Title
Periodic Orbits Picture Of Weierstrass-Like Magnetoconductance Fluctuations In Quantum Dots

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Using the semiclassical Kubo formula for conductivity, we give a periodic-orbits picture for the recently observed fractal magnetoconductance fluctuations in micron-sized phase coherent ballistic billiards. The self-similar conductance fluctuations is shown to be caused by the self-similar unstable periodic orbits which are generated through a sequence of isochronous pitchfork bifurcations of straight-line librating orbits oscillating towards harmonic saddles. The saddles are naturally created right at the point of contact with the leads or at certain places in the cavity as a consequence of the softwall confinement. Therefore our mechanism is able to explain all the self-similar magnetoconductance fluctuations in general softwall billiards. In contrast to the existing theory which claims that a classical mixed phase space is necessary, we argue that, even in the fully chaotic phase space, the self-similar magnetoconductance fluctuations should be observed as long as the self-similar unstable periodic orbits are preserved.

Without loss of generality, to make our idea clearly presented, we choose Henon-Heiles potentials as the ideal soft wall model for triangle billiard with three leads attached at its three edges. The harmonic saddles are obvious. The self-similar periodic orbits are then, created through bifurcation of the straight librating orbits oscillates towards/backward the leads. From the self similarity of the orbits, we have the following approximate scaling relation:

$$\Theta_{po}^n = \lambda \Theta_{po}^{n-1}, \quad 0 < \lambda < 1. \quad (1)$$

Here $\Theta_{po}^n$ denotes the area enclosed by the periodic orbit generated at the $n^{th}$ bifurcation.

The oscillating part of the conductivity can be approximated by the semiclassical Kubo formula in terms of periodic orbits as follows

$$\delta G_{xx}(E, B) = \frac{g e^2}{h V} \sum_{po} C_{xx}^{po} \frac{R_{po}(\tau_{\beta}) F_{po}(\tau_{\alpha})}{\| \text{Det}(M_{po} - I) \|^{1/2}} \times \cos\left(\frac{S_{po}}{\hbar} + \frac{\mu_{po}}{2} \eta\right). \quad (2)$$
Here $S_{po}$ is the action evaluated at the Fermi energy $E_F$. $\mu_{po}$ and $\tilde{M}_{po}$ are the Maslov index and the stability matrix of each periodic orbit, respectively. The temperature $T$ selects only a few shortest periodic orbits that contribute to the trace through a damping factor $R_{po}(\tau) = (T_{po}/\tau)/\sinh(T_{po}/\tau)$, where $T_{po}$ is the period of the periodic orbits and $\tau = \hbar/(\pi kT)$. Damping due to a finite mean free path is given by $F_{po}(T_s) = \exp(-\tau_{po}/2T_s)$, where $\tau_s$ is the scattering time. $g$ is a spin factor, $V$ is the volume of the system considered, and $C_{xx}$ is the velocity-velocity correlation function of the periodic orbit, defined by

$$C_{xx}^{po} = \int_0^\infty dt e^{-t/T_s} \int_0^{T_{po}} dt' v_x(t')v_x(t' + t). \tag{3}$$

On switching on a small magnetic field $B$, we can assume that only the phase of the electron is changed, and the periodic orbits (the phase-space structure) remain unchanged. Then we can expand $S_{po}$ up to the first order in $B$ as follows:

$$S_{po}(E, B) = S_{po}(E, 0) + \frac{e}{c} \Theta_{po} B. \tag{4}$$

Where $\Theta_{po}$ is the area enclosed by each periodic orbit. For weak enough magnetic field $B$, considering the contributions from $\pm \Theta_{po}$, i.e., from a pair of time-reversal symmetric orbits, we can rewrite the cosine terms as $2 \cos(\frac{S_{po}(E, 0)}{\hbar} + \frac{\mu_{po}}{2} \pi) \cos(\frac{e}{hc} \Theta_{po} B)$. Now, let us suppose that, through some kind of bifurcations, we have a sequence of periodic orbits which are self similar, and satisfy Eq. (1). Then we can expect that the fluctuations of the magnetoconductance should be characterized by many scales, with the smallest one being $\Theta_{po}$, i.e., the largest area enclosed by the periodic orbits. In Fig 1, using few shortest self-similar periodic orbits, $\delta G_{xx}(B)$ is plotted.

In the present system, assuming that the function we got can be approximated by the well-known Weierstrass function (this assumption will be made clear below in the next paragraph), we can calculate the fractal dimension as $D_F = 2 - H$, where $H$ $(0 < H \leq 1)$ is the Hurst exponent which is defined as the ratio of the logarithmic values of the scaling constants in the $y-$ and $x-$directions. $D_F$ of the fractal magnetoconductance fluctuations turns out to be equal to 1.7.

The obvious self-similarity of the function shown in Fig. 1, will tease one, at least to approximate the function by the well-known Weierstrass function. The Weierstrass function is continuous but nowhere differentiable.
Figure 1: Each row shows successive magnifications of fractal fluctuations of $\delta G_{xx}$ around $B = 0$ Tesla for different value of $\lambda$ ($\lambda = 0.163$ (a), $\lambda = 0.44$ (b)) and for $T = 0.01$ Kelvin. The vertical coordinate is scaled in arbitrary unit. The horizontal scale is in unit Tesla. The scale factor of each magnification in the horizontal direction is equal to $\lambda^{-2}$. 
It is commonly used to generate self-affine curves. In its simplest form, it can be written as follows:

\[ f(x) = \sum_{n=0}^{\infty} a_n \cos(b_n \pi x), \]

(5)

where \(0 < a \equiv a_n/a_{n-1} < 1\), and \(b \equiv b_n/b_{n-1}\) is an odd integer satisfies \(ab > 1 + \frac{3}{2}\pi\). In less mathematical words, the frequency \(b_n\) must form an arithmetic progression, to guarantee the scaling in the \(x\)-direction of the function. Whereas, the amplitude \(a_n\) should decrease quickly enough to make the function converge, and again the arithmetic decreasing progression of \(a_n\) guarantees the scaling in the \(y\)-direction of the function. As it is clear from Eq. (6), \(f(x)\) has no smallest scale. But it does have a largest scale, which corresponds to \(b_0\). As we will show below, the function we obtained from equation (1)-(4) is not similar to Eq. (6).

To be able to compare the function we obtained from the Kubo formula with the Weierstrass function, let us ignore the contributions from the non-self-similar periodic orbit as well as from the repetitions of the self-similar periodic orbits to the conductance fluctuations. Then \(b_n\) in Eq. (6) corresponds to \(\varepsilon \Theta_{po}^{(n)}\). From Eq. (4), \(b_n\) in our function certainly forms an arithmetic progression as \(n\) is increased. It guarantees the scaling in the \(x\)-direction which is obvious in Fig. 3. It also shows the presence of the Weierstrass type spectrum, \(\lambda^n\) (\(0 < \lambda < 1, n = 0, 1, 2, \ldots\)). However, in our case, we have upper bound scale, which corresponds to \(\Theta_{po}^{(0)}\) (the largest among the areas enclosed by the self-similar periodic orbits), and, as shown in Fig. 3 causes the self similarity to be truncated at certain smallest scale. This is the first difference between our function and Weierstrass function modelled as in Eq. (6).

\(a_n\) in our function can be written as

\[ a_n \sim A_p^{(n)}(\lambda, T) \cos\left(\frac{R}{h} - \mu \frac{\pi}{2}\right), \]

(6)

where \(A_p^{(n)}(\lambda, T)\) collects the contributions from all terms in the amplitude of Eq. (1), and certainly depends on \(\lambda\) and \(T\). For ballistic and phase-coherent systems, the value of \(\tau_\beta\) and \(\tau_S\) is large enough so that the damping factors \(R_{po}^{(n)}(\tau_\beta)\) and \(F_{po}^{(n)}(\tau_S)\) are close to 1. The increasing progression of \(A_p^{(n)}\) as \(n\) is increased is then guaranteed by the increasing progression of \(C_{xx}^{po,n}\). The constant \(R\) is the action at \(B = 0\), \(S_{po}^{(n)}(E, 0)\), which is assumed not to be
much varied by $n$ (For $E_F \to +E_S$, all $S_{po}^{(n)}$ converges to the same value). The Maslov index, $\mu_{po}^{(n)}$ satisfies

$$\mu_{po}^{(n)} = \mu_{po}^{(n-1)} + 2$$

with $\mu_{po}^{(0)} = 5$. Using Eq. (8) we can rewrite the cosine term in Eq. (7) as $(-1)^n \sin(R/h)$, so that we have two series of $a_n$: one is for even value of $n$ which gives series of positive number, and the other is for odd $n$ whose members are negative. This is the second difference between our function and Weierstrass function. Both absolute values of the series make an increasing progression. However, unlike in Weierstrass function, it is not trivial whether they make an arithmatic progression.

The existence of those two series turns out to be the reason why the scaling in $x$-direction in our $\delta G_{xx}(B)$ is $\lambda^{-2}$ instead of $\lambda^{-1}$ as in Weierstrass function. This can be seen easily as follows: since each term in Eq. (2) can finally be expressed as $g_n = (-1)^n A_{po}^{(n)}(\lambda, T) \sin(R/h) \cos(\lambda^n \Theta_{po}^{(n)} B)$, under the transformation $B \rightarrow \lambda^{-1} B$, the vertical direction must be scaled by some ”negative” constant to be able to get $g_{(n-1)}$. The result is that the scaled graph and the unscaled graph are not similar to each other. Therefore the horizontal direction should be scaled as $B \rightarrow \lambda^{-2} B$, and by some positive scaling in the vertical direction $g_n$ will becomes $g_{n-2}$. Under this transformation, both the unscaled and the scaled graphs are similar, as shown in Fig. 1.

Summarizing all the above analysis, for a certain range of weak magnetic field $B$ and for low enough temperature $T$, $\delta G_{xx}(B)$ satisfies, at least approximately, the self-affine scaling relationship as follows:

$$\delta G_{xx}(\lambda^{-2} B) \sim \lambda^{-2H(\lambda, T)} \delta G_{xx}(B).$$

It means $\delta G_{xx}(B)$ can be rescaled by simultaneously changing the length scale in the $x$-direction by $\lambda_x = \lambda^{-2}$ and the length scale in the $y$-direction by $\lambda_y = \lambda^{-2H(\lambda, T)}$. $H(\lambda, T) \equiv \frac{\ln \lambda_y}{\ln \lambda_x}$ is called the Hurst exponent ($0 < H \leq 1$). The scaling in the $x$-direction is obviously independent on temperature $T$ (which is assumed to be small, so that the self-affinity is still preserved). The scaling in $y$-direction, however does depend on $T$. Interestingly, our numerical calculation also shows that the Hurst exponent seems to depend on the value of the weak magnetic field $B$ also.