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遅れを含んだフォッカープランク方程式

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[要　旨]

本稿ではランジュバン方程式に「遅れ」の効果入れて拡張した確率微分方程式に対応するフォッカープランク方程式を近似的に導出する。手法としては拡張されたランジュバン方程式に近似的に対応するランダムウオークを使い、それを近似的に時間と空間について展開することによってフォッカープランク方程式を得る。このようなランダムウオークは遷移確率が一定の遅れを持った過去の位置によって決まるようなランダムウオーク (Delayed Random Walk) である。またここで得られたフォッカープランク方程式はちょうど遅れ時間と同じ時間離れた2点におけるジョイント確率分布の時間発展方程式として限られている。この式を定常状態において明示的に解き、数値計算の結果とくらべたところノイズの「大きさ」にくらべて十分に小さい「ドリフト」係数では有用な結果を確認した。

A Fokker–Planck equation with delay

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Abstract

A approximate Fokker–Planck equation which corresponds to a Langevin equation with delay is derived. Our derivation takes an indirect path using an expansion of a corresponding random walk with delayed transition probability dependence (delayed random walk). The derived Fokker–Planck equation is found to contain a derivative term by the delay parameter. An equilibrium solution is also obtained and compared with numerical simulations.

The main theme of this paper is derivation of a Fokker–Planck equation[1] for a simple Langevin equation with delay[2] to gain more insight into systems with noise and delay. Our strategy of derivation uses indirect derivation via expansion of a stochastically equivalent random walk with delayed transition probability dependence (delayed random walk)[3, 4], rather than a direct derivation from the Langevin equation. We find that the derived Fokker-Planck equation contains a derivative with respect to the delay parameter. The stationary solution of the equation is also determined and is shown to have good agreement with numerical simulation.

The Langevin equation we start with is:

$$\frac{d}{dt}X(t) = -\beta X(t-\tau) + \xi(t), \quad \langle \xi(t_1)\xi(t_2)\rangle = \delta(t_1 - t_2). \tag{1}$$

It has been shown that a delayed random walk model which has the following definition approximately corresponds to the above Langevin equation with delay [4].

$$\begin{aligned}
&P(n, t+1; s, t+1-\tau) &(2)\\
&= g(s-1)P(n-1, t; s, t+1-\tau; s-1, t-\tau)\\
&+ g(s+1)P(n-1, t; s, t+1-\tau; s+1, t-\tau)\\
&+ f(s-1)P(n+1, t; s, t+1-\tau; s-1, t-\tau)\\
&+ f(s+1)P(n+1, t; s, t+1-\tau; s+1, t-\tau),
\end{aligned}$$

$$\begin{aligned}
f(x) &= \frac{1}{2}(1+2d) \quad (x > a), \qquad \frac{1}{2}(1+\beta x) \quad (-a \le x \le a), \qquad \frac{1}{2}(1-2d) \quad (x < -a),\\
g(x) &= \frac{1}{2}(1-2d) \quad (x > a), \qquad \frac{1}{2}(1-\beta x) \quad (-a \le x \le a), \qquad \frac{1}{2}(1+2d) \quad (x < -a).
\end{aligned} \tag{3}$$

where $P(u_1, t_1; u_2, t_2)$ is the joint probability for the walker to be at u_1 and u_2 at time t_1 and t_2, respectively. $f(x)$ and $g(x)$ are transition probabilities to take a step in the negative and positive directions, respectively, at position x. Physically, this model implies that when $\tau = 0$ the transition probability for the walker to move toward the origin increases linearly at a rate of $\beta \equiv d/a$ as the distance increases from the origin to the position a, after which the transition probability is held constant. We assume that with sufficiently large a, we can ignore the probability for the walker to be outside of the range (-a, a), Within this approximation, it corresponds to the Langevin equation with delay (1).

We proceed toward a a Fokker–Planck equation by expanding the above equation using the "step operators" and its expansion as discussed in [5]. We first transform the above equation as,

$$\begin{aligned} P(n,t+1;s,t-\tau) - P(n,t;s,t-\tau) &= g(s)P(n-1,t;s,t-\tau) \\ + \quad f(s)P(n+1,t;s,t-\tau) &- (g(s)+f(s))P(n,t;s,t-\tau) \end{aligned} \tag{4}$$

where we have subtracted $P(n,t;s,t-\tau)$ from both sides and used $g(s)+f(s)=1$. In order to go from this discrete space and time model to a continuous time and space expansion, we introduce the step operators, defined by the following action on an arbitrary function h:

$$\mathcal{E}_u^+ h(u) = h(u+1), \quad \mathcal{E}_u^- h(u) = h(u-1) \tag{5}$$

In effect, $\mathcal{E}_u^+$ and $\mathcal{E}_u^-$ shift u by one. They can be expanded as

$$\mathcal{E}_u^{\pm} = 1 \pm \frac{\partial}{\partial u} + \frac{1}{2}\frac{\partial^2}{\partial u^2} \pm \dots \tag{6}$$

Using these step operators both for time and space, we can rewrite the above equation as follows:

$$(\mathcal{E}_t^+ \mathcal{E}_\tau^+ - 1)P(n,t;s,t-\tau) = [(\mathcal{E}_n^- - 1)g(s) + (\mathcal{E}_n^+ - 1)f(s)]P(n,t;s,t-\tau) \tag{7}$$

With the expansion of step operator to second order in n and the first order in t and τ, we obtain a Fokker–Planck equation.

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial \tau}\right)P(n,t;s,t-\tau) = \frac{\partial}{\partial n}[(\beta s)P(n,t;s,t-\tau)] + \frac{1}{2}\frac{\partial^2}{\partial n^2}P(n,t;s,t-\tau) \tag{8}$$

We note that this equation is for the joint probability between two points with time specifically τ apart. Also, it should be noted that we are making an implicit assumption to take τ as variables, which is not correct in exact sense. However, it leads to our approximate equation which has the delay parameter τ appears in a derivative term in a dimensionally correct way.

Let us investigate the stationary solution of the Fokker–Planck equation,

$$\frac{\partial}{\partial \tau}P_e(n,s,\tau) = \frac{\partial}{\partial n}[(\beta s)P_e(n,s,\tau)] + \frac{1}{2}\frac{\partial^2}{\partial n^2}P_e(n,s,\tau), \tag{9}$$

with a boundary condition,

$$P_e(n,s,\tau=0) = \sqrt{\frac{\beta}{\pi}}e^{-\beta n^2}\delta(n-s). \tag{10}$$

This boundary condition is chosen to satisfy the requirement that with $\tau = 0$, the stationary solution should be identical to that for the Ornstein–Uhlenbeck process.

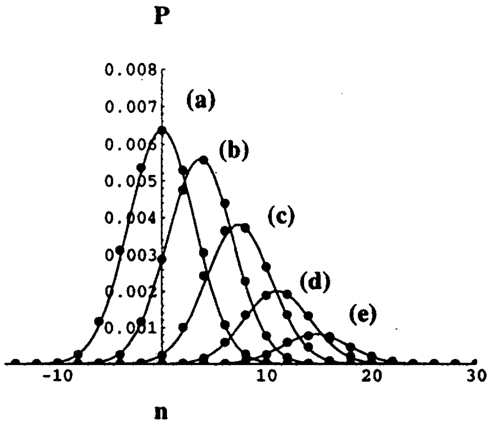

Figure 1: Comparison of the stationary probability distribution. $\beta = 0.008$, $\tau = 10$, $s = (a)0, (b)4, (c)8, (d)12, (e)16$.

The solution can be found using Fourier transforms, which involves only gaussian integrations for this equation.

$$P_e(n, s, \tau) = \left(\sqrt{\frac{\beta}{\pi}} e^{-\frac{1}{2}\beta^2 s^2 \tau - \beta n s}\right)\left(\sqrt{\frac{1}{2\pi\tau}} e^{-\frac{(n-s)^2}{2\tau}}\right). \tag{11}$$

We note that the second factor approaches the delta function with τ approaching 0 consistent with the boundary condition.

To test our approximate derivation, we have compared the solution (11) with numerical simulation with various τ. Samples are presented in Figures 1 which show good agreement with $\beta\tau \ll 1$.

Even though more thorough study of the derived equation such as investigation of the behavior of its non–stationary solution are left for the future, our systematic derivation here has provided another tool in the form of the Fokker–Planck equation to study delayed stochastic systems, particularly with respect to gaining an understanding of their probability density function. It is also of interest to consider a more direct derivation from the Langevin equation (1) given this indirect path, and this requires further investigation.

References

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