Cauchy wavelets in Terms of su(1, 1)-Coherent States

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Abstract. A kind of annihilation/creation relation related to the coherent states associated with the irreducible unitary representation of the affine group is re-interpreted as an annihilation/creation relation associated with the irreducible unitary representation of the algebra su(1, 1), by adding another generator to the two generators of the unitary representation of the affine group.

Key Words: Affine group, Irreducible unitary representation, Eigenvector, Generalized coherent state, Algebra su(1, 1), Associated Laguerre polynomial, Annihilation/creation retation, Cauchy wavelet

あらまし

アフィン群のコヒーレント状態に関係があるある種の生成消滅関係は、アフィン群の既約ユニタリ表現の２つの生成子にさらにもう一つ生成子を追加することにより、代数 su(1, 1) の既約ユニタリ表現の生成消滅関係として解釈できる。この関係により、信号処理の分野で用いられるコーシー・ウェーブレットは、代数 su(1, 1) のコヒーレント状態であると解釈できる。

1. Introduction

In quantum mechanical systems, the canonical commutation relation

$$[X,Y]=-iI \tag{1}$$

is important, and the annihilation and creation operators, coherent states, number states and the number operator are derived from this commutation relation. Here, we will treat the commutation relation

$$[X,Y]=-iX \tag{2}$$

similar to (1). The Lie group associated with this algebra is called the affine group ($ax+b$ group), and it is well known that it has two unitarily equivalent irreducible unitary representations (See Gel'fand and Naimark [1], or Aslaksen and Klauder [2]). For the commutation relation (2), in a similar manner to the case of the canonical commutation relation (1), the corresponding coherent states (affine coherent states) has been known [3] [4]. The affine coherent states are the minimum uncertainty states between one and the inverse of the other of the canonical commutation pair. In signal processing, the wavepackets corresponding to the affine coherent states are called Cauchy wavelets and they are important in the theory of the continuous wavelet transformation[7].

In this paper, in this context, we will derive not only the coherent states but also the analogues of the annihilation and creation operators, the number operator and the number states which are associated to the commutation relation (2). Moreover, we will interpret them more naturally by starting from the irreducible unitary representation of the algebra su(1,1), because the irreducible unitary representation of the affine group is closely related to the irreducible unitary representation of the algebra su(1,1) associated with the Lie group $SU(1,1)$.

2. Annihilation/creation Relation Related to Irreducible Unitary Representation of Affine Group: A heuristic approach

It has been known[2] that the irreducible unitary representation of the affine group can be reduced to the following two:

$$Y=\frac{PQ+QP}{2}, \quad X=Q \quad \text{on} \quad L^2(\mathrm{R}^+)$$

$$Y=\frac{PQ+QP}{2}, \quad X=Q \quad \text{on} \quad L^2(\mathrm{R}^-),$$

where Q denotes the position coodinate operator and P denotes the momentum operator. In the following discussions, we will treat only the former representation, because the latter representation can be discussed in almost the same way only with minor changes of the signs.

It has been known that the wavefunction of the coherent state $|\eta\rangle^k$ associated with the affine group is

$$_Q\langle x|\eta\rangle^k := \sqrt{\frac{(2\,\mathrm{Im}\,\eta)^{2k+1}}{\Gamma(2k+1)}}\; x^k\, e^{i\eta x}, \quad \eta \in \mathrm{H},$$

where H denotes the upper half plane and k is a real parameter which specify the representation. Especially, in the cases of the representations with $k > 0$, The system of the coherent states constitutes the resolution of identity

$$\frac{1}{2k}\int_{\mathrm{H}} |\eta\rangle^k\; {}^k\langle\eta| \frac{d^2\eta}{4\pi(\mathrm{Im}\,\eta)^2} = I,$$

while this does not exist in the cases of the representations with $-1/2 < k < 0$ because of the divergence of the integration.

Next, we will introduce the operator which has $|\eta\rangle^k$ as its eigenvector associated with the eigenvalue η for $\forall\eta \in \mathrm{H}$. Define

$$A_k := P + ikQ^{-1},$$

formally. Then, it is easily shown that

$$A_k|\eta\rangle^k = \eta|\eta\rangle^k \quad \forall\eta \in \mathrm{H}$$

The operator A_k has a similar property to the boson annihilation operator in the sense that it has the coherent states as its eigenvectors. However, the operator A_k does not have the property of the annihilation operator which decrease the number of the number states.

We can find heuristically an operator having the above both properties, as a function of A_k, as follows; Define

$$a_k := (A_k - iI)(A_k + iI)^{-1}. \tag{3}$$

Then

$$a_k|\eta\rangle^k = \frac{\eta - i}{\eta + i}|\eta\rangle^k \quad \forall\eta \in \mathrm{H}.$$

By letting $\zeta := \frac{\eta-i}{\eta+i}$ and $|\zeta\rangle_a^k := |\eta\rangle^k$, we have

$$a_k|\zeta\rangle_a^k = \zeta|\zeta\rangle_a^k \quad \forall\zeta \in \mathrm{D},$$

where D denote the inside of the unit disk. Moreover, by regarding $|0\rangle_a^k$ as the vacuum, define

$$|n\rangle_N^k := \frac{1}{||(a_k^*)^n|o\rangle_a^k||}(a_k^*)^n|o\rangle_a^k.$$

Then we have a kind of annihilation-creation relations

$$a_k|n\rangle_N^k = \sqrt{\frac{n}{n+2k}}\; |n-1\rangle_N^k$$

$$a_k^*|n\rangle_N^k = \sqrt{\frac{n+1}{n+2k+1}}\; |n+1\rangle_N^k.$$

Moreover, by defining

$$N_k := \frac{1}{2}(PQP + k^2Q^{-1} + Q - (2k+1)I),$$

we have

$$N_k|n\rangle_N^k = n|n\rangle_N^k$$

where N_k can be regarded as the number operator. The wavefunction of the number state is

$${}_Q\langle x|n\rangle_N^k(x) = \sqrt{\frac{2^{2k+1}n!}{\Gamma(n+2k+1)}}e^{-x}x^k S_n^{2k}(2x)$$

where $S_n^l(x)$ is the Sonine Polynomial (or the associated Laguerre polynomial) defined by

$$S_n^l(x) := \sum_{m=0}^{n} \frac{(-1)^m}{(n-m)!}\frac{\Gamma(n+l+1)x^m}{\Gamma(m+l+1)m!}.$$

(Sometimes another definition with $n+l$ instead of l is used.) The inverse Fourier transform of this wavepacket is identical to the Cauchy wavelet used in signal processing whose basic wavelet function is $\frac{(const.)}{(t+i)^{k+1}}$.

Among the operators a_k, a_k^* and N_k, the following relations hold;

$$a_k^* a_k = (N_k + 2kI)^{-1}N_k$$

$$a_k a_k^* = (N_k + (2k+1)I)^{-1}(N_k + I)$$

$$[a_k, a_k^*] = 2k(N_k + 2kI)^{-1}(N_k + (2k+1)I)^{-1}$$

$$[a_k, N_k] = a_k \tag{4}$$

$$[N_k, a_k^*] = a_k^*, \tag{5}$$

where the relations (4) and (5) have the same forms as usual boson cases.

3. Interpretation of Annihilation-Creation Relations Related to Affine Group in Terms of Algebra su(1,1)

The linear fractional transform (Möbius transform) used in (3) in Section 2 was found only heuristically. However, in this section, we will give a natural derivation of this structure from the context of the irreducible unitary representation of the algebra su(1,1) which satisfies some conditions. In this context, we will re-interpret the annihilation-creation relations introduced in Section 2 more clearly, in this section and the next section.

The irreducible unitary representation of the algebra su(1,1) is given by the triple of skew-adjoint operators (E_0, E_+, E_-) satisfying the commutation relations

$$\begin{aligned}[E_0, E_\pm] &= \pm 2E_\pm \\ [E_+, E_-] &= E_0.\end{aligned} \tag{6}$$

(Note that the unitary representation of the affine group is given here if we pay attention only to $-iE_0$ and $-iE_+$.) Here, be careful about the fact that the representation of the Lie group $SU(1,1)$ can not be always constructed from the representation of the algebra su(1,1), because $SU(1,1)$ is not a universal covering group. Since the operators E_0, E_+ and E_- have continuous spectra usually under each unitary representation, it is difficult to analyze the forms of the representation. In order to avoid this difficulty, instead of the triple (E_0, E_+, E_-), we will use the triple (L_0, L_+, L_-) defined by

$$L_0 := i(E_- - E_+)$$

$$L_\pm := (E_0 \pm i(E_+ + E_-))/2,$$

where the triple (L_0, L_+, L_-) satisfies the same commutation relations

$$\begin{aligned} [L_0, L_\pm] &= \pm 2L_\pm \\ [L_+, L_-] &= L_0. \end{aligned} \tag{7}$$

The operators L_0, L_+ and L_- are not skew-adjoint, but they satisfy

$$L_0^* = L^0, \quad L_+^* = L_-. \tag{8}$$

It has been known that giving the triple of skew-adjoint operators which satisfy (6) is equivalent to giving the triple which satisfies (7) and (8) [6]. The commutation relation $[L_0, L_\pm] = \pm 2L_\pm$ shows that the operator $L\pm$ is a kind of up/down-ladder of the eigenvector system of L_0. From this fact, and from the irreducibility and the unitarity, it is shown that only the follwing three cases are possible;

$$Case1 : \dim \ \mathrm{Ker} \ L_+ = 0, \quad \dim \ \mathrm{Ker} \ L_- = 1$$

$$Case2 : \dim \ \mathrm{Ker} \ L_+ = 1, \quad \dim \ \mathrm{Ker} \ L_- = 0$$

$$Case3 : \dim \ \mathrm{Ker} \ L_+ = 0, \quad \dim \ \mathrm{Ker} \ L_- = 0.$$

Case 2 can be reduced to Case 1 by some changes of the signs. In this paper, we will not treat Case 3, and we will discuss only Case 1 in the following. In this case, the minimum eigenvalue of L_0 exists.

Let λ be the minimum eigenvalue of L_0. From the unitarity of the representation, λ should be positive [6]. Let v_0 be the eigenvector of L_0 associated with the eigenvalue λ. Then, we have

$$\begin{aligned} L_0 v_n &= (\lambda + 2n)v_n \quad (n \geq 0), & L_- v_n &= -n(\lambda + n - 1)v_{n-1} \quad (n \geq 1) \\ L_+ v_n &= v_{n+1} \quad (n \geq 0), & L_- v_0 &= 0, \end{aligned}$$

for

$$v_n := (L_+)^n v_0.$$

Here, from the irreducibility of the representation, we can show that $\{v_n | n = 0, 1, 2, 3....\}$ constitutes a CONS. These facts show that the minimum eigenvalue λ of L_0 specify the irreducible unitary representation uniquely, in Case 1. Especially, in Case 1, the irreducible unitary representation of the Lie group $SU(1,1)$, which is not a universal covering group, can be constructed if and only if λ is a positive integer. By letting

$$|n\rangle_N := \sqrt{\frac{\Gamma(\lambda)}{n!\Gamma(\lambda+n)}}v_n, \quad N := \frac{1}{2}(L_0 - \lambda)$$

we have

$$N|n\rangle_N = n|n\rangle_N,$$

hence N and $|n\rangle_N$ are regarded as the number operator and the number state, respectively.

Next, we define $a := L_+^{-1}N = \frac{1}{2}L_+^{-1}(L_0 - \lambda)$, where the well-defined-ness is guaranteed because Ran L_+ is included in Ran N. Then, we have

$$a|n\rangle_N = \sqrt{\frac{n}{n+\lambda-1}}|n-1\rangle_N,$$

and similarly we have

$$a^*|n\rangle_N = \sqrt{\frac{n+1}{n+\lambda}}\,|n+1\rangle_N.$$

Moreover, we have the following relations;

$$\begin{aligned}
a^*a &= (N+(\lambda-1)I)^{-1}N \\
aa^* &= (N+\lambda I)^{-1}(N+I) \\
[a,a^*] &= (\lambda-1)(N+(\lambda-1)I)^{-1}(N+\lambda I)^{-1} \\
[a,N] &= a \\
[N,a^*] &= a^*.
\end{aligned}$$

Thus, with the correspondence $\lambda = 2k+1$, we have made a systematic derivation of the same type of the relations as the relations introduced at the end of Section 2, in terms of the irreducible unitary representation of the algebra su$(1,1)$.

Next we will introduce the coherent states. According to Perelomov[5], define the coherent state associate with $SU(1,1)$ by

$$\begin{aligned}
|\zeta\rangle_a &:= U\left(\frac{1}{2}e^{i\arg\zeta}\ln\frac{1+|\zeta|}{1-|\zeta|}\right)\,|0\rangle_N \\
&= \exp(\zeta L_+)\exp\left(\frac{1}{2}\ln(1-|\zeta|^2)\,L_0\right)\exp(\zeta^* L_-)\,|0\rangle_N \\
&= (1-|\zeta|^2)^{\lambda/2}\exp(\zeta L_+)\,|0\rangle_N
\end{aligned}$$

with

$$U(\xi) := \exp\left(\xi L_+ - \xi^* L_+^*\right).$$

Then, from the commutation relation $[a, \exp(\zeta L_+)] = \zeta \exp(\zeta L_+)$ which is derived from $[a, L_+] = I$, and by using the relation $\exp\left(\frac{1}{2}\ln(1-|\zeta|^2)\, L_0\right)\exp(\zeta^* L_-)\ |0\rangle_N = (1-|\zeta|^2)^{\lambda/2}|0\rangle_N$, we have

$$a|\zeta\rangle_a = \exp(\zeta L_+)\ a\ |0\rangle_N + \zeta \exp(\zeta L_+)|0\rangle_N = \zeta|\zeta\rangle_a$$

This shows that the operator a has the other property of the annihilation operator, the property that it has the coherent states as its eigenvectors.

Thus, the annihilation-creation relations and their relation to the coherent states have been defined naturally from the context of the irreducible unitary representation of the algebra su$(1,1)$. In our framework, we have not used L_- as the annihilation operator but we have used $a = L_+^{-1}N$, because L_- does not have the coherent states as the eigenvectors.

Note that the above operator a approaches to the boson annihilation operator as λ tends to infinity, in the following sense; Let $U_\lambda : \mathcal{H} \to L^2(\mathrm{R})$ be the unitary map such that $U_\lambda(|n\rangle_N) = |n\rangle_{\mathbf{boson}}$. Then, we can show that $U_\lambda a U_\lambda^*$ converges to the boson anniholation operator in the sense of the weak convergence as λ tends to infinity.

4. Concrete Representation of su(1, 1) corresponding to Results for Affine Group Given in Section 2

In this section, we will discuss what concrete representation explains the correspondence between the framework given in Section 3 in terms of the irreducible unitary representation and the heuristic results for the affine group given in Section 3.

On $L^2(\mathrm{R}^+)$, by choosing

$$E_0 = i(PQ+QP), \quad E_+ = iQ$$

$$E_- := -i(PQP + k^2Q^{-1})$$

formally, the results for the affine group given in Section 2 can be directly derived from the general framework given in Section 3. However, especially in the cases where $-1/2 < \lambda < 1/2$, we should be careful about the domain of the operator E_-, as follows; First, define the operator

$$\tilde{E}_{-,k} := -i(PQP + k^2Q^{-1}) \quad (k > -1/2)$$

on the dense subspace $\mathcal{D}_o(\tilde{E}_{-,k})$ of $L^2(\mathrm{R}^+)$ defined by

$$\mathcal{D}_o(\tilde{E}_{-,k}) := \left\{ f(x) = x^k f_0(x) \in L^2(\mathrm{R}^+) \cap C^1(\mathrm{R}^+) \;\middle|\; \begin{array}{l} (2k+1)x^k f_0'(x) + x^{k+1} f_0''(x) \in L^2(\mathrm{R}^+), \\ \limsup\limits_{s\to 0} f_0(s) < \infty, \quad x^k f_0(x) \to 0 \text{ as } x \to \infty \end{array} \right\}.$$

Then, it is confirmed that $i\tilde{E}_{-,k} = PQP + k^2Q^{-1}$ is a symmetric operator on $\mathcal{D}_o(\tilde{E}_{-,k})$, from the fact that the difference

$$\int_s^t \left(\left(PQP + k^2Q^{-1}\right) f\right)^*(x) g(x)\, dx - \int_s^t (f(x))^* \left(\left(PQP + k^2Q^{-1}\right) g\right)(x)\, dx$$

$$
\begin{aligned}
&= \left[x(f'(x))^* g(x) - x g'(x)(f(x))^*\right]_s^t \\
&= t(f'(t))^* g(t) - t g'(t)(f(t))^* \\
&\quad - \left(s g(s) \left((f'(s))^* - \frac{k}{s}(f(s))^* \right) - s(f(s))^* \left(g'(s) - \frac{k}{s} g(s) \right) \right) \\
&= t(f'(t))^* g(t) - t g'(t)(f(t))^* - \left(s g(s) s^k (f_0'(s))^* - s(f(s))^* s^k g_0'(s) \right)
\end{aligned}
$$

tends to zero as $s \to 0, t \to \infty$. Moreover, we can show that $D(\tilde{E}^*_{-,k}) \cap C^2(\mathrm{R}^+) = \overline{\mathcal{D}_o(\tilde{E}_{-,k})}$. Therefore, $\tilde{E}_{-,k}$ is a closable operator and $iE_{-,k} := i\check{E}_{-,k}$ is a self-adjoint operator. Note that the operator $E_{-,k}$ is quite different $E_{-,-k}$.

By letting $L_{+,k}, L_{-,k}, L_{0,k}, A_k, N_k$ be $L_+, L_-, L_0, \tilde{A}, N, |n\rangle_N$, respectively, with $\tilde{A} := -i(a+I)(a-I)^{-1}$ defined on the vector space whose elements are finite linear sums of $\{|n\rangle_N\}_{n=0}^{\infty}$, we have

$$
\begin{aligned}
L_{+,k} &= \frac{1}{2}\left(i(PQ+QP) - Q + PQP + k^2 Q^{-1} \right), \\
L_{-,k} &= \frac{1}{2}\left(i(PQ+QP) + Q - PQP - k^2 Q^{-1} \right), \\
L_{0,k} &= \left(PQP + k^2 Q^{-1} + Q \right), \quad A_k = P + ikQ^{-1}, \\
N_k &= \frac{1}{2}\left(PQP + k^2 Q^{-1} + Q - 1 - 2k \right),
\end{aligned}
$$

In this representation, the minimum eigenvalue of $L_{0,k}$ is $\lambda = 2k+1$. Moreover, the affine coherent state $|\eta\rangle^k$ and the related number state $|n\rangle_N^k$ are shown to be the su$(1,1)$-coherent state $|-i\frac{\zeta+1}{\zeta-1}\rangle_a$ and the su$(1,1)$-number state $n\rangle_N$, respectively.

Thus the whole results given in Section 2 have been re-interpreted in terms of the irreducible unitary representations of the algebra su$(1,1)$.

5. Conclusion

We have derived a kind of annihilation/creation relation similar to the boson annihilation/creation relation, from the irreducible unitary representation of the algebra su$(1,1)$. The proposed framework contains an annihilation/creation relation related to the affine coherent states as a special case. In signal processing, in a similar manner to the boson annihilation/creation relation applied to the operator method in Wiener-Hermite expansion of Gaussian stochastic processes, our results may be applied to an analogous operator method for the stochastic processes related to the χ^2-distribution and the wavelets, because the affine coherent states are closely related to them.

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