Equivalence Principle in Quantum Mechanics

KIMIHIDE NISHIMURA

Institute for Nuclear Study, University of Tokyo
Midori-cho, Tanashi, Tokyo, 188, Japan

ABSTRACT

Quantum mechanics in curved space obtained by rewriting a time evolution kernel in a covariant form is investigated and compared with other quantization methods. The possibility to observe a quantum-gravitational effect is also discussed.

An attempt to conform quantum mechanics to general relativity has caused controversy and criticism for many years. Usually the problem has been regarded as a "quantization" prescription in a curved manifold *i.e.*, the WKB approximation [1], the geometric quantization [2], and Ashtekàr's formulation [3]. For a spinless particle in a curved manifold with the metric g_{ij} of N-dimension, these methods give a Hamiltonian operator:

$$\hat{H} = -\frac{\hbar^2}{2m}(\Delta + \xi R), \tag{1}$$

by quantizing a classical Hamiltonian: $H(q,p) = g^{ij}(q)p_i p_j/2m$, $(i = 1, 2, \ldots, N)$, where a scalar curvature R emerges as a quantum correction of order $\hbar^2$. A numerical constant ξ is $-1/6$ for the WKB method [4]. Since the WKB method is the first order approximation of the $\hbar$-expansion, the figure is not necessarily reliable. The geometric quantization method also gives the same result for this system [2].

On the other hand, the extension of canonical commutation relations to curved space [3] can not determine ξ because of the following reason. Let us consider about a particle localized around some point in curved space. The quantity $m\frac{d}{dt}\langle q^i\rangle$'s for the state can be regarded as the expectation value of momentum operators. From this relation of Ehrenfest, we obtain $\hat{p}^i := \frac{m}{\sqrt{-1}\hbar}[q^i, \hat{H}]$ which is determined independent of a scalar curvature term in (1). Commutation relations among these operators are therefore the same for arbitrary value of ξ, and conversely, the canonical commutation relations do not determine the unique quantum Hamiltonian.

Here, we treat the problem as "covariantization" of quantum Hamiltonian rather than quantization in curved space, or the extension of canonical commutation relations, and we determine the constant. Since quantum mechanics for a spinless particle is already known in flat space, the remaining problem is the way to take effects of curved space into it.

We first start with notations. Let (Q^a, P_a)'s denote the set of canonical variables referring to a Cartesian system, and (q^i, p_i)'s to curvilinear coordinates. Two sets of variables are in the relations: $Q^a = F^a(q)$, $p_i = (\partial F^a/\partial q^i)(q)P_a$. In quantum mechanics, corresponding observables $(\hat{Q}^a, \hat{P}_a)$, $(\hat{q}^i, \hat{p}_i)$ are thought to exist. We impose canonical commutation relations : $[\hat{Q}^a, \hat{P}_b] = \sqrt{-1}\hbar\delta^a{}_b$ *etc.* , only in a Cartesian system and regard $Q^a = F^a(q)$'s as operator equations. Let $|Q\rangle$ denote a simultaneous eigen-ket of $\hat{Q}^a$'s with normalization $\langle Q|Q'\rangle = \delta^N(Q - Q')$. The eigen-kets $|q\rangle$'s for $\hat{q}^i$'s are defined by $\langle q|Q\rangle = \delta^N(Q^a - F^a(q))$. Then $|q\rangle$'s are normalized as $\langle q|q'\rangle = \frac{1}{\sqrt{g(q)}}\delta^N(q - q')$, where $g_{ij}(q) := \delta_{ab}(\partial F^a/\partial q^i)(\partial F^b/\partial q^j)$. The completeness for eigen-kets: $\hat{1} = \int d^N Q|Q\rangle\langle Q| = \int d^N q\sqrt{g(q)}|q\rangle\langle q|$ is understood. We adopt these two sets of eigen-kets as basic vectors fixing representations in the respective frames.

Then we can show the following relation for matrix elements of Hamiltonians:

$$\langle q|\hat{H}(\hat{q}, \hat{p})|q'\rangle = \langle Q|\hat{H}_D(\hat{Q}, \hat{P})|Q'\rangle\Big|_{Q^a=F^a(q), Q'^a=F^a(q')}, \tag{2}$$

where $\hat{H}(\hat{q}, \hat{p})$ and $\hat{H}_D(\hat{Q}, \hat{P})$ are the quantum Hamiltonians in curvilinear and Cartesian coordinates, respectively. This relation is obtained by simply rewriting Schrödinger's equation for $\psi(t, Q) := \langle Q|t, \psi\rangle$ to that for $\psi(t, q) := \langle q|t, \psi\rangle$.

We from now on evaluate matrix elements of an infinitesimal time evolution operator: $\exp[-\frac{\sqrt{-1}}{\hbar}\epsilon\hat{H}]$, instead of $\hat{H}$. Simple changes of variables lead to

$$\langle q|\exp[-\frac{\sqrt{-1}}{\hbar}\epsilon\hat{H}]|q'\rangle = \int \frac{d^N p'}{(2\pi\hbar)^N\sqrt{g(q')}} \exp\frac{\sqrt{-1}}{\hbar}\epsilon[p'_i v^i - g^{ij}(q')p'_i p'_j/2m], \tag{3}$$

where the complete set of momentum eigenstates in a Cartesian system have been inserted. The remaining definition

$$\begin{aligned} \epsilon v^i &:= \Delta q^i + \frac{1}{2}\Gamma^i_{jk}(q')\Delta q^j\Delta q^k \\ &+\frac{1}{6}[\Gamma^i_{jk,l}(q') + \Gamma^i_{jr}(q')\Gamma^r_{kl}(q')]\Delta q^j\Delta q^k\Delta q^l, \qquad (\Delta q^i := q^i - q'^i) \end{aligned} \tag{4}$$

is obtained if one expands the right hand side of $\epsilon p'_i v^i := P'_a(F^a(q) - F^a(q'))$ in Taylar series around q'^i and rewrites it in Christoffel symbols which can be

obtained from the definition of $g_{ij}(q)$, simply by differentiations and algebraic manipulations. Fourth and higher order terms have been neglected, because they do not affect the property of the time evolution within order ϵ .

Although the equation (3) is valid only in flat space, we from now on assume it is valid also in curved space from the viewpoint of the equivalence principle. The principle of equivalence is certainly for classical mechanics, which states laws of physics expressed in local inertial coordinates should coincide with that in flat space. According to this principle, generalization is unique only when the tensorial extension of equations has no fear to contain second order derivative terms of the metric. Our situation is, in the strict sense, beyond the scope of the principle. Nevertheless, because of the following reason, we adopt (4) for the relation in curved space .

Suppose that a classical particle moving in a curved manifold with the metric $g_{ij}(q)$ starts from a point q'^i at $t=0$ and reaches to a point q^i at $t=\epsilon$. From the equation of motion $\frac{d^2q^i}{dt^2}+\Gamma^i_{jk}(q)\frac{dq^j}{dt}\frac{dq^k}{dt}=0$, the classical orbit $q^i_c(t)$ is obtained as

$$\begin{aligned}
q^i_c(t) =& q'^i + t\dot{q}^i(0) + \frac{1}{2}t^2\ddot{q}^i(0) + \frac{1}{3!}t^3\dddot{q}^i(0) + \ldots, \\
\dot{q}^i(0) :=& v^i, \\
\epsilon^2\ddot{q}^i(0) :=& -\Gamma^i_{jk}(q')\Delta q^j\Delta q^k - \Gamma^i_{jr}(q')\Gamma^r_{kl}(q')\Delta q^j\Delta q^k\Delta q^l, \\
\epsilon^3\dddot{q}^i(0) :=& [-\Gamma^i_{jk,l}(q') + 2\Gamma^i_{jr}(q')\Gamma^r_{kl}(q')]\Delta q^j\Delta q^k\Delta q^l,
\end{aligned} \tag{5}$$

which is accurate up to order $(\Delta q)^3$. Then we can interpret v^i as an initial velocity of a particle moving from q'^i to q^i along geodesics, and it is a vectorial quantity within a required accuracy. This observation in turn suggests that the discretization prescription $\int_0^\epsilon dt p_i\dot{q}^i \to \epsilon p_i v^i$ is taken in path integral quantization. After performing momentum integrations, the kernel becomes

$$\left(\frac{m}{2\pi\sqrt{-1}\hbar\epsilon}\right)^{N/2}\exp\left[\frac{\sqrt{-1}}{\hbar}S(t+\epsilon,q;t,q')\right], \tag{6}$$

where,

$$S(t+\epsilon,q;t,q') := (\frac{m}{2\epsilon})[g_{ij}(q')\Delta q^i\Delta q^j + g_{ir}\Gamma^r_{jk}(q')\Delta q^i\Delta q^j\Delta q^k$$
$$+\{\frac{1}{3}g_{ir}(\Gamma^r_{jk,l}(q') + \Gamma^r_{js}(q')\Gamma^s_{kl}(q')) + \frac{1}{4}g_{rs}\Gamma^r_{ij}(q')\Gamma^s_{kl}(q')\}\Delta q^i\Delta q^j\Delta q^k\Delta q^l], \tag{7}$$

up to the required accuracy. $S(t,q;t',q')$ is the extremum of the action and a solution of the Hamilton-Jacobi equation. From (7) ,we have $\xi = -1/3$ which is twice as large as the previous result, where Ricci tensor is defined by $R_{ij} := R^k_{ikj} = \Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^k_{jl}\Gamma^l_{ik}$.

For a case of a relativistic particle in curved space-time with the metric $g_{\mu\nu}$ of D-dimension, our result is the following:

$$[\Box + \frac{m^2c^2}{\hbar^2} - \frac{1}{3}R]\phi(x) = 0, \tag{8}$$

where sign$(g) = (+,-,\ldots,-)$. This result is obtained by the path integral method using an auxiliary field [6], or a trick using a proper time [1]. In the context of the quantum field theory in a curved space-time, the coupling constant of a scalar curvature term is, in massless case, usually discussed from the viewpoint of the invariance under the conformal transformation: $g_{\mu\nu} \to e^{\varphi}g_{\mu\nu}$ [7], which is determined as $-\frac{1}{4}(D-2)/(D-1)$. We find the equation (8) is therefore not conformally invariant even in the massless case, while $\xi = -1/6$ from old methods is conformal in $D = 4$.

There exists criticism that "gravitational forces are quite unimportant in atomic phenomena"[5]. In other words, it has been thought that the resolution of the problem for things which can not be checked by experiments nor observations is nothing worth except for the theoretical self-consistency. Can quantum-gravitational effects never really be observed; Is quantum mechanics nothing to do with gravity ? Then next we consider about the possibility to observe quantum-gravitational effects caused by a scalar curvature term.

As we see from (8), this effect can be interpreted as a correction to an inertial mass. If the following effective mass

$$m_{\rm eff}^2 := m^2 - \frac{1}{3}(\frac{\hbar}{c})^2 R, \tag{9}$$

is defined, an increase of it becomes

$$\Delta m_{\rm eff} c^2 \simeq \frac{4\pi}{3}\ell_{\rm p}^2 \ell_{\rm c} T, \tag{10}$$

where $\ell_{\rm p}$ and $\ell_{\rm c}$ are the Planck and the Compton length respectively, and T is the trace of the energy-momentum tensor. We have used Einstein's equation of gravity. If the mass density $\rho \simeq 10^{18}{\rm kg/m^3}$ of a neutron star [8] is used as the approximated value of T, the ratio of increase for the effective mass of an electron is found to be $\Delta m_{\rm eff}/m_e \simeq 10^{-35}$, which may hardly be observed.

For a massless particle, neutrinos for example, the effective mass is given by

$$m_\nu = \ell_{\rm p}\sqrt{\frac{8\pi\hbar T}{3c}}. \tag{11}$$

Then we find $m_\nu \simeq 5.4 \times 10^{-18}{\rm eV}$ for $\rho \simeq 1.2 \times 10^5{\rm kg/m^3}$ which is the mass density at the center of the sun, and $m_\nu \simeq 10^{-12} \sim 10^{-11}{\rm eV}$ for the density of a neutron star. Neutrinos are thought to be most suitable for observation of quantum-gravitational effects, because they suffer only weak interactions besides gravitational interactions. In this case, the observability depends on how much influence neutrinos' mass makes on nuclear interactions in high density stars, and the neutrino flux from it. Further investigations are therefore needed.

REFERENCES

1. L.S. Schulman, *Techniques and Applications of Path Integration*, John Wiley & Sons, Inc. (1981). References therein.
2. J. Sniatycki, *Geometric Quantization and Quantum Mechanics*, Springer-Verlag, (1980).
3. A. Ashtekar, Appendix C in *Lectures on Non-Perturbative Canonical Gravity*, World Scientific, (1991).
4. B.S. DeWitt, Rev. Mod. Phys. **29**, 377 (1957).
5. P.A.M. Dirac, *The Principles of Quantum Mechanics* , Oxford Univ. press (1930).
6. A. Cohen, G. Moore, P. Nelson and J. Polchinski, Nucl. Phys. **B267** 143 (1986).
7. N.D. Birrell and P.C.W. Davies,*Quantum Fields in Curved Space*, Cambridge Univ. press, (1982).
8. S. Weinberg, *Gravitation and Cosmology*, John Wiley & Sons, Inc, (1972).