P. Wiegmann 氏の報告は英文で書かれており，以下にそのアブストラクトを示す.

Hierarchical Structure of Azbel-Hofstader Problem and its Asymptotycally Exact Solution

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incommensurate な場合の Harper 方程式（cosine potential を持つ格子上の一次元シュレーディンガー方程式）の漸近的に厳密な解を与える．波動関数は string 多項式の無限次の積で与えられ，この多項式の根は Bethe 方程式を満たしている．string 仮説に基づいて解は分類され，string 仮説は漸近的に厳密な解を与えると同時に，スペクトルの階層構造を与える．

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Hierarchical Structure of Azbel-Hofstader Problem and its asymptotycally exact solution

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The problem of two-dimensional electrons in a periodic potential and a uniform magnetic field (Azbel-Hofstadter problem [1, 2]) has a rich history and numerous applications. It is equivalent (in the Landau gauge) to a one-dimensional quasiperiodic difference equation:

$$\psi_{n+1} + \psi_{n-1} + 2\lambda \cos(k_y + 2\pi n\eta)\psi_n = E\psi_n \tag{1}$$

with two competing incommensurate periods 1 and $1/\eta$. Here η is the flux of the magnetic field through the unit cell in units of the flux quantum h/e. This equation is also known as the Harper equation, the almost (or discrete) Mathieu equation, etc. It has been applied to quasicrystals, localization/delocalization transition[3, 4, 5], quantum Hall effect[6] and even DNA chains[7]. For recent review of models of Hofstadter type see [8] and references therein.

The spectrum of this equation is complex. If the flux η is a rational number P/Q, there is a common period Q in the equation (1) and the spectrum consists of Q bands. In the incommensurate limit, when η is an irrational number ($P \to \infty$, $Q \to \infty$) the spectrum becomes an infinite Cantor set[1] [1, 9] with total bandwidth (Lebesgue measure of the spectrum) $4|\lambda - 1|$ [3, 10, 11]. At $|\lambda| = 1$ the spectrum becomes a purely singular continuous—uncountable but measure zero set of points[12] (for review see [13, 5]). There is numerical evidence that in this case the spectrum and wave functions are multifractal [14].

The Azbel-Hofstadter problem as a typical quasiperiodic system generates a complex spectrum. Similar complex spectra were found in a number of dynamical systems. Since the empirical observations of Hofstadter [2], evidence has been mounting that these spectra are regular and universal rather than erratic or "chaotic". They have a deterministic hierarchical structure.

The scaling description of multifractal sets, generated by a quasiperiodic equation, an

[1] closed, nowhere dense set which has no isolated points

important and challenging problem, is far from being understood.

Recently, it has been shown that the Harper equation, belongs to the class of integrable models of quantum field theory. Despite being just a one particle problem, it has been "solved" by the Bethe Ansatz [15]. This creates a perspective for an analytical solution of the problem and eventually for the employment of methods of conformal field theory for finding scaling properties of multifractal sets.

In this lecture we present a first step towards solving the Bethe Ansatz equations for the most interesting critical case $|\lambda| = 1$[2] . We show that the topology of the strange sets, generated in the problem is determined by the Chern numbers of the spectral curve i.e. by the Hall conductances. Even more so—at every finite step of the hierarchy the spectrum is integrable. We have found an anticipated match between Hall conductances and dimensions of representations of the quantum group $U_q(sl_2)$. The latter are Takahashi-Suzuki numbers or the lengths of the strings of the Bethe Ansatz solution. This correspondence suggests a natural hierarchical tree, which, we believe, is relevant for general quasiperiodic systems.

In the Bethe Ansatz language, each state is characterized by a particular string content. Proceeding along the tree toward the incommensurate limit corresponds to addition of strings. This picture is somewhat reminiscent of the discrete renormalization group approach [21].

We were concentrated on the topological aspect of the string hypothesis. It alone gives the explicit asymptotically exact form of some wave functions for irrational flux. Multi-fractal properties of these functions, although not exact, are in good agreement with numerical results.

We show that in the incommensurate limit roots of the Bethe Ansatz equations are

[2] The initial progress toward solving Bethe equations of [15] has been made in [16] where the explicit analytical solution for zero energy level as well as some numerical results for midband levels are reported.

grouped in complexes called *strings*, so that each state is a composition of strings. Strings are well known for standard integrable models of quantum field theory, for instance XXZ Heisenberg chain [17]. They become exact only for a macroscopic system, where the number of particles and the size of the system are sent to infinity. The role of thermodynamic limit for the AH problem is obviously the incommensurability—$P, Q \to \infty$, $P/Q \to \eta =$ irrational. The common period Q plays the role of the size of the system. Indeed, we present numerical evidence that the "string hypothesis" remains valid—strings become exact in the incommensurate limit $Q \to \infty$.

A first important application of the string structure of solutions is a detailed *hierarchical tree* of the spectrum, i.e. an algorithm for generating this Cantor set spectrum[18]. The hierarchical tree gives the topology of the set. We show that the string decomposition of a state is tied to the holonomy of the wave function, i.e. the Hall conductance of the state, and therefore must be of an algebraic geometrical nature. The set of Hall conductances generates the spectral flow, which in its turn describes the hierarchical tree.

However, the major ends of the strings are loose. The string hypothesis solves the Bethe Ansatz equations with an accuracy $\mathcal{O}(Q^{-2})$, i.e. is asymptotically exact in the incommensurate limit $Q \to \infty$. However, the most interesting quantitative characteristics of the spectrum are actually in the finite size corrections of the order of $\mathcal{O}(Q^{-2})$ to the bare value of strings. Among them are the anomalous dimensions of the spectrum. We believe that it is possible to find them analytically via a more detailed study of the Bethe Ansatz equations, beyond just the analysis of singularities. This is a technically involved but a fascinating problem. The ultimate solution of the problem, however, would be through the application of conformal field theory approach, which has been proven to be effective for finding the finite size corrections of integrable systems, without the actual solution of the Bethe Ansatz.

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