_\mu}{d\tau})_Q$. Then the equation of motion of a point particle is written as

$$(\frac{\delta p_\mu}{d\tau})_Q = F_\mu, \tag{4.3a}$$

where $F_\mu$ is called the force and transformed as

$$F_\mu \to F'^\mu =< e^\nu{}_\mu, F_\nu > \tag{4.3b}$$

under the transformtion (1).

From (4.1),the form of $F_\mu$ is expressed as

$$F_\mu = \frac{1}{m} < F_{\mu\rho} g^{\rho\lambda}, p_\lambda > + G_\mu(q), \tag{4.4}$$

where $G_\mu$ is a vector depending only on $q^\beta$'s and its property will be investigated later. The quantum vector property of the first term in r.h.s. of (4.4) is easily confirmed;

$$\begin{aligned} < g^{\lambda\rho} F_{\rho\mu}, p_\lambda > \to & < \xi_\alpha{}^\lambda e^\beta{}_\mu g^{\alpha\rho} F_{\rho\beta}, < e^\nu{}_\lambda, p_\nu >> \\ & = < e^\beta{}_\mu g^{\nu\rho} F_{\rho\beta}, p_\nu > \\ & = < e^\beta{}_\mu, < g^{\nu\rho} F_{\rho\beta}, p_\nu >> . \end{aligned}$$

After some inspection, we see that one possible form of $(\frac{\delta p_\mu}{d\tau})_Q$ is given as

$$(\frac{\delta p_\mu}{d\tau})_Q = \frac{dp_\mu}{d\tau} + \frac{1}{2m} p_\alpha (\partial_\mu g^{\alpha\beta}) p_\beta + \frac{\hbar^2}{2m} \{ \frac{1}{2} \partial_\mu \partial_\beta (\Gamma_\alpha g^{\alpha\beta}) + \frac{1}{4} \partial_\mu (\Gamma_\alpha \Gamma_\beta g^{\alpha\beta}) \} \tag{4.5}$$

where $\Gamma_\alpha := \Gamma^\beta_{\alpha\beta} = g^{-1} \partial_\alpha g/2, \quad g := \det(g_{\mu\nu})$.
Due to $\partial_\mu g^{\alpha\beta} = -\Gamma^\alpha_{\mu\nu} g^{\nu\beta} - \Gamma^\beta_{\mu\nu} g^{\alpha\nu}$, the quantum absolute derivative reduces to the classical one, (4.2). Under the integrability condition (3.7c) one can prove directly,

$$(\frac{\delta p_\mu}{d\tau})_Q \to ((\frac{\delta p_\mu}{d\tau})_Q)' = < e^\nu{}_\mu, (\frac{\delta p_\nu}{d\tau})_Q >, \tag{4.6}$$

the derivation of which is given in Appendix.

Using the commutation relations, one can prove the rotation-free property of $G_\mu$;

$$\partial_\mu G_\nu - \partial_\nu G_\mu = 0, \tag{4.7a}$$

because

$$\begin{aligned} i\hbar(\partial_\mu G_\nu - \partial_\nu G_\mu) &= [p_\nu, G_\mu] - [p_\mu, G_\nu] \\ &= [p_\nu, \frac{dp_\mu}{d\tau} + \frac{1}{2m} p_\alpha (\partial_\mu g^{\alpha\beta}) p_\beta - \frac{1}{m} < F_{\mu\lambda} g^{\lambda\rho}, p_\rho >] - (\mu \leftrightarrow \nu) \\ &= \frac{d[p_\nu, p_\mu]}{d\tau} + \frac{i\hbar}{m} < \{ F_{\nu\alpha} \partial_\mu g^{\alpha\beta} + \partial_\nu (F_{\mu\alpha} g^{\alpha\beta}) \} - (\mu \leftrightarrow \nu), p_\beta > \qquad (4.7b) \\ &= \frac{i\hbar}{m} < (\partial_\alpha F_{\nu\mu} + \partial_\mu F_{\alpha\nu} + \partial_\nu F_{\mu\alpha}) g^{\alpha\beta}, p_\beta > \\ &= 0 \qquad [\text{due to } (3.5)]. \end{aligned}$$

5. *Final remarks and conclusion* (*i*) By taking into consideration the point transformation property of relevant quantities in quantum-mechanical sense, we gave the equation

of motion of a point particle expressed as

$$(\frac{\delta p_\mu}{d\tau})_Q = F_\mu, \tag{5.1}$$

where the quantities in both sides are transformed as quantum vectors under the transformation (1). Various expressions equivalent to (4.5) are of course possible. The vector field $A_\mu(q)$ satisfies (3.5), so that $A_\mu(q)$ has no source such as magnetic monopole.

($ii$) It seems worthy to notice that, by using the covariant kinetic energy term

$$K = \frac{1}{2m} g^{-\frac{1}{4}} p_\alpha g^{\frac{1}{2}} g^{\alpha\beta} p_\beta g^{-\frac{1}{4}}, \tag{5.2a}$$

we obtain

$$\begin{aligned}\frac{1}{i\hbar}[p_\mu, K] =& \frac{1}{m} < F_{\mu\alpha} g^{\alpha\beta}, p_\beta > - \frac{1}{m} p_\alpha (\partial_\mu g^{\alpha\beta}) p_\beta \\ & - \frac{\hbar^2}{2m} \{ \frac{1}{2} \partial_\mu \partial_\beta (\Gamma_\alpha g^{\alpha\beta}) + \frac{1}{4} \partial_\mu (\Gamma_\alpha \Gamma_\beta g^{\alpha\beta}) \};\end{aligned} \tag{5.2b}$$

therefore,

$$(\frac{\delta p_\mu}{d\tau})_Q = \frac{dp_\mu}{d\tau} - \frac{1}{i\hbar}[p_\mu, K] + \frac{1}{m} < F_{\mu\alpha} g^{\alpha\beta}, p_\beta > . \tag{5.2c}$$

Setting l.h.s. to be equal to the force (4.4), one obtains

$$\frac{dp_\mu}{d\tau} = \frac{1}{i\hbar}[p_\mu, K + V(q)], \tag{5.3a}$$

where $V(q)$ is a scalar function, satisfying

$$G_\mu(q) = -\partial_\mu V(q). \tag{5.3b}$$

Thus we see the equation of motion (4.3a) determined in accordance with the basic commutaion relation (2.1) as well as the (quantum) transformation is equal to Hamilton equation of motion, where the $\tau$-development is assumed to be determined by Hamiltonian $H = K + V(q)$. Note that

$$\frac{1}{i\hbar}[q^\mu, H] = \frac{1}{2m} < g^{\mu\beta}, p_\beta >= \dot{q}^\mu. \tag{5.4}$$

($iii$) In Tanimura's formulation of the general relativistic case [3], transformation property of relevant quantities are obscure, and further we are apt to think that the derived equation of motion for a point particle may be rather general in the framework of quantum theory and not necessary to have a conncetion with Lagrangian or Hamiltonian. Our conclusion is, however, that the extended quantum transformation property under the point transformation gives a strong condition to the equaion of motion allowed in the quantum-mechanical framwork, and this equation of motion (5.1) reduces to the Hamilton-type equaton (5.3a).

The authors would like to express their thanks to Dr.N.M.Chepilko (Institute for Physics, Kiev), Dr.K-I.Sato(Science University of Tokyo, Oshyamanbe, Hokkaido) and also the members of the group of the Elementary Particle Physics in Hokkaido University.

*Appendix———— Proof of the 'vector' property of* $(\frac{\delta p_\mu}{d\tau})_Q$

We consider the transformation property of each term in

$$(\frac{\delta p_\mu}{d\tau})_Q = \frac{dp_\mu}{d\tau} + \frac{1}{2m}p_\alpha(\partial_\mu g^{\alpha\beta})p_\beta + \frac{\hbar^2}{2m}\{\frac{1}{2}\partial_\mu\partial_\beta(\Gamma_\alpha g^{\alpha\beta}) + \frac{1}{4}\partial_\mu(g^{\alpha\beta}\Gamma_\alpha\Gamma_\beta)\}. \qquad (A.1)$$

under $q^\mu \longrightarrow q'^\mu = q'^\mu(q)$. At first, we obtain

$$\begin{aligned}\frac{d}{d\tau} < e^\nu{}_\mu, p_\nu > - < e^\nu{}_\mu, \frac{dp_\nu}{d\tau} > &= < \frac{de^\nu{}_\mu}{d\tau}, p_\nu > \\ &= \frac{1}{2m}\{p_\rho(g^{\lambda\gamma}\partial_\gamma e^\rho{}_\mu + g^{\rho\gamma}\partial_\gamma e^\lambda{}_\mu)p_\lambda - \frac{\hbar^2}{2}\partial_\lambda\partial_\nu(\partial_\rho e^\nu{}_\mu \cdot g^{\rho\lambda})\}.\end{aligned} \qquad (A.2)$$

Next, as to the second term in r.h.s. of (A.1), we obtain

$$\begin{aligned}&\frac{1}{2m}\{p'_\alpha(\partial'_\mu g'^{\alpha\beta})p'_\beta - < e^\nu{}_\mu, p_\alpha(\partial_\nu g^{\alpha\beta})p_\beta >\} \\ &= \frac{1}{2m}\{(p_\mu e^\rho{}_\alpha + \frac{i\hbar}{2}\partial_\rho e^\rho{}_\alpha)\partial'_\mu g'^{\alpha\beta} \cdot (e^\lambda{}_\beta p_\lambda - \frac{i\hbar}{2}\partial_\lambda e^\lambda{}_\beta) - < e^\nu{}_\mu, p_\alpha(\partial_\nu g^{\alpha\beta})p_\beta >\} \\ &= \frac{1}{2m}\{-p_\rho(g^{\gamma\lambda}\partial_\gamma e^\rho{}_\mu + g^{\gamma\rho}\partial_\gamma e^\lambda{}_\mu)p_\lambda + \frac{\hbar^2}{2}\partial_\lambda(\partial_\rho e^\nu{}_\mu \cdot \partial_\nu g^{\rho\lambda}) \\ &- \frac{\hbar^2}{2}\partial_\lambda(\partial_\rho e^\rho{}_\alpha \cdot \partial'_\mu g'^{\alpha\beta} \cdot e^\lambda{}_\beta) + \frac{\hbar^2}{4}\partial_\rho e^\rho{}_\alpha \cdot \partial'_\mu g'^{\alpha\beta} \cdot \partial_\lambda e^\lambda{}_\beta\}.\end{aligned} \qquad (A.3)$$

As to the last term in r.h.s. of (A.1), with the use of $\Gamma'_\alpha = \Gamma_\gamma e^\gamma{}_\alpha + \partial_\gamma e^\gamma{}_\alpha$, we obtain

$$\begin{aligned}&\frac{\hbar^2}{2m}\{\frac{1}{2}\partial'_\mu\partial'_\beta(\Gamma'_\alpha g'^{\alpha\beta}) + \frac{1}{4}\partial'_\mu(g'^{\alpha\beta}\Gamma'_\alpha\Gamma'_\beta) - e^\nu{}_\mu[\frac{1}{2}\partial_\nu\partial_\beta(\Gamma_\alpha g^{\alpha\beta}) + \frac{1}{4}\partial_\nu(g^{\alpha\beta}\Gamma_\alpha\Gamma_\beta)]\} \\ &= \frac{\hbar^2}{2m}\partial'_\mu\{-\frac{1}{4}\partial_\gamma e^\gamma{}_\alpha \cdot \partial_\lambda e^\lambda{}_\beta \cdot g'^{\alpha\beta} + \frac{1}{2}\partial_\lambda(e^\lambda{}_\beta\partial_\gamma e^\gamma{}_\alpha \cdot g'^{\alpha\beta})\}.\end{aligned} \qquad (A.4)$$

Summing (A.2), (A.3) and (A.4), we obtain

$$\begin{aligned}(\frac{\delta p_\mu}{d\tau})'_Q - < e^\nu{}_\mu, (\frac{\delta p_\mu}{d\tau})_Q > = \frac{\hbar^2}{2m}\frac{1}{4}\{&-2\partial_\lambda(\partial_\rho\partial_\nu e^\nu{}_\mu \cdot e^\rho{}_\alpha e^\lambda{}_\beta g'^{\alpha\beta}) \\ &- 2\partial_\lambda(\partial_\rho e^\rho{}_\alpha \cdot \partial'_\mu g'^{\alpha\beta} \cdot e^\lambda{}_\beta) + \partial_\rho e^\rho{}_\alpha \cdot \partial'_\mu g'^{\alpha\beta} \cdot \partial_\lambda e^\lambda{}_\beta \\ &+ 2\partial'_\mu\partial_\lambda(\partial_\rho e^\rho{}_\alpha \cdot g'^{\alpha\beta} \cdot e^\lambda{}_\beta) - \partial'_\mu(e^\rho{}_\alpha \cdot g'^{\alpha\beta} \cdot \partial_\lambda e^\lambda{}_\beta)\};\end{aligned} \quad (A.5)$$

the first term in the curly bracket of (A.5) is

$$-2\partial'_\beta[\partial'_\alpha(\partial_\nu e^\nu{}_\mu) \cdot g'^{\alpha\beta}] - 2\partial'_\alpha(\partial_\nu e^\nu{}_\mu) \cdot \partial_\lambda e^\lambda{}_\beta \cdot g'^{\alpha\beta}, \quad (A.6a)$$

and the remaining 4 terms in the curly bracket of (A.5) is

$$\partial'_\mu[\partial_\lambda e^\lambda{}_\beta \cdot \partial_\nu e^\nu{}_\alpha] \cdot g'^{\alpha\beta} + 2\partial'_\beta[\partial'_\mu(\partial_\nu e^\nu{}_\alpha) \cdot g'^{\alpha\beta}]; \quad (A.6b)$$

thus (A.6a)+(A.6b)=0, because

$$\begin{aligned}\partial'_\mu(\partial_\nu e^\nu{}_\alpha) - \partial'_\alpha(\partial_\nu e^\nu{}_\mu) &= \partial_\nu(e^\lambda{}_\mu\partial_\lambda e^\nu{}_\alpha) - \partial_\nu(e^\lambda{}_\alpha\partial_\lambda e^\nu{}_\mu) \\ &= \partial_\nu(\partial'_\mu e^\nu{}_\alpha - \partial'_\alpha e^\nu{}_\mu) \\ &= 0, \quad [\text{duc} \quad \text{to} \quad (3.7\text{c})].\end{aligned} \quad (A.6c)$$

Therefore we see

$$(\frac{\delta p_\mu}{d\tau})_Q = < e^\nu{}_\mu, (\frac{\delta p_\nu}{d\tau})_Q > . \quad (A.7)$$

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