Title	Group Theoretical Aspects of Quantizing on Manifolds
Author(s)	MCMULLAN, DAVID
Citation	物性研究 (1996), 67(3): 329-332
Issue Date	1996-12-20
URL	http://hdl.handle.net/2433/95961
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

Group Theoretical Aspects of Quantizing on Manifolds

DAVID MCMULLAN

School of Mathematics and Statistics, University of Plymouth
Drake Circus, Plymouth, Devon PL4 8AA, United Kingdom

The aim of this talk is to give a brief review of how the familiar process of quantizing linear systems can be extended to more complicated ones. Problems associated with quantizing a general classical system will be discussed, although the emphasis will be on theories with a high degree of symmetry so that group theoretic methods can be exploited.

Introduction

Quantum mechanics as it is understood nowadays can be identified with those developements in quantum theory which came after the revolutionary paper of Heisenberg [1]. That paper starts with the statement—"The present paper seeks to establish a basis for theoretical quantum mechanics founded exclusively upon relationships between quantities which in principle are observable." This clearly articulated view that a quantum description of the world must be formulated in terms of observable quantities was then taken up by Dirac [2] who realized that it is actually the algebra of observables that should be taken as fundamental. The basic physical task then is to decide on an appropriate algebra of observables to describe a given system. This is not always as easy as one might expect since in a quantum theory we are attempting to describe processes that are far removed from our every day experiences. However, since the measurement process must always produce some classically recognized effect, the standard approach to this is to start with a classical system from which one extracts an algebra of observables. The quantum theory associated with the representations of this algebra is then said to be the quantization of the classical system.

At first this seems a rather unnatural approach to finding quantum theories. After all, we are restricting the quantum theory to contain only classically inspired structures. Yet, as we shall see, even such an intrinsically quantum mechanical effect as spin can be understood from this approach.

Quantization

The kinematical arena for a classical system is a phase space M (the points of which correspond to the classical pure states). This is an even dimensional manifold, usually identified with the cotangent bundle T^*Q of a configuration space Q. The fact that we take the cotangent bundle rather than, say, the trivial bundle $Q \times \mathbb{R}^n$, where n is the dimension of Q, is because on T^*Q there is an intrinsic construction of the Poisson bracket $\{,\}$, giving us a globally well defined algebra of functions on the phase space which we identify as the algebra of classical observables. One can always *locally* introduce position q^α $(\alpha = 1, \ldots, n)$ and momentum p_α functions such that $\{q^\alpha, q^\beta\} = \{p_\alpha, p_\beta\} = 0$ and $\{q^\alpha, p_\beta\} = \delta^\alpha_\beta$, but not globally unless the phase space is itself $\mathbb{R}^{2n}$. It is this fact that makes quantum theory on manifolds so interesting.

For a general configuration space Q we can construct various useful examples of classical observables that are globally well defined. For example, any function on Q can be pulled back to T^*Q to give a position observable. Also any diffeomorphism of Q can be extended to a point transformation of T^*Q, and hence is generated by an observable. If the diffeomorphism is described by the vector field X on Q, then the momentum observable associated with it, p_X, is a function linear in momentum on T^*Q which satisfies the Poisson algebra: $\{p_X, p_Y\} = p_{[X,Y]}$ and $\{p_X, f\} = X(f)$ for all position observables f.

The quantization procedure can then be (informally) described as the process of replacing the (smooth) functions on the phase space by operators in such a way that the Poisson bracket goes into the commutator. Dirac seems to have been aware that this prescription only holds for simple function (a proof that it does not work in general, for any system, was given by van Hove [3]). So to quantize a classical system we need to first find a sub-algebra of the classical observables that has a representation in terms of operators acting on a Hilbert space.

There is no canonical way of finding this sub-algebra of classical observables; one must be guided by the mathematical neccessity that the observables are globally well defined and the physical requirement that any observable of interest is included.

For example, the position and momentum observables introduced above form a Poisson algebra which can be identified with the Lie algebra of $\mathrm{Diff}(Q) \ltimes C^\infty(Q)$, the semi-direct product of the diffeomorphism group of Q and the smooth functions on Q, where for $\phi \in \mathrm{Diff}(Q)$ and $f \in C^\infty(Q)$ we have the action $(\phi f)(q) = f(\phi^{-1}q)$. There is no obstruction to finding representations of this algebra—only our ignorance (the representation theory of the diffeomorphism group is still very much in its infancy). It is also not clear physically that such a large algebra of observables should be taken as basic. Thus we need to cut down further the number of observables used to quantize the theory.

For the linear system with configuration space $Q = \mathbb{R}^3$, and hence phase space $M = \mathbb{R}^6$, we can take the globally well defined position and momentum observables (q^α, p_α) as basic. Their algebra of Poisson brackets is then the Heisenberg algebra whose representation theory is well known—yielding the familiar Schrödinger representation where the quantum states are wave functions on the configuration space $\mathbb{R}^3$ and the position operator acts by multiplication while the momentum operator acts as a derivative. This result is viewed by many as the paradigm example of what to expect from the quantization process. It is important, though, to realize that this is a very special system and that, in particular, the uniqueness of the quantization is not at all typical. So what happens on a more general configuration space? In general we don't know. However, if Q is a homogeneous space then it is possible, for a restricted class of dynamics, to generalize this simple example and find a manageable basis of observables upon which the quantization can be based.

Quantizing on a coset space

A systematic approach to quantizing systems whose configuration space is a homogeneous space was developed by Mackey in the 60's and summarized in his book [4]. This involved the introduction of a system of imprimitivity to characterize the basic observables—the quantizations then corresponding to the representation of this system of imprimitivity. This important work seems to have been largely ignored by most physicists, and it was not untill the work of Isham [5], and more recently [6] and [7], that the relevance of these ideas have become more apparent.

If the configuration space Q can be identified with a coset space G/H, for some Lie groups G and H, then the G-action gives a basis (at each point) of the vector fields on Q and hence of the momenta. Thus the infinite dimensional Diff(Q)-algebra of momentum observables can be reduced to a much more attractive G-algebra of observables. If we, in addition, keep the full algebra of functions on Q as basic observables then we are naturally lead to a system of imprimitivity. However, we can also exploit the G-action to find a basis of position observables. To see this let V be a finite dimensional vector space on which G acts and such that there is an orbit diffeomorphic to Q. Then the coordinates of the embedding of Q in V give continuous functions on Q with the desired properties.

Following this route to quantization we end up with a basis of observables for this system whose Poisson algebra can be identified with the Lie algebra of the (canonical) group $\mathrm{G}_c = \mathrm{G} \ltimes V$. The irreducible representations of G_c then give the quantizations of the system and they are found by the method of induced representations. In contrast to the linear system discussed above, standard results in representation theory tell us that now *many* quantizations of these coset systems are possible, the different quantizations being labelled by the irreducible representations of the little group H.

For example, the D-dimensional sphere can be identified with the coset space $\mathrm{SO}(D+1)/\mathrm{SO}(D)$, and one can use the defining representation of $\mathrm{SO}(D+1)$ to identify the vector space V with $\mathbb{R}^{D+1}$. The canonical group for the D-sphere is thus $\mathrm{SO}(D+1)\ltimes\mathbb{R}^{D+1}$, which is simply the Euclidean group $\mathrm{E}(D+1)$. In particular, for the two-sphere we see that there are infinitly many possible quantizations labelled by the integer characterizing the irreducible representations of the little group SO(2).

Our discussion so far has been purely kinematical. To get a better feel for the physical significance of these different quantizations we need to introduce some dynamics. It is important to realize that the dynamics must be compatible with the choice of basic variables, and this essentially restricts us classically to free motion with respect to the natural G-invariant metric on Q. However, in the quantum theory one finds [6, 7] that a Yang-Mills gauge field is induced, minimally coupled to the particle, which in its turn introduces new compact degrees of freedom. Thus for the example of the two-sphere we find that there is an effective Dirac monopole at the centre of the sphere whose charge labels the quantum sector of the theory. More strikingly, on the four-sphere, we get a BPST-instanton and chiral spin degrees of freedom [8].

It is clear that in this group theoretic approach to quantization many choices must be made which have a direct bearing on the resulting quantum theory. For example, the identification of the homogeneous space Q with the coset G/H is far from unique. This, in its turn, reflects the freedom to choose what observables are to be taken as basic in the theory. So for the linear system on $\mathbb{R}^3$ the choice of position and momentum as the basic observables is equivalent to the identification of $\mathbb{R}^3$ as the trivial coset space $\mathbb{R}^3/\{1\}$, with the little group H being the identity. Another possibility is to identify $\mathbb{R}^3$ with the coset E(3)/SO(3), or their covering groups. With this identification we are taking position, momentum and angular momentum as basic variables. Classically we know that the angular momentum is simply a function of position and momentum, but that information is *not* encoded in the abstract canonical group algebra. With this new canonical group many quantizations of the linear system are now possible, labelled by the spin of the system.

Discussion

Coset spaces can be considered as toy models for the structures found in field theories. There we want to know how the geometry of, say, the Yang-Mills configuration space effects the quantization. Recall that $\mathcal{U}$, the space of Yang-Mills fields, is similar in many ways to the linear space discussed above. However, the true degrees of freedom in Yang-Mills theory is the "coset space" $\mathcal{U}/\mathcal{G}$ of gauge fields modulo the group $\mathcal{G}$ of gauge transformation. This is an infinite dimensional manifold with many structures in common with the simple coset spaces we have been studying. In particular, there are non-trivial D-spheres of various dimensions in this configuration space that should directly effect the quantization process. Although this expectation seems reasonable, the immediate problem we face is that the representation theory that underlined this account of quantization does not exist for these infinite dimensional systems.

Of course, any discussion of quantum field theory suffers from a lack of mathematical sureness. This is reflected in the fact that the operator formalism of observables and states is not the most useful language to discuss field theory. Rather, it has proven more productive to use the mathematically unsound but physically appealing language of path integrals. For all their intuitive (and hence classical) appeal, path integrals are, by construction, very much at odds with Heisenberg's stated philosophy. As a consequence there does not exist, as yet, an intrinsic path integral account of the rich quantizations we have seen for coset spaces. This is a serious defect in our understanding of the quantum theory on manifolds, and one that I feel obstructs the direct extension of these ideas to field theory.

Acknowledgements: I wish to thank the Royal Society and JSPS for providing the financial support for my visit to Japan.

References

[1] W. Heisenberg, *Zeits. Phys.* **33** (1925), 897.

[2] P.A.M. Dirac, "Lectures on Quantum Mechanics", Yeshiva, New York, 1964.

[3] L. van Hove, *Mem. de l'Acad. Roy. de Belgique (Classe des Sci.)* **XXVI** (1951) 61.

[4] G.W. Mackey, "Induced Representations of Groups and Quantum Mechanics", Benjamin, New York, 1969.

[5] C.J. Isham, in "Relativity, Groups and Topology II", (B.S. DeWitt and R. Stora, Eds) North-Holland, Amsterdam, 1984.

[6] N.P. Landsman and N. Linden, *Nucl. Phys.* **B365** (1991) 121.

[7] Y. Ohnuki and S. Kitakado, *Mod. Phys. Lett.* **A7** (1992) 2477; *J. Math. Phys.* **34** (1993) 2827.

[8] D. McMullan and I. Tsutsui, "On the emergence of gauge structures and generalized spin when quantizing on a coset space", *Ann. Phys. (N.Y.)* (1994) in press.