MKdV Equation and Submanifold Quantum Mechanics.

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Abstract.
In this note, I will show that the differential operator appearing in the submanifold quantum mechanics reflects the geometrical properties of the system and it is regarded as a natural or algebraic object; the Dirac operator defined over a thin elastic rod agrees with the Lax operator of the MKdV equation and the new index theorem related to the submanifold geometry is found.

§1. Introduction.

It is known that the physical state of an electron in the hydrogen atom is regarded as a vector in the representation space of the so(3). The Lie algebra so(3) and the algebra of the differential operators in the hydrogen atom are homomorphic.

The differential equation is, in general, an analytic object. However the differential operator appearing in the quantum mechanics should also be regarded as an algebraic object due to Heisenberg theory. Heisenberg showed that an operator in the quantum mechanics is written by a matrix, even though its row and column may be infinite, while Schrödiger expressed it in terms of the differential operator. The matrix in the Heisenberg theory should be regarded as an element of a representation of an algebra.

After unification of the fields in the mathematics, the algebraic structures for an (physical) object sometimes give the same invariant. For example, the algebraic structures of the fermionic (analytic) differential operator with a gauge field and of the gauge transformation give the same invariant due to the Atiyah-Singer index theorem and the theory of an anomaly. Furthermore physicists deal with the local properties in the system rather than the global one; although the U(1)-gauge field sometimes gives a trivial index, there appears a U(1) chiral anomaly locally.

Thus when a differential operator appears in physics, one has a question whether it is a natural object or not; does the differential operator reflect the algebraic structure of the system, e.g. the geometrical properties?

On the other hand in this decade, the submanifold quantum mechanics is developed [1]. After we quantize a particle in $\mathbb{R}^n$, we confine it in a submanifold embedded in $\mathbb{R}^n$. Then there appears the effective potential related to the affine curvature of the submanifold.

Our purpose in this note is to show that the differential operator appearing in the submanifold quantum mechanics reflects the property of the base submanifold. Furthermore I will investigate the local properties of the relation between them.

Along the submanifold quantum mechanics scheme, we will consider the Dirac operator defined over a thin elastic rod [2-5]. Then we find that the algebraic relation of the fermionic operators restores the equation of the motion of the elasitca, i.e. the MKdV equation; the Dirac operator, given through the submanifold quantum mechanics scheme, and the operator, preserving the adiabatic condition, are regarded as the Lax pair of the MKdV soliton. Through the investigation of the Dirac operator, we rediscover the various properties of the soliton theory of the MKdV equation. If one quantizes the classical Dirac field, he naturally finds the Jimbo-Miwa-Sato theory or the Hirota bilinear equation [2-7].

Furthermore, as the global property of the system, we obtain the Atiyah-Singer-type index theorem related to the topology of the submanifold; by dealing with the Dirac operator, we find the sum of the signed crossing points.

§2. Classical Submanifold Field Theory.

The submanifold quantum mechanics is constructed as follows [1];

1) We construct the quantum equation in $\mathbb{R}^n$ with the ordinary metric.

2) We embed (exactly speaking, immerse) a differential submanifold in the $\mathbb{R}^n$.

3) We define the coordinate system along the submanifold in its tubular neighborhood.

4) We express the quantum equation in terms of the curved coordinate so that the normal component in the equation is expressed by the momentum operator of the normal direction.

5) We restrict the quantum equation along the submanifold.

Physically speaking, these processes are naturally performed when we introduce the confinement potential along the submanifold with the same thin thickness.

Along the scheme [2-5], we obtain the Dirac operator on a thin rod in $\mathbb{R}^2$ as

$$\mathcal{D}_0 := \partial_0, \qquad \mathcal{D}_1 := \partial_1 + \mathrm{i}v\gamma^0, \tag{2-1}$$

where $v := k/2$ and k is the curvature of the rod. q^0 is the time coordinate and $q^1 \equiv s$ is the arclength of the rod. Here we neglect the effective mass term. The Dirac equation for the classical field is

$$\not{\mathcal{D}}\psi = 0. \tag{2-2}$$

This equation is valid even if the base space, the rod, is slowly moving. If the rod is an elastica, its curvature $k(q^1) = 2v(q^1)$ obeys the MKdV equation [2-5],

$$\partial_t v + 6v^2\partial_1^3 v + \partial_1^3 v = 0 \tag{2-3}$$

where t is the time of the elastica and an adiabatic parameter in the fermionic system.

Then the equation (2-2) agrees with the Lax's eigen equation of the MKdV equation; $E\phi = L\phi$, $L := \gamma^0\gamma^1 D_1$ if $\psi = \mathrm{e}^{\mathrm{i}Eq^0}\phi$. The adiabatic condition for the sufficiently long elastica is identified with

$$\partial_t E = 0, \qquad \partial_t L = [L, B] \quad \text{and} \quad \partial_t\phi = B\phi. \tag{2-4}$$

Here B is the partner of L in the Lax pair of the MKdV equation;

$$B = -4\mathrm{i}\partial_s^3 - 3\mathrm{i}\partial_s(v^2 - \mathrm{i}(\partial_s v)\gamma^0 - 3\mathrm{i}(v^2 - \mathrm{i}(\partial_s v)\gamma^0)\partial_s. \tag{2-5}$$

(2-4) restores the equation of the motion of the base space (2-3). Here due to (2-4) B operator can be regarded as the generator of the geometric phase [2-4].

Thus we can find the physical meaning of the Lax pairs and physically reconstruct the inverse scattering method for the MKdV equation [2-5].

§3. Quantum Submanifold Field Theory.

In this section, we will quantize the fermionic field [3,5]; we will consider the infinite product of the eigenvalue of the operator. The partition function is defined as

$$Z[v] = \det\not{\mathcal{D}}. \tag{3-1}$$

For the infinitesimal gauge transformation,

$$v(s,t) \to v'(s,t) = v(s,t) + \partial_s \alpha, \tag{3-2}$$

the partition function becomes

$$\begin{aligned} Z[v'] &= \det(\not{\mathcal{D}} + \mathrm{i}\gamma^2 \partial_s \alpha) \\ &= \det \not{\mathcal{D}} \mathrm{e}^{\mathrm{i}\Phi[\alpha]}. \end{aligned} \tag{3-3}$$

The phase Φ is determined through the transformation in the infinite dimensional fermionic functional space. From the first expression in (3-3), we define the current,

$$\partial_s < j_2 > := \frac{\delta}{\delta \alpha} Z[v']|_{\alpha=0}. \tag{3-4}$$

In terms of proper reguralization, from the second expression in (3-3) we obtain the geometrical term. The phase factor is obtained as the jacobian of the infinite grassmannian space related to the gauge transformation (3-2). This jacobian is intrisically the same as that in the Jimbo-Miwa-Sato theory [6,7].

Due to the equal of the both expressions, we obtain an identity [3];

$$< j_2 > = \frac{1}{2\pi} v. \tag{3-5}$$

This is a kind of the anomalous terms i.e. an anomaly. This form is the same as the well-known bosonaization. The rhs is the MKdV soliton and the lhs is expressed by the linear differential system in the infinite fermionic space. In terms of this scheme, we can reconstruct the Jimbo-Miwa-Sato theory on the MKdV equation [5-7]. ((3-5) is valid for more general v but the restriction of the function space, say, is sometimes more interest than its generalization.)

Then we obtain the index theorem for the submanifold,

$$\int < j_2 > \mathrm{d}s = \frac{1}{2\pi}(\varphi(l) - \varphi(0)). \tag{3-6}$$

where φ is the tangential angle along the closed elastica and l is its length. The rhs of (3-6) gives the sum of the signed crossing points and is an integer.

§4. Discussion.

We have shown that the differential operator in the submanifold quantum system reflects the global and the local geometrical properties. Hence although one may think that the construction of the operator appearing in the submanifold quantum mechanics is so artificial, it should be regarded as a natural and algebraic object [2-5].

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Reference.

1. R. C. T. da Costa, Phys. Rev. A **23** (1981) 1982, refs. in [2-5].
2. S. Matsutani and H. Tsuru, Phys. Rev **A 46** (1992) 1144.
3. S. Matsutani, Prog. Theor. Phys. **91** (1994) 1005.
4. S. Matsutani, Phys. Lett. A **189** (1994) 27.
5. S. Matsutani, preprint.
6. M. Sato, RIMS Lecture note 5. (1989). (in Japanese)
7. M. Jimbo and T. Miwa, Publ. RIMS. Kyoto Univ. **19** (1983), 943.