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Quantum Mechanics of a Particle Confined to a Curved Surface

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§1. Can One Hear the Twist of a Brass[Kac]?

Consider a wavicle [Eddington] of unit mass constrained to a closed curved wire ("quantum loop") by a physical confining potential (e.g., electrostatic binding potential). In the thin-wire limit in which the thickness of the wire is much less than the radius of curvature of the closed curve C, where C is the mathematical curve to which the wire reduces as its thickness tends to zero, the effective Schrödinger equation describing the wavicle's motion along C takes the form [Takagi & Tanzawa]

$$i\hbar\frac{\partial}{\partial t}\psi(s,t)=\{-\frac{\hbar^2}{2}(\frac{\partial}{\partial s}-iA(s))^2-\frac{\hbar^2}{8}\kappa^2(s)\}\psi(s,t), \tag{1a}$$

where s is the distance measured along C, $\kappa(s)$ is the local curvature of C, and $A(s)$ is a function which is locally arbitrary but is globally constrained by the condition

$$\int_C dsA(s)=m\int_C ds\tau(s), \tag{1b}$$

where $\tau(s)$ is the local torsion of C and $m\hbar$ is the angular-momentum of the wavicle along the local tangent of C; this angular momentum comes not from the motion along C but from the motion within the cross-section of the wire. Note that $\kappa(s)$ and $\tau(s)$ are locally fixed by the 3-geometry of C, while $A(s)$ is only globally fixed. We may summarize Eq.(1) as expressing a 3-geometry-induced Aharonov-Bohm-like effect. This is reminiscent of an analogous effect by a dislocation in solids [Kawamura; Araki et al]. Equation (1) constitutes a straightforward transcription of the parallel-transport anholonomy to a wavicle in a closed wire, and provides an example of Berry's phase. We shall not elaborate on the derivation of Eq.(1) here, since its higher-dimensional generalization is going to be discussed by the next speaker. Prof. K. Fujii.

§2. An Attractive Dune

Equation (1a) tells us that the curvature of C gives rise to an effective attraction. As explained by Prof. Nagaoka this morning, a corresponding effect occurs if a wavicle is physically constrained to a curved membrane. In the thin-membrane limit, the wavicle feels the attractive potential of the form

$$-\frac{\hbar^2}{8}(\kappa_1 - \kappa_2)^2, \tag{2}$$

where κ_1^{-1} and κ_2^{-1} are the local principal radii of curvature of the surface $\sum$ to which the membrane reduces as its thickness tends to zero. Note that the curvature in question is not intrinsic but extrinsic.

One would wonder why the extrinsic curvature gives rise to attraction rather than repulsion. In my opinion, unforturnately, no intuitive answer exists to this question. For, a straightforward generalization of Eq.(2) to the case of a (N-1)-dimensional surface (as the limit of thin membrane) in the N-dimensional Euclidean space $\mathbf{E}^N$ gives the effective potential [Jensen & Koppe]

$$V_c = -\frac{\hbar^2}{8}\{2Tr(\mathbf{K}^2) - (Tr\mathbf{K})^2\}, \tag{3}$$

where $\mathbf{K}$ is the extrinsic-curvature tensor and Tr denotes trace; the subscript c to V signifies that it is obtained by constraing the wavicle to the membrane by a physical *confining* potential. As we have seen, V_c is negative semi-definite for $N = 2$ and 3. But its sign is indefinite for $N \geq 4$. Indeed, if $\kappa_j = 0$ for $j \geq 3$, V_c reduces to (2), while if $\kappa_j = \kappa$ for all j,

$$V_c = \frac{\hbar^2}{8}(N-1)(N-3)\kappa^2. \tag{4}$$

We note, however, that Matsutani [Matsutani] has proposed an idea which he hopes might explain the sign of V_c.

§3. Dirac vs Dirac Theory

For a practical application to a real membrane, e.g., a surface of a semi-conductor, Eq.(2) is the whole story. On the other hand, to a mathematically oriented person interested only in the effective Hamiltonian on a curved surface $\sum$ (a mathematical construct without a thickness), the above procedure of taking the thin-membrane limit would appear clumsy. He would rather seek a method which is free from such *mathematical redundancy* as the shape of the physical confining potential, etc.. An appropriate method has been developed, as is well-known, by Dirac[Dirac].

Dirac's method has actually been applied to the present problem [Ogawa et al]. Curiously the obtained effective potential is positive semi-definite;

$$V_D = \frac{\hbar^2}{8}(Tr\mathbf{K})^2, \tag{5}$$

where the subscript D stands for Dirac. Where does this apparent contradiction come from? The following [Ikegami et al] offers a partial answer, I hope, to this question.

The result (5) was obtained by faithfully following Dirac's prescription, namely by incorporating the mathematical condition of confinement

$$f(\mathbf{x}) = 0, \tag{6}$$

which defines the surface $\sum$, into the Lagrangian, computing the primary and secondary constraints by use of *Poickets* (i.e., Poisson brackets), where all the constraints turn out to be of the second class, introducing *Dirackets* (i.e., Dirac brackets), and finally quantizing the theory by replacing Dirackets by commutators. An inspection of the procedure shows that the result is almost unique; choice of operator ordering more general than that of Ogawa et al only multiplies (5) by a numerical factor.

There is one important point though. As far as the classical mechanics is concerned, the condition (6) is indistinguishable from

$$\dot{f}(\mathbf{x}) = 0 \tag{7}$$

where the overdot denotes the time derivative, provided that the intial (and only initial) condition is so chosen that $f(\mathbf{x}(t=0)) = 0$. Nothing prevents one from following Dirac's prescription with (7). One then finds [Ikegami et al]

$$V_{\dot{D}} = -\frac{\hbar^2}{8}\{2\xi Tr(\mathbf{K}^2) - \eta(Tr\mathbf{K})^2\}, \tag{8}$$

where ξ and η are real constants depending on the choice of operator ordering. (It is possible to concoct a "physical argument" in favour of the particular ordering which gives $\xi = \eta = 1$, but the status of the "physical argument" is not quite clear. At any rate this is not a main issue here.) The subscript $\dot{D}$ signifies the fact that Dirac's prescription has been combined with the condition (7).

It is to be emphasized that $\sum$ is a surface embedded in $\mathbf{E}^N$ and that the wavicle is supposed to obey the standard quantum mechanics in $\mathbf{E}^N$ apart from the constraint (6) or (7).

We have thus arrived at the following conclusion: the $\dot{D}$ theory leaves the possibility of reproducing V_c, while the D theory does not. It is not difficult to see why this is so. Let $\sum(n)$ be the surface parallel to and at the distance n from $\sum$ so that $\sum(0) = \sum$. Let $\mathbf{K}(n)$ be the extrinsic curvature of $\sum(n)$. Since $\sum(n)$ are embedded in the flat space,

$$
\begin{aligned}
Tr(\mathbf{K}^2) &= \partial Tr\mathbf{K}(n)/\partial n \\
&= [\frac{\partial}{\partial n}, Tr\mathbf{K}(n)],
\end{aligned}
\tag{9}
$$

where the right-hand side is to be evaluated at $n = 0$. From this relation, one can correctly guess that the term proportional to $Tr(\mathbf{K}^2)$ arises from the non-commutativity of the normal momentum $P_\perp$ (i.e., the component of the momentum perpendicular to $\sum$) and the mean extrinsic curvature. Now the most important difference between the D and $\dot{D}$ theories is that $P_\perp$ is *not* a dynamical variable in the former where $P_\perp \equiv 0$, while it is in the latter; there is no chance for $Tr(\mathbf{K}^2)$ to arise in the D theory, hence no chance to reproduce V_c. Hence, although the conditions (6) and (7) are equivalent in classical mechanics, they entail completely different effective potentials in quantum mechanics. Compare also Ogawa's article [Ogawa].

Originally I became interested in the present problem because of the difference in sign between (2) and (5). In retrospect, however, what is important is the difference in the functional forms between (2) [or (8)] and (5); $Tr\mathbf{K}$ and $\partial Tr\mathbf{K}(n)/\partial n$ carry distinct informations. (Recall that in the Friedmanncosmology the former has to do with the cosmic size and the latter with the cosmic expansion rate.)

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