Coherent states, Path integral, and Semiclassical approximation

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1 Introduction

Physical systems in actual situation are so complicated that there usually need some approximation. In path integral the most promissing one seems to be a semiclassical approximation. And there are some systems in which the WKB approximation gives the exact result such as a harmonic oscillator.

Recently a new possibility of exactness of the WKB approximation is discussed in connection with the theorem of Duistermaat-Heckman about $SU(2)$ spin system or Cp^N system. But there are a number of unsatisfactory points and mysteries in the discussion: starting action being already semiclassical, subtlety of the use of continuum path integral under the coherent state representation, and so called overspecification problem and so on.

Motivated these we study the exactness of the WKB approximation under the path integral formula in the case of $SU(2)$ and $SU(1,1)$ with the aid of generalized coherent states, and in this papar we concentrate the WKB exactness in our method.

2 Coherent states and path integral formulae

$su(2)$ algebra reads

$$[J_+, J_-] = 2J_3, \quad [J_3, J_\pm] = \pm J_\pm, \tag{1}$$

where $J_\pm \equiv J_1 \pm iJ_2$. We take a representation as usual,

$$J_3 \,|J,\ M\rangle = M\,|J,\ M\rangle\,,\ J_\pm\,|J,\ M\rangle = \sqrt{(J\mp M)(J\pm M+1)}\,|J,\ M\pm 1\rangle\,,\ (|M|\leq J). \tag{2}$$

Let us define

$$|\xi\rangle \equiv \left(\langle J,-J|\exp^{\xi^* J_-}\exp^{\xi J_+}|J,-J\rangle\right)^{-1/2}\exp^{\xi J_+}|J,-J\rangle \quad , \tag{3}$$

where we have introduced $|J,\ -J\rangle$ as the fiducial vector. They satisfy

$$\frac{2J+1}{\pi}\int\frac{d\xi^* d\xi}{(1+|\xi|^2)^2}\,|\xi\rangle\,\langle\xi| \equiv \int d\mu(\xi^*,\xi)\,|\xi\rangle\,\langle\xi| = \mathbf{1}_J \quad , \tag{4}$$

where $d\xi^* d\xi \equiv d\mathrm{Re}(\xi)d\mathrm{Im}(\xi)$ and $\mathbf{1}_J$ $(\equiv \Sigma_{M=-J}^{J}|J,M\rangle\langle J,M|)$ is the identity operator in $2J+1$-dimensional irreducible representation.

Using the above state we discuss the path integral formula for a Hamiltonian $H = hJ_3 \in su(2)$ because within the trace we can always diagonalize it by use of $SU(2)$ rotation. Then the trace formula becomes

$$Z(T) \equiv \mathrm{Tr}e^{-iHT} = \mathrm{Tr}\lim_{N\to\infty}(1-i\epsilon H)^N = \lim_{N\to\infty} Z_N, \tag{5}$$

$$Z_N \equiv \prod_{j=1_{\mathrm{PBC}}}^{N}\int d\mu\left(\xi_j^*,\xi_j\right)\exp\left[iJ\left\{2i\ln\left(\frac{1+\xi_j^*\xi_j}{1+\xi_j^*\xi_{j-1}}\right)+\epsilon h\frac{1-\xi_j^*\xi_{j-1}}{1+\xi_j^*\xi_{j-1}}\right\}\right] \quad . \tag{6}$$

Here we have repeatedly inserted the resolution of unity (4) into the second relation in (5) and PBC means periodic boundary condition. This trace is equivalent to the character formula;

$$Z(T) = \mathrm{Tr}\exp(-ihJ_3T) = \frac{\sin((J+1/2)hT)}{\sin(hT/2)}. \tag{7}$$

In the following we shall consider the case that J becomes large; where the saddle point of the exponent in (6), leaving that finite, is important. With satisfying PBC, they are

$$\xi_j^c = 0 \equiv \xi_c^{(+)}\ , \quad \xi_j^c = \infty \equiv \xi_c^{(-)} \quad . \tag{8}$$

We can also argue $SU(1,1)$ system in the same way as $SU(2)$. Its algebra reads

$$[K_+,K_-] = 2K_3, \quad [K_3,K_\pm] = \pm K_\pm; \quad K_\pm \equiv \pm(K_1 \pm iK_2)\ ; (K_+)^\dagger = -K_-, \tag{9}$$

We confine ourselves in a discrete series to write

$$\begin{aligned} \mathbf{K}^2|K,\ M\rangle &= K(1-K)|K,\ M\rangle, \quad K_3|K,\ M\rangle = M|K,\ M\rangle, \\ K_\pm|K,\ M\rangle &= \pm\sqrt{(M\pm K)(M\mp K\pm 1)}|K,M\pm 1\rangle, \\ M &= K,\ K+1,\ K+2,\ \dots \quad , \quad K = 1/2,\ 1,\ 3/2,\ 2,\ \dots \quad . \end{aligned} \tag{10}$$

Adopting $|K,\ K\rangle$ as the fiducial vector, we construct the normalized states to replace the fiducial vector and $J_+(J_-)$ by $K_+(K_-)$ in (3). Then the trace formula becomes

$$Z(T) = \mathrm{Tr}e^{-iHT} = \lim_{N\to\infty} Z_N, \tag{11}$$

$$Z_N \equiv \prod_{j=1_{\mathrm{PBC}}}^{N}\int d\mu\left(\xi_j^*,\xi_j\right)\exp\left[iK\left\{2i\ln\left(\frac{1-\xi_j^*\xi_{j-1}}{1-\xi_j^*\xi_j}\right)-\epsilon h\frac{1+\xi_j^*\xi_{j-1}}{1-\xi_j^*\xi_{j-1}}\right\}\right]. \tag{12}$$

with the Hamiltonian of the form $H = hK_3$. In this case the equation of motion has only one solutions which satisfy the finiteness of the exponent as well as the periodic boundary condition

$$\xi_j^c = 0 \quad . \tag{13}$$

3 Exactness of the WKB approximation

The WKB approximation is valid as the saddle point method when $J(K)$ becomes large in $SU(2)$ ($SU(1,1)$) case. To make the expansion transparent, let us put

$$\xi_j = \sqrt{\kappa} z_j, \tag{14}$$

in each trace formula such as (6). Here $\kappa \equiv 1/2J+1$ $(1/2K-1)$ for $SU(2)(SU(1,1))$. Plugging (14) into(6) and expanding the logarithm, we have

$$\begin{aligned} Z_N \equiv \; & e^{ihJT} \prod_{j=1}^{N} \int \frac{dz_j^* dz_j}{\pi} \exp\Bigg[-(z_j^* z_j - e^{-i\epsilon h} z_j^* z_{j-1}) \\ & + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \kappa^n \left[\left\{ \left(z_j^* z_j\right)^n + \left(e^{-i\epsilon h} z_j^* z_{j-1}\right)^n \right\} \right. \\ & \left. - \frac{n}{n+1} \left\{ \left(z_j^* z_j\right)^{n+1} - \left(e^{-i\epsilon h} z_j^* z_{j-1}\right)^{n+1} \right\} \right] \Bigg] \quad , \end{aligned} \tag{15}$$

where we have discarded $O(\epsilon^2)$ terms to arrange the expression. Now write Z_N as

$$Z_N = e^{ihJT} \sum_{n=0}^{\infty} \frac{\kappa^n}{n!} Z_{N,n}. \tag{16}$$

After some lengthy calculations we can prove

$$Z_{N,n} = \frac{\delta_{n,0}}{1 - e^{-ihT}} \quad . \tag{17}$$

About other critical points we obtain the similar results, so we find

$$\begin{aligned} Z_N &= \frac{e^{ihJT}}{1-e^{-ihT}} + \frac{e^{-ihJT}}{1-e^{ihT}} = \frac{\sin((J+1/2)hT)}{\sin(hT/2)} \quad : \quad SU(2) \,, \\ Z_N &= \frac{e^{-ihKT}}{1-e^{-ihT}} \quad : \quad SU(1,1). \end{aligned} \tag{18}$$

Therefore the conclusion is that *the WKB is exact under coherent state path integrals for $SU(2)$ and $SU(1,1)$(in the discrete series).*

4 Discussion

We consider the WKB exactness in other points of view. Making a change of variables

$$\xi_j \quad \mapsto \quad \xi_j e^{-ij\epsilon h} \quad , \tag{19}$$

and taking into account the measure being local $U(1)$ invariant, (6) is rewritten by only one integration variable as

$$Z(T) = e^{ihJT} \int d\mu\,(\xi^*, \xi)\, \langle \xi | \xi e^{-ihT} \rangle \ , \tag{20}$$

After carrying out the trivial integration of the phase of ξ, we obtain the character formula again:

$$Z(T) = e^{ihJT}(2J+1) \int_0^\infty \frac{du}{(1+u)^2} \left(\frac{1+e^{-ihT}u}{1+u} \right)^{2J} = \frac{\sin((J+1/2)hT)}{\sin(hT/2)} \quad . \tag{21}$$

Further let us take another point of view: Again making a change of variable

$$u \quad \mapsto \quad z = -\ln\left(\frac{1+e^{-ihT}u}{1+u} \right), \tag{22}$$

gives us the relation

$$\begin{aligned} Z(T) &= \frac{\sin((J+1/2)hT)}{\sin(hT/2)} = \frac{e^{ihJT}}{1-e^{-ihT}} + \frac{e^{-ihJT}}{1-e^{ihT}} \\ &= \mathrm{Tr} e^{-ih(a^\dagger a - J)T} + \mathrm{Tr} e^{ih(a^\dagger a - J)T} \ . \end{aligned} \tag{23}$$

Hence it is now clear that the character formula can be expressed as the sum of partition functions of two harmonic oscillators with frequency $\pm h$, which correspond the contributions of each saddle point.

The final comment is on the difference between the continuum path integral and (6); namely if we had used the continuum path integral formula, we would get $\sin(JT)/\sin(T/2)$ instead of the correct one; $\sin((J+1/2))\,T/\sin(T/2)$. The models which we have been considering are very much alike to (three dimensional) Chern-Simons theory. So the issue that $J \to J + 1/2$(Weyl shift) may correspond to the Coxeter shift, $k \to k+2$ (where k denotes a level) in the Chern-Simons case. Thus if it would be possible to perform the integration in discretized version of the Chern-Simons theory, we could get the correct value $k+2$, which will be an interesting subject in the future.

Another task for us is to generalize our discussion to the case of Grassmannian manifold.