Zero-mode, winding number and commutators of abelian sigma model in (1+1) dimensions*
— 日本語のまえおきつき —

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日本語による序

この研究会報告を読まれるような方なら、当然ご承知のことを述べますが、初心に帰って動機を説明したいと思います。本題だけに関心のある方は、どうぞこの序文をとばして、この後にある英文をお読み下さい。ここでは私の philosophy を述べさせて頂きます。

ユークリッド空間上の粒子の量子論は、正準交換関係から出発して、状態ベクトルの空間 (ヒルベルト空間) を作ることによって定式化されます。あとは、このヒルベルト空間でいろいろな物理量の固有値問題を解いたり、遷移確率の計算をしたりするわけです。交換関係は古典力学のポアソン括弧に対応するわけですが、それだけでは量子論は完成しなくて、ヒルベルト空間を用意しなければならない、演算子 (operator) に被演算子 (operand) を与えなければならない、数学の言葉で言うと、代数を表現しなければならない、ここが量子論に独特の手続きになっているわけです。

ここに問題が現れます。ヒルベルト空間はちゃんと作れるのか、また作れたとしてもみな同じなのか、異なるヒルベルト空間があって物理の計算結果に違う答えを出してしまうことはないのか、という問いです。

いわゆる正準交換関係 $[x,p]=i\hbar$ の場合は、この点は解決されていて、既約表現空間はユニタリ同値の意味でただ一つしか存在しないというフォン・ノイマンの定理があります。ですから、波動関数表現を使っても、調和振動子表現を使っても、物理的には同等になるわけです。抽象的な演算子が、微分演算子で表されたり、行列で表されたりしてもオッケーという保証があるのです。

かといって、ヒルベルト空間の存在と一意性の問題は決して杞憂ではなく、特に場の量子論では現実的な問題になります。場の量子論では、フォック空間しか表現空間の構成が知られていませんが、対称性の自発的破れでは、非同値なフォック空間が無限個できてしまうことが見つかっています。これは、場が無限自由度の力学系であることに関連した現象です。

さて、いま一度有限自由度の力学に戻って、ただしユークリッド空間上の粒子ではなく、多様体上の粒子を考えてみましょう。多様体というものが考案された理由の一つに、一般的な古典力学の舞台としてはユークリッド空間はちょっと制限が強すぎる、一枚の rigid な座標系で覆われていなければならないというのは人為的な要求ではないか、もっと柔軟性に富んだ枠組みがほしい、という要

*研究会では「等質空間の量子力学」と題して講演しましたが、改題させて頂きます。

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求があったわけです。とくに作用原理は座標系に依らないで力学を記述しますから、対称性の見通しがいいし、問題を解くときにどんな座標系が便利か教えてくれます。回転対称な問題だから、極座標がよい、とか。ややおおげさな言い方をすれば、観測者に依らない自然のパターンを物理法則というのであり、座標系の取り方に依らない空間の性質を扱うことを幾何学と呼ぶわけです。したがって幾何学は、物理法則を記述する言葉を提供するのです。これは数学者のクラインが気がついていたことだし、アインシュタインが一般相対性原理として前面に押し出したことです。もちろん具体的な計算をするときは、座標系を入れなければいけません。しかし計算の手続きというのは計算者に依存するものであり、物理そのものではありません。はじかれているそろばんのたまや、コンピュータの中の電子スイッチのオン・オフが実際に起きている物理現象の中に対応物を持つわけではありません。これらはたんに推論の手続きであります。

philosophy はここらで切り上げて、具体的な話をしましょう。ここで一次元の円周 S^1 の上を動く粒子を考えましょう。古典論なら、腕の長さの変わらない振り子を考えることになります。S^1 には一価連続な座標は存在せず、例えば角度座標 θ は $2\pi n$ の多価性を持ちます。これを量子化しようとすると、ちとやっかいなことになります。たんに共役運動量 P を持ってきて

$$[\theta, P] = i$$

とやっても、これは一次元ユークリッド空間上の量子力学と変わらないわけで、ヒルベルト空間を作っても、$|\theta\rangle = |\theta + 2\pi\rangle$ とはなりません。状態ベクトルが S^1 上に乗っていないのです。そこで大貫氏は、θ のかわりに $U = e^{i\theta}$を位置を示す演算子として使うことを考えました。θ は多価でも、$U = e^{i\theta}$なら一価です。そうすると交換関係は次のようになります。

$$[U, P] = -U$$

で、この後は、θ のことは忘れて、Uと P だけありとして、この代数を表現するヒルベルト空間を作っていく。そうすると、連続無限個の非同値なヒルベルト空間が存在することを、彼は作って示したのです。これは大変なことで(私は大変だと思いました)、たかが一自由度の力学系でも、S^1のように非自明なトポロジーを持つ空間では、量子論は一意ではない、といっているのです。さらに、大貫氏と北門氏は任意の次元の球面上の量子論の定義を与え、これを構成し、やはり無限個のヒルベルト空間を作って見せました。さらにスピン自由度やゲージ場が、この構成から自動的に現れることを示しています。詳しくは、文献 [1], [3] を参照して下さい。

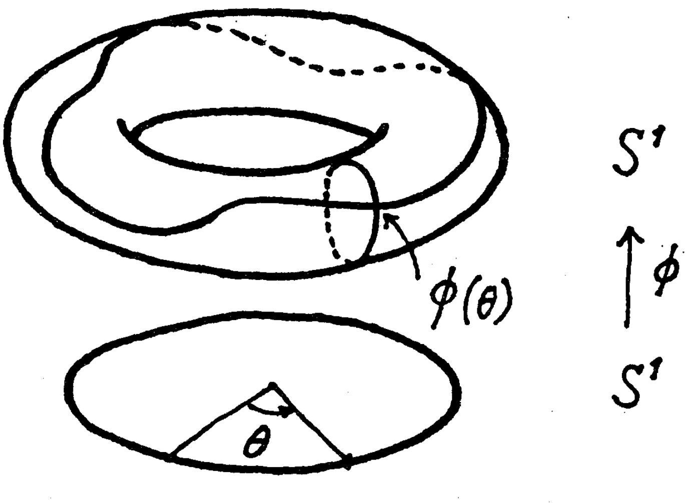
私は彼らの議論を、場の理論に、つまり無限自由度の場合に拡張することを考えました。その動機を説明します。非線形シグマ模型というのが場の理論の模型の一つとして知られていますが、これは対称性の自発的破れに伴う南部・ゴールドストンボソンを記述するものです。対称性がある群 G からその部分群 Hに落ちたとき、縮退した真空は等質空間 G/Hと同じ次元、同じトポロジーを持ちます。ゴールドストンモードは、この多様体に値を持つ場で記述され、ラグラジアンは対称性から一

意に決まる、というのが非線形シグマ模型です。ゴールドストンの場は、実数や複素数に値を持つスカラー場やスピノル場と違いまして、非自明なトポロジーの多様体に値を持つ場なわけです。もちろん作用積分は、座標系に依らずに書き下されます。問題は量子化で、従来の正準形式ではどうしても G/H に座標系を入れてしまう、おまけに正準交換関係ではそのトポロジーを反映できない、摂動論ではグローバルな性質は忘れ去られてしまう、ということになります。それでも、低エネルギーでの有効理論としてはかまわないのかもしれません。と思っていると、ヴェス・ズミノ・ウィッテン項のように、場のトポロジーを考えなければ、整合した量子論を作れない場合があったりします。

話は少しずれるかもしれませんが、非可換ゲージ理論のグリボフ問題や、ストリングのゼロモードやトポロジーの変化を扱うときにも、大域的な性質をどうやって量子論の中で整合的に扱うかという場面に出くわすのです。ゲージ理論の側面は、この研究会で David McMullan 氏が述べていますし、ストリング理論の面は、橘氏が議論されています。

そこで私の取り上げる問題は、自明でないトポロジーの多様体に値を持つ場の理論を、大域的性質を尊重しつつ量子化せよ、量子化とは表現空間まで作ることだが、それは一意的か、それとも異なる表現が、したがって異なる物理が存在するのか、ということになります。特に非線形シグマ模型は、等質空間という、比較的きれいな多様体ですから、まあなんとか扱えるかもしれない。そこでまず、等質空間上の有限自由度の量子力学を作ってみました。これは文句のつけようがないほど完成していて [2]、筒井氏の講演にあるとおりです。しかし、場の理論への拡張となると、一筋縄ではいきそうにない。そこで次には、(1+1) 次元時空で、しかもアーベル群の場合を考えてみることにしました。その結果が以下の英文論文なわけです。研究会の時点ではやりかけであったことを、ある程度の結果を出して報告しています。

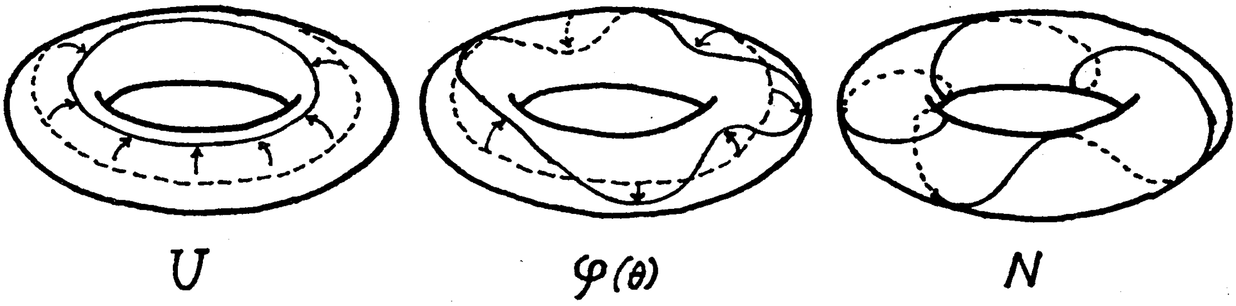
まえおきが長くなりますが、許しを乞うて、ここで扱う系を紹介します。考える空間 (いわゆる実空間です) は一次元円周 S^1 で、考える場は $\phi : S^1 \to S^1 \cong U(1)$ です。つまり、$U(1)$ シグマ模型を扱います。この場は、絵としてはこんなふうに描かれます。



その自由度は次のように、三つに分解されます。式で書くと

$$\phi(\theta) = Ue^{i(\varphi(\theta)+N\theta)}$$

Uは$U(1)$の元であり、$\varphi(\theta)$は$\varphi : S^1 \to \boldsymbol{R}$という連続関数で$\int_0^{2\pi}\varphi(\theta)d\theta = 0$を満たすもの、$N$は整数です。絵でかけば、



と、順に、Uは collective mode、あるいは zero-mode を表し、$\varphi(\theta)$は fluctuation mode を表し、Nは巻き数 (winding number) を表しています。で、これらが、いかにもぐるぐる回ったり、震えたり、巻きついたりしているのが見えるように、代数を設定するわけです。そしたら、やっぱり連続無限個の非同値なヒルベルト空間ができてしまった、というのが結果の一つ。それから、実は代数には中心拡大 (central extension) という変形の余地があって、これを取り入れると、さらに変形された交換関係が出てきます。このときは zero-mode と winding number が独立にならず、奇妙な交換関係を満たします。私はそれを twist relation と呼びました。

ここまで読まれた方どうもありがとうございます。とにかく、量子論には表現論がつきまとうため、というより、量子論では表現論が本質的であるため、トポロジーがちょっと非自明になっただけで、古典論では思いもかけなかったことがたやすく起こるのだ、ということをわかって頂ければ、この序文の目的は達せられたわけです。

さあ、あとはがんばって本文を読もう!!

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本研究を励まして下さった筒井氏に感謝します。また、神戸大学の坂本氏、橘氏には有意義な議論をして頂いたことを感謝します。

Zero-mode, winding number and commutators of abelian sigma model in (1+1) dimensions

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It is known that there exist an infinite number of inequivalent quantizations on a topologically nontrivial manifold even if it is a finite-dimensional manifold. In this paper we consider the abelian sigma model in (1+1) dimensions to explore a system having infinite degrees of freedom. The model has a field variable $\phi : S^1 \to S^1$. An algebra of the quantum field is defined respecting the topological aspect of this model. A central extension of the algebra is also introduced. It is shown that there exist an infinite number of unitary inequivalent representations, which are characterized by a central extension and a continuous parameter α ($0 \leq \alpha < 1$). When the central extension exists, the winding operator and the zero-mode momentum obey a nontrivial commutator.

1 Introduction

In both field theory and string theory there are several models which have manifold-valued variables. For instance, the nonlinear sigma model has a field variable $\phi : \boldsymbol{R}^3 \to G/H$, where G/H is a homogeneous space. This manifold G/H is closely related to vacua associated with spontaneous symmetry breaking. As another example, the toroidal compactification model of closed bosonic string has a variable $X : S^1 \to T^n$. To study global aspects of these models in quantum theory, we should have a quantization scheme respecting topological nature. However in the scheme of usual canonical quantization and perturbation method, the global aspects are obscure.

On the other hand it is known [1], [2] that there exist an infinite number of inequivalent quantizations on a topologically nontrivial manifold even if it is a finite-dimensional manifold. Unfortunately it remains difficult to extend those quantization schemes to include field theory.

In this paper we consider the abelian sigma model in (1+1) dimensions to explore a

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system having infinite degrees of freedom. In the context of classical theory, a field variable of the model is a map from S^1 to S^1. An algebra of the quantum field is defined respecting the topological aspect of this model. Special attention is paid for the zero-mode and the winding number. A central extension of the algebra is also introduced. Representation spaces of the algebra are constructed using the Ohnuki-Kitakado representation and the Fock representation. It is shown that there exist an infinite number of unitary inequivalent representations, which are parametrized by a continuous parameter α $(0 \leq \alpha < 1)$. It is expected that this model gives a physical insight to nonlinear sigma models of field theory and orbifold models of string theory.

2 Ohnuki-Kitakado representation

Here we briefly review quantum mechanics of a particle on S^1 considered by Ohnuki and Kitakado [1]. They assume that a unitary operator $\hat{U}$ and a self-adjoint operator $\hat{P}$ satisfy the commutation relation

$$[\hat{P}, \hat{U}] = \hat{U}. \tag{2.1}$$

An irreducible representation of the above algebra is defined to be quantum mechanics on S^1. The operators $\hat{U}$ and $\hat{P}$ are called a position operator and a momentum operator, respectively. It is shown below that this naming is reasonable.

A representation space is constructed as follows. Let $|\alpha\rangle$ be an eigenvector of $\hat{P}$ with a real eigenvalue α; $\hat{P}\,|\alpha\rangle = \alpha\,|\alpha\rangle$. Assume that $\langle\alpha|\alpha\rangle = 1$. The commutator (2.1) implies that the operator $\hat{U}$ increases an eigenvalue of $\hat{P}$ by a unit. If we put

$$|n+\alpha\rangle := \hat{U}^n\,|\alpha\rangle \quad (n = 0, \pm 1, \pm 2, \cdots), \tag{2.2}$$

it is easily seen that

$$\hat{P}\,|n+\alpha\rangle = (n+\alpha)\,|n+\alpha\rangle, \tag{2.3}$$

$$\hat{U}\,|n+\alpha\rangle = |n+1+\alpha\rangle. \tag{2.4}$$

Unitarity of $\hat{U}$ and self-adjointness of $\hat{P}$ imply that

$$\langle m+\alpha|n+\alpha\rangle = \delta_{mn}. \tag{2.5}$$

Let H_α denote the Hilbert space defined by completing the space of finite linear combinations of $|n+\alpha\rangle$ $(n = 0, \pm 1, \pm 2, \cdots)$. By (2.3) and (2.4), H_α becomes an irreducible representation space of the algebra (2.1).

H_α and H_β are unitary equivalent if and only if the difference $(\alpha - \beta)$ is an integer. Consequently there exists an inequivalent representation for each value of the parameter α

ranging over $0 \leq \alpha < 1$. At this point, quantum mechanics on S^1 is in contrast to quantum mechanics on $\boldsymbol{R}$. For the one on $\boldsymbol{R}$, it is well-known that the algebra of the canonical commutation relations has a unique irreducible representation upto unitary equivalence.

To clarify the physical meaning of the parameter α, they [1] study eigenstates of the position operator $\hat{U}$. If we put

$$|\lambda\rangle := \sum_{n=-\infty}^{\infty} e^{-in\lambda} |n+\alpha\rangle, \tag{2.6}$$

it follows that

$$\hat{U}\,|\lambda\rangle = e^{i\lambda}|\lambda\rangle, \tag{2.7}$$

$$|\lambda + 2\pi\rangle = |\lambda\rangle, \tag{2.8}$$

$$\langle\lambda|\lambda'\rangle = 2\pi\,\delta(\lambda - \lambda'). \tag{2.9}$$

In the last equation it is assumed that the δ-function is periodic with periodicity 2π. It is also easily seen that

$$\exp(-i\mu\hat{P})|\lambda\rangle = e^{-i\alpha\mu}\,|\lambda + \mu\rangle, \tag{2.10}$$

which says that $\hat{P}$ is a generator of translation along S^1. It should be noticed that an extra phase factor $e^{-i\alpha\mu}$ is multiplied. These states $|\lambda\rangle$ $(0 \leq \lambda < 2\pi)$ define a wave function $\psi(\lambda)$ for an arbitrary state $|\psi\rangle \in H_\alpha$ by $\psi(\lambda) := \langle\lambda|\psi\rangle$. This definition gives an isomorphism between H_α and $L^2(S^1)$ that is a space of square-integrable functions on S^1. A bit calculation shows that the operators act on the wave function as

$$\hat{U}\psi(\lambda) := \langle\lambda|\hat{U}|\psi\rangle = e^{i\lambda}\psi(\lambda), \tag{2.11}$$

$$\hat{P}\psi(\lambda) := \langle\lambda|\hat{P}|\psi\rangle = \left(-i\frac{\partial}{\partial\lambda} + \alpha\right)\psi(\lambda). \tag{2.12}$$

In the last expression the parameter α is interpreted as the vector potential for magnetic flux $\Phi = 2\pi\alpha$ surrounded by S^1. Physical significance of α is further discussed in the reference [3].

3 Algebra

3.1 Fundamental algebra

Next we would like to extend Ohnuki-Kitakado's quantum mechanics on S^1 to a field-theoretical model. We shall propose an algebra of the model. To motivate definition of the algebra we remind another expression of the canonical commutation relations. If we put

$\hat{V}(a) := \exp(-i\sum_j a_j \hat{p}_j)$ for real numbers $a = (a_1, \cdots, a_n)$, $\hat{V}(a)$ is a unitary operator and satisfies

$$\hat{x}_j \hat{x}_k = \hat{x}_k \hat{x}_j, \tag{3.1}$$

$$\hat{V}(a)^\dagger \hat{x}_j \hat{V}(a) = \hat{x}_j + a_j, \tag{3.2}$$

$$\hat{V}(a) \hat{V}(b) = \hat{V}(a+b). \tag{3.3}$$

Geometrical meaning of the above algebra is obvious; positions $\hat{x}$'s are simultaneously measurable and movable by the displacement operator $\hat{V}(a)$; displacement operators satisfy associativity.

Quantum mechanics on S^1 is easily generalized to the one on n-dimensional torus $T^n = (S^1)^n$. For this purpose we introduce unitary operators $\hat{U}_j$ and self-adjoint operators $\hat{P}_j (j = 1, \cdots, n)$. Put $\hat{V}(\mu) := \exp(-i\sum_j \mu_j \hat{P}_j)$ for $\mu = (\mu_1, \cdots, \mu_n) \in \boldsymbol{R}^n$. Naive generalization of (2.1) leads the following relations;

$$\hat{U}_j \hat{U}_k = \hat{U}_k \hat{U}_j, \tag{3.4}$$

$$\hat{V}(\mu)^\dagger \hat{U}_j \hat{V}(\mu) = e^{i\mu_j} \hat{U}_j, \tag{3.5}$$

$$\hat{V}(\mu) \hat{V}(\nu) = \hat{V}(\mu + \nu). \tag{3.6}$$

Representations of this algebra are constructed by tensor products of Ohnuki-Kitakado representations $H_{\alpha_1} \otimes \cdots \otimes H_{\alpha_n}$. Therefore irreducible representations are parametrized by n-tuple parameter $\alpha = (\alpha_1, \cdots, \alpha_n)$.

Now we turn to the abelian sigma model in $(1+1)$ dimensions. The space-time is $S^1 \times \boldsymbol{R}$ and the target space is S^1, on which the group $U(1)$ acts. In the classical sense the model has a field variable $\phi \in Q = \mathrm{Map}(S^1; S^1)$. Let $\Gamma = \mathrm{Map}(S^1; U(1))$ a group by pointwise multiplication. The group Γ acts on the configuration space Q by pointwise action; for $\gamma \in \Gamma$ and $\phi \in Q$ let us define $\gamma \cdot \phi \in Q$ by

$$(\gamma \cdot \phi)(\theta) := \gamma(\theta) \cdot \phi(\theta) \quad (\theta \in S^1), \tag{3.7}$$

where θ denotes a point of the base space. In the right-hand side the multiplication indicates the action of $U(1)$ on S^1.

To quantize this system let us assume that $\hat{\phi}(\theta)$ is a unitary operator for each point $\theta \in S^1$ and $\hat{V}(\gamma)$ is a unitary operator for each element $\gamma \in \Gamma$. Moreover we assume the following algebra

$$\hat{\phi}(\theta) \hat{\phi}(\theta') = \hat{\phi}(\theta') \hat{\phi}(\theta), \tag{3.8}$$

$$\hat{V}(\gamma)^\dagger \hat{\phi}(\theta) \hat{V}(\gamma) = \gamma(\theta) \hat{\phi}(\theta), \tag{3.9}$$

$$\hat{V}(\gamma) \hat{V}(\gamma') = e^{-ic(\gamma,\gamma')} \hat{V}(\gamma \cdot \gamma') \quad (\gamma, \gamma' \in \Gamma). \tag{3.10}$$

At the last line a function $c : \Gamma \times \Gamma \to \boldsymbol{R}$ is called a central extension, which satisfies the cocycle condition

$$c(\gamma_1, \gamma_2) + c(\gamma_1\gamma_2, \gamma_3) = c(\gamma_1, \gamma_2\gamma_3) + c(\gamma_2, \gamma_3) \pmod{2\pi}. \tag{3.11}$$

If $c \equiv 0$, the algebra (3.8)-(3.10) is a straightforward generalization of (3.4)-(3.6) to a system with infinite degrees of freedom. We call the algebra (3.8)-(3.10) the fundamental algebra of the abelian sigma model.

To clarify geometrical implication of the algebra we shall decompose the degrees of freedom of $\phi \in Q$ and $\gamma \in \Gamma$. In the classical sense we may rewrite $\phi : S^1 \to S^1 \cong U(1)$ by

$$\phi(\theta) = U\, e^{i(\varphi(\theta)+N\theta)}, \tag{3.12}$$

where $U \in U(1)$, $N \in \boldsymbol{Z}$ and φ satisfies the no zero-mode condition;

$$\mathrm{Map}_0(S^1; \boldsymbol{R}) := \{\varphi : S^1 \to \boldsymbol{R} \,|\, C^\infty,\ \int_0^{2\pi} \varphi(\theta)\, d\theta = 0\}. \tag{3.13}$$

The decomposition (3.12) says that $Q \cong S^1 \times \mathrm{Map}_0(S^1; \boldsymbol{R}) \times \boldsymbol{Z}$. Geometrical meaning of this decomposition is apparent; U describes the zero-mode or collective motion of the field ϕ; φ describes fluctuation or local degrees of freedom of ϕ; N is nothing but the winding number. Topologically nontrivial parts are U and N.

Similarly $\gamma : S^1 \to U(1)$ is also rewritten as

$$\gamma(\theta) = e^{i(\mu+f(\theta)+m\theta)}, \tag{3.14}$$

where $\mu \in \boldsymbol{R}$, $f \in \mathrm{Map}_0(S^1; \boldsymbol{R})$ and $m \in \boldsymbol{Z}$. The action (3.7) of γ (3.14) on ϕ (3.12) is decomposed into

$$U \;\to\; e^{i\mu} U, \tag{3.15}$$

$$\varphi(\theta) \;\to\; \varphi(\theta) + f(\theta), \tag{3.16}$$

$$N \;\to\; N + m. \tag{3.17}$$

So the first component of γ (3.14) translates the zero-mode; the second one gives a homotopically trivial deformation; the third one increases the winding number.

As a nontrivial central extension for γ (3.14) and

$$\gamma'(\theta) = e^{i(\nu+g(\theta)+n\theta)}, \tag{3.18}$$

we define

$$c(\gamma, \gamma') := \frac{1}{4\pi} \int_0^{2\pi} (f'(\theta) g(\theta) - f(\theta) g'(\theta))\, d\theta + m\nu - n\mu. \tag{3.19}$$

This central extension is the simplest but nontrivial one which is invariant under the action of Diff(S^1); $c(\gamma\circ\omega,\gamma'\circ\omega)=c(\gamma,\gamma')$ for any $\omega\in$ Diff(S^1). The group Γ associated with such an invariant central extension is called a Kac-Moody group. The relation (3.10) means that $\hat{V}$ is a unitary representation of the Kac-Moody group. For classification of central extensions see the literature [4].

3.2 Algebra without central extension

According to decomposition of classical variables (3.12) and (3.14), quantum operators are also to be decomposed. For simplicity we consider the fundamental algebra (3.8)-(3.10) without the central extension, that is, here we restrict $c\equiv 0$.

Corresponding to (3.12) we introduce a unitary operator $\hat{U}$, self-adjoint operators† $\hat{\varphi}(\theta)$ for each $\theta\in S^1$ constrained by

$$\int_0^{2\pi}\hat{\varphi}(\theta)\,d\theta=0, \tag{3.20}$$

and a self-adjoint operator $\hat{N}$ satisfying

$$\exp(2\pi i\,\hat{N})=1, \tag{3.21}$$

which is called the integer condition for $\hat{N}$. We demand that the quantum field $\hat{\phi}(\theta)$ is expressed by them as

$$\hat{\phi}(\theta)=\hat{U}\,e^{i(\hat{\varphi}(\theta)+\hat{N}\theta)}. \tag{3.22}$$

Next, corresponding to (3.14) we introduce a self-adjoint operator $\hat{P}$, self-adjoint operators $\hat{\pi}(\theta)$ for each $\theta\in S^1$ constrained by

$$\int_0^{2\pi}\hat{\pi}(\theta)\,d\theta=0, \tag{3.23}$$

and a unitary operator $\hat{W}$. When γ is given by (3.14), the operator $\hat{V}(\gamma)$ is defined by

$$\hat{V}(\gamma)=e^{-i\mu\hat{P}}\exp\left[-i\int_0^{2\pi}f(\theta)\,\hat{\pi}(\theta)\,d\theta\right]\hat{W}^m. \tag{3.24}$$

Using these operators the fundamental algebra is now rewritten as

$$[\,\hat{P},\hat{U}\,]=\hat{U}, \tag{3.25}$$

$$[\,\hat{\varphi}(\theta),\hat{\pi}(\theta')\,]=i\Big(\delta(\theta-\theta')-\frac{1}{2\pi}\Big), \tag{3.26}$$

$$[\,\hat{N},\hat{W}\,]=\hat{W}, \tag{3.27}$$

†Expressing rigorously $\hat{\varphi}(\theta)$ is an operator-valued distribution.

and all other commutators vanish. In (3.26) it is understood that the δ-function is defined on S^1. These commutators are equivalent to

$$e^{i\mu\hat{P}}\,\hat{U}\,e^{-i\mu\hat{P}} = e^{i\mu}\,\hat{U}, \tag{3.28}$$

$$\exp\left[i\int_0^{2\pi} f(\theta)\hat{\pi}(\theta)\,d\theta\right]\,\hat{\varphi}(\theta)\,\exp\left[-i\int_0^{2\pi} f(\theta)\hat{\pi}(\theta)\,d\theta\right] = \hat{\varphi}(\theta) + f(\theta), \tag{3.29}$$

$$\hat{W}^\dagger\,\hat{N}\,\hat{W} = \hat{N} + 1. \tag{3.30}$$

This algebra realize (3.15)-(3.17) by means of (3.9). Observing the relation (3.30) we call $\hat{N}$ and $\hat{W}$ the winding number and the winding operator, respectively. Then remaining task is to construct representations of the algebra.

3.3 Algebra with central extension

Before constructing representations we reexpress the fundamental algebra with the central extension (3.19) respecting the decomposition (3.12) and (3.14). The decomposition (3.22) of $\hat{\phi}$ still works. On the other hand the decomposition (3.24) of $\hat{V}$ should be modified a little. We formally introduce an operator $\hat{\Omega}$ by

$$\hat{W} = e^{-i\hat{\Omega}}. \tag{3.31}$$

Although $\hat{W}$ itself is well-defined, $\hat{\Omega}$ is ill-defined. If $\hat{\Omega}$ exists, (3.27) would imply $[\hat{N}, \hat{\Omega}] = i$, which is nothing but the canonical commutation relation. Therefore $\hat{N}$ should have a continuous spectrum, that contradicts the integer condition (3.21). Consequently $\hat{\Omega}$ must be eliminated after calculation. Bearing the above remark in mind, we replace (3.24) by

$$\hat{V}(\gamma) = \exp\left[-i\Big(\mu\hat{P} + \int_0^{2\pi} f(\theta)\,\hat{\pi}(\theta)\,d\theta + m\,\hat{\Omega}\Big)\right]. \tag{3.32}$$

For the central extension (3.19) it is verified that the following commutation relations should be added to (3.25)-(3.27) to satisfy the fundamental algebra;

$$[\,\hat{\Omega}, \hat{P}\,] = 2i, \tag{3.33}$$

$$[\,\hat{\pi}(\theta), \hat{\pi}(\theta')\,] = -\frac{i}{\pi}\,\delta'(\theta - \theta'). \tag{3.34}$$

Using (3.31) Eq. (3.33) implies

$$[\,\hat{P}, \hat{W}\,] = -2\,\hat{W}, \tag{3.35}$$

which says that the zero-mode momentum $\hat{P}$ is decreased by two units when the winding number $\hat{N}$ is increased by one unit under the operation of $\hat{W}$. This is an inevitable consequence of the central extension. We call this phenomenon “twist”. Using (3.33) the decomposition (3.32) results in

$$\hat{V}(\gamma) = e^{-i\mu m}\,\exp\left[-i\Big(\mu\hat{P} + \int_0^{2\pi} f(\theta)\,\hat{\pi}(\theta)\,d\theta\Big)\right]\,\hat{W}^m. \tag{3.36}$$

Here we summarize a temporal result; with the notations (3.22) and (3.36) and the constraints (3.20), (3.21) and (3.23), the algebra (3.25), (3.26), (3.27), (3.34) and (3.35) is equivalent to the fundamental algebra (3.8), (3.9) and (3.10) including the central extension (3.19). Noticeable effects of the central extension are the twist relation (3.35) and the anomalous commutator (3.34). These features also affect representation of the algebra as seen in the following sections.

4 Representations

4.1 Without the central extension

Now we proceed to construct representations of the algebra defined by (3.25)-(3.27) and other vanishing commutators with the constraints (3.20), (3.21) and (3.23).

Remember that $\hat{P}$ and $\hat{N}$ are self-adjoint and that $\hat{U}$ and $\hat{W}$ are unitary. Both of the relations (3.25) and (3.27) are isomorphic to (2.1). Hence the Ohnuki-Kitakado representation works well for them. $\hat{P}$ and $\hat{U}$ act on the Hilbert space H_α via (2.3) and (2.4). $\hat{N}$ and $\hat{W}$ act on another Hilbert space H_β via

$$\hat{N}\,|n+\beta\rangle = (n+\beta)\,|n+\beta\rangle, \tag{4.1}$$

$$\hat{W}\,|n+\beta\rangle = |n+1+\beta\rangle. \tag{4.2}$$

The value of α is arbitrary. However β is restricted to be an integer if we impose the condition (3.21).

For $\hat{\varphi}$ and $\hat{\pi}$ the Fock representation works. We define operators $\hat{a}_n$ and $\hat{a}^\dagger$ by

$$\hat{\varphi}(\theta) = \frac{1}{2\pi}\sum_{n\neq 0}\sqrt{\frac{\pi}{|n|}}\,(\hat{a}_n\,e^{in\theta} + \hat{a}_n^\dagger\,e^{-in\theta}), \tag{4.3}$$

$$\hat{\pi}(\theta) = \frac{i}{2\pi}\sum_{n\neq 0}\sqrt{\pi|n|}\,(-\hat{a}_n\,e^{in\theta} + \hat{a}_n^\dagger\,e^{-in\theta}). \tag{4.4}$$

In the Fourier series the zero-mode $n=0$ is excluded because of the constraints (3.20) and (3.23). It is easily verified that the commutator (3.26) is equivalent to

$$[\,\hat{a}_m, \hat{a}_n^\dagger] = \delta_{mn} \quad (m,n = \pm1, \pm2, \cdots) \tag{4.5}$$

with the other vanishing commutators. Hence the ordinary Fock space F gives a representation of $\hat{a}$'s and $\hat{a}^\dagger$'s.

Consequently the tensor product space $H_\alpha \otimes F \otimes H_0$ gives an irreducible representation of the fundamental algebra without the central extension. The inequivalent ones are parametrized by α $(0 \le \alpha < 1)$.

A remark is in order here; the coefficients in front of $\hat{a}$'s in (4.3) and (4.4) are chosen to diagonalize the Hamiltonian of free field

$$\begin{aligned}\hat{H} &:= \frac{1}{2}\int_0^{2\pi}\Big[\Big(\frac{1}{2\pi}\hat{P}+\hat{\pi}(\theta)\Big)^2+\Big(\partial\hat{\varphi}(\theta)+\hat{N}\Big)^2\Big]\,d\theta \\ &= \frac{1}{2}\Big(\frac{1}{2\pi}\hat{P}^2+2\pi\hat{N}^2\Big)+\sum_{n\neq 0}|n|\Big(\hat{a}_n^\dagger\,\hat{a}_n+\frac{1}{2}\Big). \end{aligned}\tag{4.6}$$

This Hamiltonian corresponds to the Lagrangian density

$$\mathcal{L}=\frac{1}{2}\,\partial_\mu\phi^\dagger\,\partial^\mu\phi. \tag{4.7}$$

Interacting field theory will be briefly discussed later.

4.2 With the central extension

Next we shall construct representations of the algebra defined by (3.25), (3.26), (3.27), (3.34) and (3.35) and other vanishing commutators with the constraints (3.20), (3.21) and (3.23). The way of construction is similar to the previous one.

Taking account of the twist relation (3.35), the representation of $\hat{P}$, $\hat{U}$, $\hat{N}$ and $\hat{W}$ are given by

$$\hat{P}\,|\,p+\alpha;\,n\rangle=(p+\alpha)\,|\,p+\alpha;\,n\rangle, \tag{4.8}$$

$$\hat{U}\,|\,p+\alpha;\,n\rangle=|\,p+1+\alpha;\,n\rangle, \tag{4.9}$$

$$\hat{N}\,|\,p+\alpha;\,n\rangle=n|\,p+\alpha;\,n\rangle, \tag{4.10}$$

$$\hat{W}\,|\,p+\alpha;\,n\rangle=|\,p-2+\alpha;\,n+1\rangle. \tag{4.11}$$

The inner product is defined by

$$\langle p+\alpha;\,m\,|\,q+\alpha;\,n\rangle=\delta_{pq}\,\delta_{mn}\quad(p,q,m,n\in Z). \tag{4.12}$$

The Hilbert space formed by completing the space of linear combinations of $|\,p+\alpha;\,n\rangle$ is denoted by T_α. (T indicates "twist".)

Let us turn to $\hat{\varphi}$ and $\hat{\pi}$. Considering the anomalous commutator (3.34), after a tedious calculation we obtain a Fourier expansion

$$\hat{\varphi}(\theta)=\sum_{n\neq 0}\frac{1}{\sqrt{2|n|}}\,(\hat{a}_n\,e^{in\theta}+\hat{a}_n^\dagger\,e^{-in\theta}), \tag{4.13}$$

$$\hat{\pi}(\theta)=\frac{i}{2\pi}\sum_{n=1}^{\infty}\sqrt{2n}\,(-\hat{a}_n\,e^{in\theta}+\hat{a}_n^\dagger\,e^{-in\theta}) \tag{4.14}$$

and the commutation relations (4.5). It should be noticed that only positive n's appear in the expansion of $\hat{\pi}$ even though both of positive and negative n's appear in $\hat{\varphi}$. The algebra

(4.5) is also represented by the Fock space F. Hence the tensor product space $T_\alpha \otimes F$ gives an irreducible representation of the fundamental algebra with the central extension for each value of $\alpha\,(0 \le \alpha < 1)$.

5 Summary and discussion

In this paper we have defined the algebra of the abelian sigma model in $(1+1)$ dimensions and constructed representation spaces. In the context of classical theory, this model has a field variable $\phi \in Q = \mathrm{Map}(S^1; S^1)$. The degrees of freedom are decomposed as

$$Q \cong S^1 \times \mathrm{Map}_0(S^1; \boldsymbol{R}) \times \boldsymbol{Z} \tag{5.1}$$

by (3.12). The right-hand side is a direct product of topological spaces. The first component represents the zero-mode; the second one describes the fluctuation mode; the third one corresponds the winding number. Topological nature is concentrated in the first and the third components. On the other hand the group $\Gamma = \mathrm{Map}(S^1; U(1))$ acts on Q transitively. Its covering group $\tilde{\Gamma}$ is also decomposed as

$$\tilde{\Gamma} \cong \boldsymbol{R} \times \mathrm{Map}_0(S^1; \boldsymbol{R}) \times \boldsymbol{Z} \tag{5.2}$$

by (3.14). The right-hand side is a direct product of topological groups. We assign the algebra (3.8)-(3.10) to Q and $\tilde{\Gamma}$. According to (5.1) and (5.2), the algebra is decomposed into (3.25)-(3.27). When the central extension (3.19) is included, the anomalous commutator (3.34) and the twist relation (3.35) must be added. An irreducible representation space is constructed by tensor product of two Ohnuki-Kitakado representations with one Fock representation. We obtain inequivalent ones parametrized by a parameter $\alpha\,(0 \le \alpha < 1)$. The anomalous commutator eliminates negative-modes from $\hat{\pi}(\theta)$ by means of (4.14). The twist relation causes (4.11); the winding operator $\hat{W}$ increases the winding number by one unit and simultaneously decreases the zero-mode momentum by two units.

Physical implication of our model is not clear yet. Roles of the parameter α, the anomalous commutator and the twist relation are to be examined further. As a model with interaction, the sine-Gordon model

$$\mathcal{L} = \frac{1}{2}\partial_\mu \psi(x) \partial^\mu \psi(x) + \kappa^2 \cos(\psi(x)) \tag{5.3}$$

may be interesting. Substitution of $\phi = e^{i\psi}$ defines the corresponding Hamiltonian

$$\hat{H} \;:=\; \frac{1}{2}\int_0^{2\pi} \left[\Big(\frac{1}{2\pi}\hat{P} + \hat{\pi} \Big)^2 + \partial\hat{\phi}^\dagger \, \partial\hat{\phi} - \kappa^2 (\hat{\phi} + \hat{\phi}^\dagger) \right] d\theta$$

$$
\begin{aligned}
&= \frac{1}{2}\Big(\frac{1}{2\pi}\hat{P}^2 + 2\pi\hat{N}^2\Big) + \sum_{n\neq 0}|n|\Big(\hat{a}_n^\dagger\hat{a}_n + \frac{1}{2}\Big) \\
&\quad -\frac{\kappa^2}{2}\int_0^{2\pi}\Big(\hat{U}e^{i(\hat{\varphi}(\theta)+\hat{N}\theta)} + \hat{U}^\dagger e^{-i(\hat{\varphi}(\theta)+\hat{N}\theta)}\Big)d\theta. \qquad (5.4)
\end{aligned}
$$

The last term yields highly nonlinear complicated interaction. It is known [5] that this model has a topological soliton, which behaves like a fermion. It is expected that our formulation may shed light on the soliton physics of sigma models.

From both points of view, field theory and string theory, it is hoped to extend our model to nonabelian cases and to higher dimensions. Our model has a field configuration manifold $Q = \mathrm{Map}(S^1; S^1)$. The most general model has $Q = \mathrm{Map}(M; N)$. (1)An immediate extension is a choice $M = S^1$ and $N = T^n = (S^1)^n$. This is the toroidal compactification model of string theory [6]. (2)A rather easy extension is a choice $M = S^n$ or T^n and $N = S^1$. For $M = S^n (n \geq 2)$ there is no winding number. Yet we should pay attention to the zero-mode. There may be nontrivial central extensions. (3)Another nontrivial extension is a choice $M = S^n$ or T^n and $N = G$, that is a Lie group. This corresponds to a chiral Lagrangian model. For a nonabelian group G, we know neither existence nor uniqueness of the decomposition

$$
\Gamma := \mathrm{Map}(S^n; G) \cong G \times \mathrm{Map}_0(S^n; \mathrm{Lie}(G)) \times \pi_n(G), \qquad (5.5)
$$

where π_n denotes the n-th homotopy group and

$$
\mathrm{Map}_0(S^n; \mathrm{Lie}(G)) := \{\, g : S^n \to \mathrm{Lie}(G) \,|\, C^\infty, \int_{S^n} g = 0\}. \qquad (5.6)
$$

Even if it exists, it may not be a direct product of topological groups because of nonabelian nature. (4)A highly nontrivial one is a choice $N = G/H$, that is a homogeneous space. This model is a nonlinear sigma model. Quantum mechanics on G/H is already well-established [2]. However extension to field theory remains difficult. (5)Another interesting one is a choice $M = S^1$ and $N = T^n/P$, that is an orbifold. This is nothing but the orbifold model of string theory [7]. It is found [7] that zero-mode variables obey peculiar commutators and nontrivial quantization is obtained for string theory on an orbifold with a background 2-form. It is expected that our model may work as a simplified model to understand such a complicated behavior.

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