Periodic Orbits Picture Of Weierstrass-Like Magnetoconductance Fluctuations In Quantum Dots

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Using the semiclassical Kubo formula for conductivity, we give a periodic-orbits picture for the recently observed fractal magnetoconductance fluctuations in micron-sized phase coherent ballistic billiards. The self-similar conductance fluctuations is shown to be caused by the self-similar unstable periodic orbits which are generated through a sequence of isochronous pitchfork bifurcations of straight-line librating orbits oscillating towards harmonic saddles. The saddles are naturally created right at the point of contact with the leads or at certain places in the cavity as a consequence of the softwall confinement. Therefore our mechanism is able to explain all the self-similar magnetoconductance fluctuations in general softwall billiards. In contrast to the existing theory which claims that a classical mixed phase space is necessary, we argue that, even in the fully chaotic phase space, the self-similar magnetoconductance fluctuations should be observed as long as the self-similar unstable periodic orbits are preserved.

Without loss of generality, to make our idea clearly presented, we choose Henon-Heiles potensial as the ideal soft wall model for triangle billiard with three leads attached at its three edges. The harmonic saddles are obvious. The self-similar periodic orbits are then, created through bifurcation of the straight librating orbits oscillates towards/backward the leads. From the self similarity of the orbits, we have the following approximate scaling relation:

$$\Theta_{po}^{n} = \lambda \Theta_{po}^{n-1}, \quad 0 < \lambda < 1. \tag{1}$$

Here Θ_{po}^{n} denotes the area enclosed by the periodic orbit generated at the n^{th} bifurcation.

The oscillating part of the conductivity can be approximated by the semi-classical Kubo formula in terms of periodic orbits as follows

$$\delta G_{xx}(E,B) = \frac{ge^2}{hV} \sum_{po} C_{xx}^{po} \frac{R_{po}(\tau_\beta) F_{po}(\tau_s)}{\|Det(\tilde{M}_{po} - I)\|^{1/2}} \times \cos\left(\frac{S_{po}}{\hbar} + \frac{\mu_{po}}{2}\pi\right). \tag{2}$$

Here S_{po} is the action evaluated at the Fermi energy E_F. μ_{po} and $\tilde{M}_{po}$ are the Maslov index and the stability matrix of each periodic orbit, respectively. The temperature T selects only a few shortest periodic orbits that contribute to the trace through a damping factor $R_{po}(\tau_\beta) = (T_{po}/\tau_\beta)/\sinh(T_{po}/\tau_\beta)$, where T_{po} is the period of the periodic orbits and $\tau_\beta = \frac{\hbar}{\pi k T}$. Damping due to a finite mean free path is given by $F_{po}(\tau_s) = \exp(-T_{po}/2\tau_s)$, where τ_s is the scattering time. g is a spin factor, V is the volume of the system considered, and C_{xx} is the velocity-velocity correlation function of the periodic orbit, defined by

$$C^{po}_{xx} = \int_0^\infty dt e^{-t/\tau_s} \int_0^{T_{po}} dt' v_x(t') v_x(t'+t). \tag{3}$$

On switching on a small magnetic field B, we can assume that only the phase of the electron is changed, and the periodic orbits (the phase-space structure) remain unchanged. Then we can expand S_{po} up to the first order in B as follows:

$$S_{po}(E,B) = S_{po}(E,0) + \frac{e}{c}\Theta_{po} B. \tag{4}$$

Where Θ_{po} is the area enclosed by each periodic orbit. For weak enough magnetic field B, considering the contributions from $\pm\Theta_{po}$, i.e., from a pair of time-reversal symmetric orbits, we can rewrite the cosine terms as $2\cos(\frac{S_{po}(E,0)}{\hbar} + \frac{\mu_{po}}{2}\pi)\cos(\frac{e}{\hbar c}\Theta_{po}B)$. Now, let us suppose that, through some kind of bifurcations, we have a sequence of periodic orbits which are self similar, and satisfy Eq. (1). Then we can expect that the fluctuations of the magnetoconductance should be characterized by many scales, with the smallest one being Θ^0_{po}, i.e., the largest area enclosed by the periodic orbits. In Fig.1, using few shortest self-similar periodic orbits, $\delta G_{xx}(B)$ is plotted.

In the present system, assuming that the function we got can be approximated by the well-known Weierstrass function (this assumption will be made clear below in the next paragraph), we can calculate the fractal dimension as $D_F = 2 - H$, where H $(0 < H \leq 1)$ is the Hurst exponent which is defined as the ratio of the logaritmic values of the scaling constants in the $y-$ and $x-$directions. D_F of the fractal magnetoconductance fluctuations turns out to be equal to 1.7.

The obvious self-similarity of the function shown in Fig. 1, will tease one, at least to approximate the function by the well-known Weierstrass function. The Weierstrass function is continuous but nowhere differentiable.

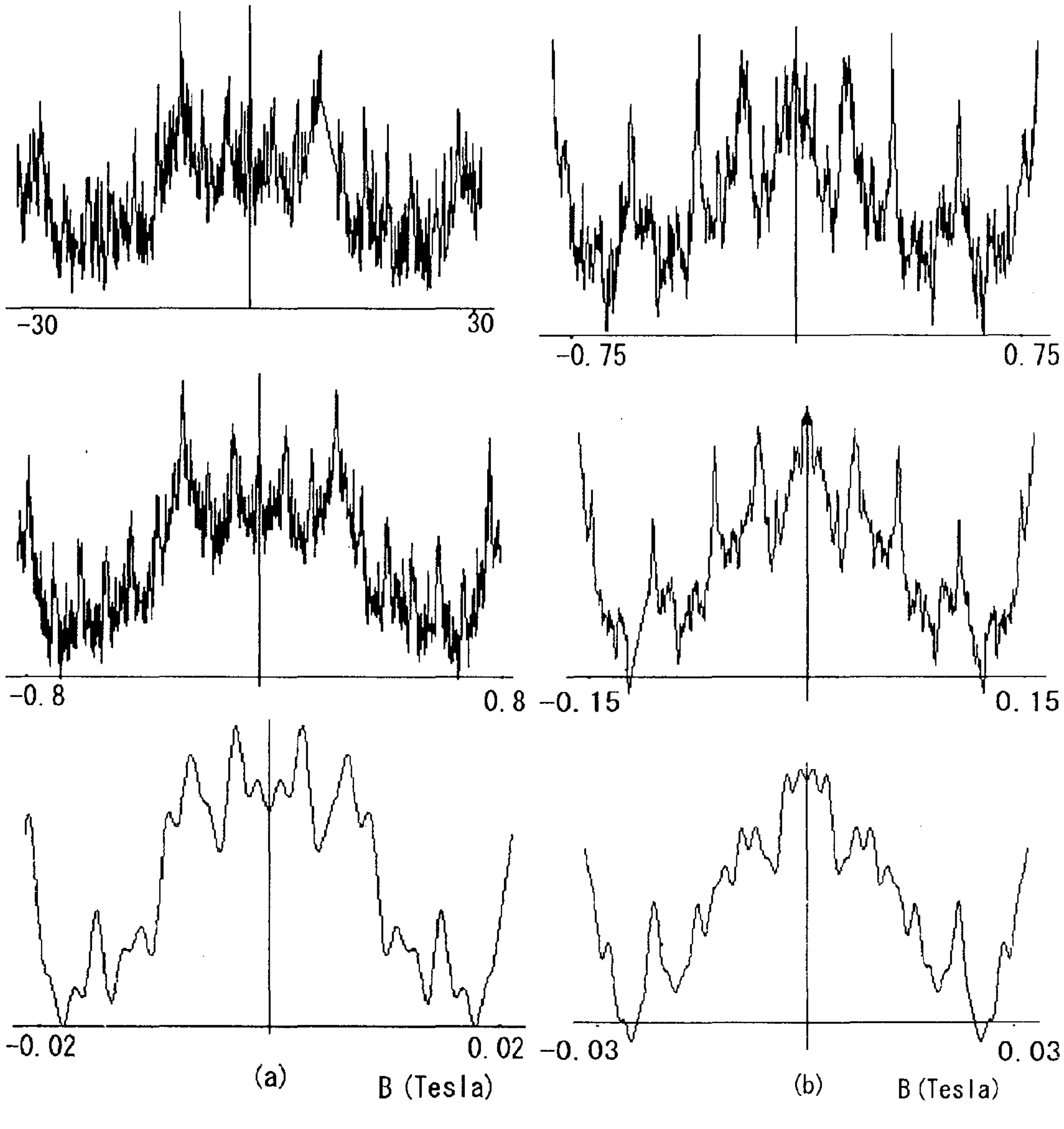

Figure 1: Each row shows succesive magnifications of fractal fluctuations of δG_{xx} around $B = 0$ Tesla for different value of λ ($\lambda = 0.163$ (a), $\lambda = 0.44$ (b)) and for $T = 0.01$ Kelvin. The vertical coordinate is scaled in arbitrary unit. The horizontal scale is in unit Testla. The scale factor of each magnification in the horizontal direction is equal to λ^{-2}.

It is commonly used to generate self-affine curves. In its simplest form, it can be written as follows:

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(b_n \pi x), \tag{5}$$

where $0 < a \equiv a_n/a_{n-1} < 1$, and $b \equiv b_n/b_{n-1}$ is an odd integer satisfies $ab > 1 + \frac{3}{2}\pi$. In less mathematical words, the frequency b_n must form an arithmatic progression, to guarantee the scaling in the $x-$direction of the function. Whereas, the amplitude a_n should decrease quickly enough to make the function converge, and again the arithmatic decreasing progression of a_n guarantees the scaling in the $y-$direction of the function. As it is clear from Eq. (6), $f(x)$ has no smallest scale. But it does have a largest scale, which corresponds to b_0. As we will show below, the function we obtained from equation (1)-(4) is not similar to Eq. (6).

To be able to compare the function we obtained from the Kubo formula with the Weierstrass function, let us ignore the constributions from the non-self-similar periodics orbit as well as from the repetitions of the self-similar periodic orbits to the conductance fluctuations. Then b_n in Eq. (6) corresponds to $\frac{e}{c}\Theta_{po}^{(n)}$. From Eq. (4), b_n in our function certainly forms an arithmatic progression as n is increased. It guarantees the scaling in the $x-$direction which is obvious in Fig. 3. It also shows the presence of the Weierstrass type spectrum, λ^n $(0 < \lambda < 1,\ n = 0, 1, 2, \ldots)$. However, in our case, we have upper bound scale, which corresponds to $\Theta_{po}^{(0)}$ (the largest among the areas enclosed by the self-similar periodic orbits), and, as shown in Fig. 3 causes the self similarity to be truncated at certain smallest scale. This is the first difference between our function and Weierstrass function modelled as in Eq. (6).

a_n in our function can be written as

$$a_n \sim A_{po}^{(n)}(\lambda, T) \cos(\frac{R}{\hbar} - \mu\frac{\pi}{2}), \tag{6}$$

where $A_{po}^{(n)}(\lambda, T)$ collects the constributions from all terms in the amplitude of Eq. (1), and certainly depends on λ and T. For ballistic and phase-coherent systems, the value of τ_β and τ_S is large enough so that the damping factors $R_{po}^{(n)}(\tau_\beta)$ and $F_{po}^{(n)}(\tau_S)$ are close to 1. The increasing progression of $A_{po}^{(n)}$ as n is increased is then guaranteed by the increasing progression of $C_{xx}^{po,n}$. The constant R is the action at $B = 0$, $S_{po}^{(n)}(E, 0)$, which is assumed not to be much varied by n (For $E_F \to +E_S$, all $S_{po}^{(n)}$converges to the same value). The Maslov index, $\mu_{po}^{(n)}$ satisfies

$$\mu_{po}^{(n)} = \mu_{po}^{(n-1)} + 2 \tag{7}$$

with $\mu_{po}^{(0)} = 5$. Using Eq. (8) we can rewrite the cosine term in Eq. (7) as $(-1)^n \sin(R/\hbar)$, so that we have two series of a_n: one is for even value of n which gives series of positive number, and the other is for odd n whose members are negative. This is the second difference between our function and Weierstrass function. Both absolute values of the series make an increasing progression. However, unlike in Weierstrass function, it is not trivial whether they make an arithmatic progression.

The existence of those two series turns out to be the reason why the scaling in $x-$direction in our $\delta G_{xx}(B)$ is λ^{-2} instead of λ^{-1} as in Weierstrass function. This can be seen easily as follows: since each term in Eq. (2) can finally be expressed as $g_n = (-1)^n A_{po}^{(n)}(\lambda, T)\sin(R/\hbar)\cos(\lambda^n \Theta_{po}^{(n)} B)$, under the transformation $B \to \lambda^{-1}B$, the vertical direction must be scaled by some "negative" constant to be able to get $g_{(n-1)}$. The result is that the scaled graph and the unscaled graph are not similar to each other. Therefore the horizontal direction should be scaled as $B \to \lambda^{-2}B$, and by some positive scaling in the vertical direction g_n will becomes g_{n-2}. Under this transformation, both the unscaled and the scaled graphs are similar, as shown in Fig. 1.

Summarizing all the above analysis, for a certain range of weak magnetic field B and for low enough temperature T, $\delta G_{xx}(B)$ satisfies, at least approximately, the self-affine scaling relationship as follows:

$$\delta G_{xx}(\lambda^{-2}B) \sim \lambda^{-2H(\lambda,T)} \delta G_{xx}(B). \tag{8}$$

It means $\delta G_{xx}(B)$ can be rescaled by simultaneusly changing the length scale in the $x-$direction by $\lambda_x = \lambda^{-2}$ and the length scale in the $y-$direction by $\lambda_y = \lambda^{-2H(\lambda,T)}$. $H(\lambda, T) \equiv \frac{\ln \lambda_y}{\ln \lambda_x}$ is called the Hurst exponent ($0 < H \leq 1$). The scaling in the $x-$direction is obviously independent on temperature T (which is assumed to be small, so that the self-affinity is still preserved). The scaling in $y-$direction, however does depend on T. Interestingly, our numerical calculation also shows that the Hurst exponent seems to depend on the value of the weak magnetic field B also.