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この論文では，位置演算子Qと運動量演算子Pの有理関数$g(P)$を用いた演算子$Q+ig(P)$の固有ベクトル系を，形式的に直交化する方法，すなわち，Qと$g(P)$の最小不確定状態を直交化する方法を提案する。この種の直交化は，空間をテンソル積により拡張することによりQと$g(P)$を「可換化」し，付加された空間の真空状態に射影することをもとにしている。特に，連続ウェーブレット変換に対応する$g(P)=-kP^{-1}$の場合について詳細に調べる。

On the 'orthogonalization' of the minimum uncertainty states between the position and the rational function of the momentum

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Abstract

In this paper, a method of the extension of a Hilbert space is proposed for a formal orthogonalization of the eigenvector system of the operator $Q + ig(P)$ where Q denotes the position-coordinate operator and the and $g(P)$ is a kind of rational function of the momentum operator, in other words, for the orthogonalization of the minimum uncertainty states between Q and $g(P)$. This kind of orthogonalization is based on the 'commutabilization' between Q and $g(P)$ by the space extension by tensor product and the projection into the analogue of the vacuum vector in the additional space. Especially, the special case with $g(P) = -kP^{-1}$, which is corresponding to a kind of Naimark extension of the continuous wavelet transformtion, is investigeted in detail.

1 Introduction

In quantum mechanics, two observables X and Y which do not commute do not have their simultaneous eigenvectors. As is well known, the uncertainty product is minimized for the minimum uncertainty states which are eigenvectors of $X + icY$. The minimum unceratinty state systems of this kind are non-orthogonal over-complete systems because $X + icY$ is not hermitian. For example, the position Q and the momentum P which satisfy $[Q, P] = iI$ (where I denotes the identity op.) do not have the simultaneous eigenvectors though the lower bound of the uncertaity relation is achieved by the coherent states and a special class of the squeezed states which are not orthogonal systems. However, for this cannonical pair, it is well known that these systems can be orthogonalized formally by the extension of the Hilbert space from $\mathcal{H}$ into $\mathcal{H} \otimes \mathcal{H}_1$ by tensor product because $Q \otimes I_1 - I \otimes Q_1$ and $P \otimes I_1 + I \otimes P_1$ commute and they have the simultaneous eigenvectors. It is also known well that the coherent state system in $\mathcal{H}$ is corresponding to their simultaneous eigenvector system in $\mathcal{H} \otimes \mathcal{H}_1$ by projecting it into the vacuum vector in $\mathcal{H}_1$. (About these relations, there are many references. See, for example, [16].)

In this paper, the extension of these relations for a class of non-canonical pairs, the pair of Q and $g(P) = \sum_n \ (c_n P + d_n I)^{-1}(a_n P + b_n I)$ (where the coefficients a_n, b_n, c_n, d_n are constant real), are discussed. After a brief comment of the minimum uncertainty state of this pair, how to 'commutabilize' this pair by the space extension by the tensor product and how to relate the simultaneous eigenvector system to the minimum uncertainty state system in the original space are proposed in Section 3. Next, the spacial case with $g(P) = -ikP^{-1}$ (where k is a positive integer and P^{-1} denotes the inverse operator of P, or i-times integration operator for the wavefunction in the position representation) is investigated. This case is closely related to the wavelet transform[1-4] used in signal processing. The wavefunctions of the minimum uncertainty states of this pair are identical to Cauchy wavelets[9,10,14,15]. It is also known well that they are a kind of the GCS[8] associated with the affine group[6,7]. In section 4, for this case, the simulataneous eigenfunction and the analogue of the vacuum in the additional space are investigated in datail.

2 Minimum uncertainty state of Q and $g(P)$

For the momentum operator P and the identity operator I on the original Hilbert space $\mathcal{H}$, define

$$g(P) \equiv \sum_{n=1}^{N}(a_nP+b_nI)\ (c_nP+d_nI)^{-1} \tag{1}$$

with real-valued scalar constants a_n, b_n, c_n, d_n $(n = 1, 2, ...N)$. (We can restrict the domain of $g(P)$ into the subspace $\{\ \psi \mid (1/y)\ \langle\psi|(y-\frac{d_n}{c_n})\rangle_P$ is finite for any $n\}$ of $\mathcal{H}$ so that $g(P)$ may be bounded, though $g(P)$ is not bounded on the whole $\mathcal{H}$.) Indeed, this operator does not commute with the position operator, and, from the CCR $[Q, P] = iI$, we have $[Q, g(P)] = ig'(P)$ where $g'(x)$ denotes $\frac{dg(x)}{dx}$. This results in the uncertainty relation

$$\langle(\Delta Q)^2\rangle\ \langle(\Delta g(P))^2\rangle \geq \frac{1}{4}[\langle g'(P)\rangle]^2. \tag{2}$$

The equality of this relation holds for the eigenstates of the non-hermitian operator $A \equiv Q+icg(P)$, which has complex eigenvalue whose real part indicates approximate position of the localization of the wavepacket. By the scale change, with loss of generality, assuming $c = 1$ below, re-define

$$A \equiv Q + ig(P). \tag{3}$$

Define the eigenfunction of this operator, in the position representation and the momentum representation, respectively

$$\phi_\alpha^{(m)}(q) \equiv\ {}_Q\langle q|\alpha\rangle_A^{(m)}, \qquad \Phi_\alpha^{(m)}(p) \equiv\ {}_P\langle p|\alpha\rangle_A^{(m)} \tag{4}$$

where the eigenvalue may be degenerated and then the label (m) attatched to the eigenvector denotes the corresponding momentum interval I_m within which the function $g(x)$ is continuous. (I_m denotes the m-th interval between the poles of $g(x)$.) Then, from the characteristic equation,

$$\Phi_\alpha^{(m)}(p) = \begin{cases} (const)\cdot e^{-G(p)-i\alpha p} & (p \in I_m) \\ 0 & (\text{otherwise}) \end{cases} \tag{5}$$

where $G(x)$ denotes $\int g(p)dp$.[17] (NB: the non-uniqueness of the integral constant does not trouble anything because it affects only the normalization constant of the eigenfunction.) By making the inverse Fourier transformation, we can show that the position shift of eigenfunction in the position representation is just corresponding to the shift of the real part of the eigenvalue.

In a similar manner to the coherent state case, this eigenvector system have the over-completeness

$$\sum_m \int |\alpha\rangle_A^{(m)}\ {}_A^{(m)}\langle\alpha|\ \ w_m(Re\ \alpha)\ \ d^2\alpha = I. \tag{6}$$

For the derivation of this and the relation between $w_m(Re\ \alpha)$ and the function $g(x)$, see [17].

3 Genaral method of 'Commutabilization' of the pair

Now we are showing how to commutabilize Q and $g(P)$ for the orthogonalization of the minimum uncertainty states of them. Prepare N additional degrees of freedom $\ell = 1, 2, ..., N$ and the Hilbert spaces $\mathcal{H}_\ell$'s with the same type of $\mathcal{H}$, and let

$$\mathcal{H}_{add.} \equiv \bigotimes_{\ell=1}^{N} \mathcal{H}_\ell \tag{7}$$

be the additional Hilbert space corresponding to these degrees of freedom. By making the tensor product with this additional space, we extend the original space $\mathcal{H}$ into $\mathcal{H} \otimes \mathcal{H}_{add.}$. For the position operator Q_ℓ and the momentum operator P_ℓ and on each $\mathcal{H}_\ell$, define

$$B_\ell \equiv \frac{1}{2}\{Q_\ell, P_\ell\}. \tag{8}$$

Then, (with the notation I denoting the identity,) we have

$$[Q_\ell, P_\ell] = i\ I_\ell, \qquad [B_\ell, P_\ell] = i\ P_\ell, \qquad [B_\ell, Q_\ell] = -i\ Q_\ell. \tag{9}$$

On the extended space $\mathcal{H} \otimes \mathcal{H}_{add.}$, define the hermitian operators

$$\tilde{Q} \equiv Q \otimes I_{add.} + \sum_{\ell=1}^{N} (c_\ell P + d_\ell I)^{-1} \otimes (\bigotimes_{j=1}^{N} J_{\ell j}) \tag{10}$$

$$\text{and} \quad \tilde{G} \equiv g(P) \otimes I_{add.} + \sum_{\ell=1}^{N} (c_\ell P + d_\ell I)^{-1} \otimes (\bigotimes_{j=1}^{N} K_{\ell j}), \tag{11}$$

$$\text{with} \quad J_{\ell j} \equiv \begin{cases} c_j B_j - (a_j d_j - b_j c_j) Q_j & (\text{if } j = \ell) \\ I_j & (\text{otherwise}) \end{cases}, \quad K_{\ell j} \equiv \begin{cases} P_j & (\text{if } j = \ell) \\ I_j & (\text{otherwise}) \end{cases}. \tag{12}$$

Then, from (1) and (9)-(12), we can prove

$$[\tilde{Q}, \tilde{G}] = 0. \tag{13}$$

Since $\tilde{Q}$ and $\tilde{G}$ are hermitean, this commutability implies that they have an orthogonal simultaneous eigenvector system, where the simultaneous eigenvector $|\tilde{q}, \tilde{g}\rangle$ satisfies

$$\tilde{Q}|\tilde{q}, \tilde{g}\rangle = \tilde{q}|\tilde{q}, \tilde{g}\rangle, \qquad \tilde{G}|\tilde{q}, \tilde{g}\rangle = \tilde{g}|\tilde{q}, \tilde{g}\rangle \tag{14}$$

$$\langle \tilde{q}, \tilde{g}|\tilde{q}_*, \tilde{g}_*\rangle = 0 \quad \text{for} \quad \tilde{q} \neq \tilde{q}_* \text{ or } \tilde{g} \neq \tilde{g}_*. \tag{15}$$

This simulataneous eigenvector system can be related to the minimum uncertainty state system of Q and $g(P)$ in the original space, as follows; On the extended space $\mathcal{H} \otimes \mathcal{H}_{add.}$, define

$$\tilde{A} \equiv \tilde{Q} + i\tilde{G}. \tag{16}$$

Then, this operator is not hermitian, and it has the common eigenvectors to the above simultaneous eigenvectors, as is shown from (14),

$$\tilde{A}|\tilde{q}, \tilde{g}\rangle = (\tilde{q} + i\tilde{g})\ |\tilde{q}, \tilde{g}\rangle. \tag{17}$$

By the definition of $\tilde{Q}$ and $\tilde{G}$, and from (3) and (16), we have

$$\tilde{A} = A \otimes I_{add.} + \sum_{\ell=1}^{N} (c_\ell P + d_\ell I)^{-1} \otimes (\bigotimes_{j=1}^{N} F_{\ell j}^{\dagger}) \quad with \quad F_{\ell j} \equiv J_{\ell j} - iK_{\ell j}. \tag{18}$$

By using the eigenvector $|0\rangle_{F_{\ell\ell}}$ of $F_{\ell\ell} = c_\ell B_\ell - (a_\ell d_\ell - b_\ell c_\ell)Q_\ell - iP_\ell$ with eigenvalue 0, define the analogue of the 'vacuum' vector

$$|0\rangle_F \equiv \bigotimes_{\ell=1}^{N} |0\rangle_{F_{\ell\ell}} \tag{19}$$

in $\mathcal{H}_{add.}$. (The existance of such an analogue of the vacuum will be exemplified in the next section.) Then, by the definition, we have

$$_F\langle 0| \, (\bigotimes_{j=1}^{N} F_{\ell j}^{\dagger}) = 0. \tag{20}$$

From (17),(18) and (20), we have

$$(A \otimes I_{add.}) \, (I \otimes |0\rangle_F \, _F\langle 0|)|\tilde{q}, \tilde{g}\rangle = (I \otimes |0\rangle_F \, _F\langle 0|)\tilde{A}|\tilde{q}, \tilde{g}\rangle = (\tilde{q} + i\tilde{g})(I \otimes |0\rangle_F \, _F\langle 0|)|\tilde{q}, \tilde{g}\rangle. \tag{21}$$

This relation implies that the vector $(I \otimes |0\rangle_F \, _F\langle 0|)|\tilde{q}, \tilde{g}\rangle$ is an eigenvector of the operator $(A \otimes I_{add.})$ with the eigenvalue $(\tilde{q} + i\tilde{g})$. Hence, we have

$$(I \otimes |0\rangle_F \, _F\langle 0|)\tilde{A}|\tilde{q}, \tilde{g}\rangle = (\sum_m (const)_m \, |\tilde{q} + i\tilde{g}\rangle_A^{(m)}) \otimes |0\rangle_F. \tag{22}$$

The result is interpreted as follows; The minimum uncertainty state of Q and $g(P)$ is obtained from the simultaneous eigenvector of $\tilde{Q}$ and $\tilde{G}$ by projecting it into the analog of the 'vacuum' $|0\rangle_F$ in the additional space $\mathcal{H}_{add.}$.

4 Special case : Cauchy wavelet

As a special case with $N = 1$, $a_1 = d_1 = 0$, $b_1/c_1 = -k$, (k :positive integer), in other words, the case for the pair of Q and $(-k)P^{-1}$, the above result is just corresponding to the relation between Cauchy wavelet and the simulatneous eigenvectors in the extended space. We are going to discuss this case in detail. As is already known[10,14], the eigenfunction of the operator

$$A_k \equiv Q - ikP^{-1} \tag{23}$$

with the complex eigenvalue α is

$$_Q\langle q|\alpha\rangle_{A_k} \equiv h_k^{(\alpha)}(q) = \frac{G_k^{(\alpha)}}{(q-\alpha)^{k+1}} \tag{24}$$

with a normalization constant $G_k^{(\alpha)}$. (Often this function is called Cauchy wavelet.) This is a complex-valued wavepacket-like square-integrable function almost localized around $q = Re\ \alpha$.

The 'width' of the wavepacket is proportional to $Im\ \alpha$, and the number of large wave peaks is approximately proportional to $\sqrt{k}$.[14] This eigenfunction has a interesting property

$$h_k^{(b+ia)}(t) = \frac{1}{|a|^{1/2}} h_k^{(i)}(\frac{t-b}{a}) , \tag{25}$$

which is directly derived from (24). This property is quite same as the wavelets used in signal processing[1-4]. The over-complete eigenfunction system of A_k is related to the following space extension.

On $\mathcal{H} \otimes \mathcal{H}_1$, define

$$\tilde{Q} \equiv Q \otimes I_1 - P^{-1} \otimes (Q_1 - B_1) \ , \qquad \tilde{G} \equiv P^{-1} \otimes I_1 - P^{-1} \otimes P_1 . \tag{26}$$

Then, in the manner discussed in the previous section, we can show that they commute and they have the simultaneous eigenvectors. Define the wavefunctions of the simultaneous eigenvector, in the position representation and the momentum representation, respectively,

$$\psi_{\tilde{q},\tilde{g}}(q, q_1) \equiv (\ _Q\langle q| \otimes \ _{Q_1}\langle q_1|)\ |\tilde{q}, \tilde{g}\rangle, \qquad \Psi_{\tilde{q},\tilde{g}}(p, p_1) \equiv (\ _P\langle p| \otimes \ _{P_1}\langle p_1|)\ |\tilde{q}, \tilde{g}\rangle \ . \tag{27}$$

By noting that the simultaneous eigenvector $|p\rangle_P \otimes |p_1\rangle_{P_1}$ of the momenta is an eigenvector of the operator $\tilde{G}$ associated with the eigenvalue $\frac{1-p_1}{p}$ and that the system of the simultaneous eigenvectors of the momenta is complete in $\mathcal{H} \otimes \mathcal{H}_1$, we can show that $\Psi_{\tilde{q},\tilde{g}}(p, p_1)$ is written, with an appropriate function $S_{\tilde{q},\tilde{g}}(p_1)$, as

$$\Psi_{\tilde{q},\tilde{g}}(p, p_1) = S_{\tilde{q},\tilde{g}}(p_1)\ \delta(\frac{1-p_1}{\tilde{g}} - p) \ . \tag{28}$$

By the substitution of this relation into the differntial equation corresponding to the characteristic function of $\tilde{Q}$, we have the differential equation for $S_{\tilde{q},\tilde{g}}(p_1)$

$$(\frac{d}{dp_1} - \frac{1}{2\tilde{g}} + i\tilde{q})\ S_{\tilde{q},\tilde{g}}(p_1) = 0 \ . \tag{29}$$

The general solution of this is easily shown to be

$$S_{\tilde{q},\tilde{g}}(p_1) = K\ e^{(-\frac{1}{2\tilde{g}} + i\tilde{q})p_1} \ . \tag{30}$$

By substituting this solution into (29), we have the simultaneous eigenfunction in the momentum representation

$$\Psi_{\tilde{q},\tilde{g}}(p, p_1) = K\ e^{(\frac{1}{2\tilde{g}} - i\tilde{q})p_1} \delta(\frac{1-p_1}{\tilde{g}} - p) \ , \tag{31}$$

and by the inverse Fourier transformation we have the simultaneous eigenfunction in the position-coordinate representation

$$\psi_{\tilde{q},\tilde{g}}(q, q_1) = K e^{\frac{1}{2\tilde{g}} + i\frac{q}{\tilde{g}}} \delta(-\tilde{q} + \frac{q}{\tilde{g}} + q_1) \ . \tag{32}$$

Both are localized on the straight lines, and both function are (exponentially weighted) sinusoidal on these lines. These properties are very similar to those of the simultaneous eigenfunctions of

$Q \otimes I_1 - I \otimes Q_1$ and $P \otimes I_1 + I \otimes P_1$. However, there are some differences in the slope of the straight line and in the restriction of the p_1−intercept.

Next, we are investigating the wavefunction of $|0\rangle_F$ which is analogous to the vacuum in the additional space. In this case, $F = F_{1k} \equiv Q_1 - B_1 + ikP_1$. Define $\chi_{1k}(t_1) \equiv {}_{Q_1}\langle q_1|0\rangle_{F_{1k}}$. Then, we have the differential equation for $\chi_{1k}(q_1)$

$$((iq_1 + k)\frac{d}{dq_1} + (q_1 + \frac{i}{2}))\ \ \chi_{1k}(q_1) = 0\ . \tag{33}$$

By solving this, we have

$$\chi_{1k}(q_1) = (const.) \cdot (q_1 - ik)^{-(k+\frac{1}{2})} e^{iq_1}\ . \tag{34}$$

By letting the phase factor of the constant to be $e^{-i(\frac{k}{2}+\frac{1}{4})\pi}$, it is easily shown the shape of this eigenfunction is very similar to that of the boson vacuum, or a Gaussian packet, under an appropriate scale change, especially for large k. This resemblance is quite parallel to the resemblance between the eigenfunction of A_k and the coherent states discussed in [14]..

5 Conclusions

A concrete method for relating the minimum uncertainty state system of a kind of non-cannonical pair to the orthogonal simultaneous eigenfunction system in the space extended by the tensor product has been proposed. For the pair of Q and P^{-1} which is closely related to the continuous wavelet transform, the wavefunctions of the simultaneous eigenvectors of the 'commutabilized' pair on the extended space and those of the analogue of the 'vacuum' in the extended space have been investigated in detail.

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