Directed Polymers in Random Impurities

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Abstract

Directed polymers in random environment can be thought of as a model of statistical mechanics in which paths of stochastic processes interact with a quenched disorder (impurities), depending on both time and space. We review here main results which have been obtained during the last fifteen years. The material covers the diffusive behavior of the polymers in weak disorder phase studied by J. Imbrie, T. Spencer, E. Bolthausen, R. Song and X. Y. Zhou [ImSp88, Bol89, SoZh96], and localization of the paths in strong disordered phase recently obtained by P. Carmona, Y. Hu, and the authors of the present article [CaHu02, CSY03].

不純物を含む溶媒に浸された高分子の形状を記述する、と考えられている確率モデルを紹介したい。S_n を d 次元格子 $\mathbb{Z}^d$ 上の単純ランダムウォークとする。つまり S_n は各時刻 $n = 1, 2, \ldots$ で独立に、$2d$ 個の隣接格子点を等確率に選んで移動するランダムな格子点である。このとき、軌道 $\{(S_n, n)\}_{n \geq 1}$ は時空 $d + 1$ 次元の高分子（或は界面）のモデルとしてしばしば数理物理学に登場する。ここではこの軌道が S_n とは別に用意された確率変数と相互作用する場合に軌道の統計を考察する。これは物理的には不純物を含む媒質内での高分子の形状を観察し、不純物が与える影響を調べることに相当する。数値実験や発見的議論を含めて現在までに知られている、或は予想されている主な事柄は以下の通りである；

(a) $d \geq 3$ で媒質の不純度が弱ければ S_n の揺らぎは純粋媒質の場合と定性的に同じ；即ち通常のランダムウォークと同様の中心極限定理に従う。

(b) 媒質の不純度が強ければ不純物の配置に応じて (S_n, n) の限られた軌道に確率が集中し（局在）、しかも通常のランダムウォークより揺れ幅が大きい。

(c) $d = 1, 2$ では媒質の不純度が弱くても上記 (b) の局在が起こる。

ここでは上記事柄についての物理的背景も交えた概説から始め、数学的に厳密な幾つかの結果を紹介する。

1 Introduction

1.1 Physical background

We start with an informal description of the situation we will discuss in these notes. Imagine a hydrophilic polymer chain wafting in water. Due to the thermal fluctuation, the shape of the polymer should be understood as a random object. We now suppose that the water contains randomly placed hydrophobic molecules as impurities, which repel the hydrophilic monomers which the polymer consists of. The question we address here is:

$$\text{How does the impurities affect the global shape of the polymer chain?} \qquad (1.1)$$

We try to answer this question in a mathematical framework. However, as is everywhere else in mathematical physics, it is very difficult to do so without compromising with a rather simplified picture of the real world. Here, our simplification goes as follows. We first suppress entanglement and U-turns of the polymer; we shall represent the polymer chain as a graph $\{(j, \omega_j)\}_{j=1}^n$ in $\mathbb{N} \times \mathbb{Z}^d$, so that the polymer is supposed to live in $(d + 1)$-dimensional discrete lattice and to stretch in the direction of the first coordinate. Each point $(j, \omega_j) \in \mathbb{N} \times \mathbb{Z}^d$ on the graph stands for the position of j-th monomer in this picture. Secondly, we assume that, the transversal motion $\{\omega_j\}_{j=1}^n$ performs a simple random walk in $\mathbb{Z}^d$, if the impurities are absent. We then define the energy of the path $\{(j, \omega_j)\}_{j=1}^n$ by

$$-\beta \sum_{j=1}^n \eta(j, \omega_j) \,, \qquad (1.2)$$

where $\beta = 1/T > 0$ is the inverse temperature and $\{\eta(n, x) : n \geq 1, \ x \in \mathbb{Z}^d\}$ is a real i.i.d. random variables, with $\eta(n, x)$ describing the presence (or strength) of impurities at site (n, x). The typical shape $\{(j, \omega_j)\}_{j=1}^n$ of the polymer is then given by the one that minimizes the energy (1.2). Let us suppose for example that $\eta(n, x)$ takes two different values $+1$ ("presence of a water molecule at (n, x)") and -1 ("presence of the hydrophobic impurity at (n, x)"). Then, the energy of the polymer is increased by $+\beta$ each time a monomer is in contact with the impurity ($\eta(j, \omega_j) = -1$). Therefore, the typical shape of the polymer for each given configuration of $\{\eta(j, x)\}$ is given by the one which tries to avoid the impurities as much as possible.

The purpose of these notes is to introduce rigorous results which answer (1.1) roughly[1] as follows.

(a) If $d \geq 3$ and β small enough, the impurities does not affect the global shape of the polymer (*the weak disorder phase*).

(b) If either (i): $d \leq 2$ and $\beta \neq 0$ or (ii): $d \geq 3$ and β large enough, then, the impurities changes the global shape of the polymer drastically (*the strong disorder phase*).

1.2 Simple random walk model for directed polymers

We now put the informal description given in section 1.1 into a mathematical framework. As we mentioned before, the framework can be thought of as a model in statistical mechanics.

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[1] To be precise, there are some exceptional choice of the distribution of $\eta(n, x)$ for which the impurities does not affect the global shape of the polymer for $d \geq 3$ even if β is large, cf. Example 2.1.1 below.

However, no prior knowledge of statistical mechanics is needed in this paper. The model we consider here is defined as a random walk in a random environment. We first fix the notations for the random walk and the random environment. Then, we introduce the polymer measure.

- *The random walk:* $(\{\omega_n\}_{n\geq 0}, P)$ is a simple random walk on the d-dimensional integer lattice $\mathbb{Z}^d$. More precisely, we let Ω be the path space $\Omega = \{\omega = (\omega_n)_{n\geq 0}; \omega_n \in \mathbb{Z}^d, n \geq 0\}$, $\mathcal{F}$ be the cylindrical σ-field on Ω, and, for all $n \geq 0$, $\omega_n : \omega \mapsto \omega_n$ be the projection map. We consider the unique probability measure P on $(\Omega, \mathcal{F})$ such that $\omega_1 - \omega_0, \ldots, \omega_n - \omega_{n-1}$ are independent and

$$P\{\omega_0 = 0\} = 1, \quad P\{\omega_n - \omega_{n-1} = \pm\delta_j\} = (2d)^{-1}, \quad j = 1, 2, \ldots, d,$$

where $\delta_j = (\delta_{kj})_{k=1}^d$ is the j-th vector of the canonical basis of $\mathbb{Z}^d$. In the sequel, $P[X]$ denotes the P-expectation of a r.v.(random variable) X on $(\Omega, \mathcal{F}, P)$.

- *The random environment:* $\eta = \{\eta(n, x) : n \in \mathbb{N}, x \in \mathbb{Z}^d\}$ is a sequence of r.v.'s which are real valued, non-constant, and i.i.d.(independent identically distributed) r.v.'s defined on a probability space $(H, \mathcal{G}, Q)$ such that

$$Q[\exp(\beta\eta(n, x))] < \infty \quad \text{for all } \beta \in \mathbb{R}. \tag{1.3}$$

Here, and in the sequel, $Q[Y]$ denotes the Q-expectation of a r.v. Y on $(H, \mathcal{G}, Q)$.

- *The polymer measure:* For any $n > 0$, define the probability measure μ_n on the path space $(\Omega, \mathcal{F})$ by

$$\mu_n(d\omega) = \frac{1}{Z_n} \exp\left(\beta \sum_{1\leq j\leq n} \eta(j, \omega_j)\right) P(d\omega), \tag{1.4}$$

where $\beta > 0$ is a parameter (the inverse temperature) and

$$Z_n = P\left[\exp\left(\beta \sum_{1\leq j\leq n} \eta(j, \omega_j)\right)\right] \tag{1.5}$$

is the normalizing constant (the partition function).

The polymer measure μ_n can be thought of as a Gibbs measure on the path space $(\Omega, \mathcal{F})$ with the Hamiltonian (1.2). We stress that the random environment η is contained in both Z_n and μ_n without being integrated out, so that they are r.v.'s on the probability space $(H, \mathcal{G}, Q)$. The polymer is attracted to sites where the random environment is positive, and repelled by sites where the environment is negative.

Remark 1.2.1 This model was originally introduced in physics literature [HuHe85] to mimic the phase boundary of Ising model subject to random impurities. Later on, the model reached the mathematics community [ImSp88, Bol89], where it was reformulated as above.

Here are two standard choices for the distribution of $\eta(n, x)$.

Example 1.2.1 *Bernoulli environment* ([Bol89, ImSp88, SoZh96]); This is the case with

$$Q\{\eta(n, x) = -1\} = p > 0, \quad Q\{\eta(n, x) = +1\} = 1 - p > 0.$$

In the physical picture described in section 1.1, $\eta(n, x) = -1$ (resp. $\eta(n, x) = +1$) can be interpreted as the presence of a hydrophobic impurity (resp. a water molecule) at site (x, n).

Example 1.2.2 *Gaussian environment* ([CaHu02]); This is the case in which $\eta(n, x)$ is a standard normal random variable;

$$Q\{\eta(n, x) \in dt\} = \frac{1}{\sqrt{2\pi}} \exp(-t^2/2)dt.$$

2 Some typical results for the simple random walk model

In this section, we present some typical results for the simple random walk model. Here, we focus on the conceptual issues on these results and do not go into the proofs.

We now introduce an important quantity for this model, which appears in the assumptions of the results we present. Let $\lambda(\beta)$ be the logarithmic moment generating function of $\eta(n, x)$,

$$\lambda(\beta) = \ln Q[\exp(\beta\eta(n, x))], \quad \beta \in \mathbb{R}. \tag{2.1}$$

The function $\lambda(\beta)$ can be explicitly computed for some typical choice of the distribution of $\eta(n, x)$. For example, $\lambda(\beta) = \ln(pe^{-\beta} + (1-p)e^{\beta})$ for the Bernoulli environment (Example 1.2.2) and $\lambda(\beta) = \frac{1}{2}\beta^2$ for the Gaussian environment (Example 1.2.2).

2.1 The weak disorder phase

The results we present in this subsection show that the impurities do not change the transeversal fluctuation of the polymer if $d \geq 3$ and β is small enough. We first recall the following fact about the return probability π_d for the simple random walk:

$$\pi_d \stackrel{\text{def.}}{=} P\{\omega_n = 0 \text{ for some } n \geq 1\} \begin{cases} = 1 & \text{if } d \leq 2, \\ < 1 & \text{if } d \geq 3. \end{cases} \tag{2.2}$$

More precisely, it is known that $\pi_{d+1} < \pi_d$ for all $d \geq 3$ [OlSo96, Lemma1] and that $\pi_3 = 0.3405...$ [Sp76, page 103]. In particular, $\pi_d \leq 0.3405...$ for all $d \geq 3$.

Theorem 2.1.1 *(The diffusive behavior; [ImSp88, Bol89, SoZh96]) Suppose that $d \geq 3$ (hence $\pi_d < 1$) and that*

$$\gamma_1(\beta) \stackrel{\text{def.}}{=} \lambda(2\beta) - 2\lambda(\beta) < \ln(1/\pi_d). \tag{2.3}$$

Then,

$$\lim_{n \nearrow \infty} \mu_n[|\omega_n|^2]/n = 1 \quad Q\text{-a.s.} \tag{2.4}$$

Note that $\gamma_1(\beta)$ is increasing on $[0, \infty)$ and $\gamma_1(0) = 0$ so that the condition in (2.3) does hold if β is small.

Example 2.1.1 Consider the Bernoulli environment (Example 1.2.1). In this case, it is not difficult to see from direct computations that $\lim_{\beta \nearrow \infty} \gamma_1(\beta) = -\ln(1-p)$. This shows that (2.3) holds for all $\beta \geq 0$ if $p < 1 - \pi_d$.

Example 2.1.2 Consider the Gaussian environment (Example 1.2.2). Then, $\gamma_1(\beta) = \beta^2$ and hence (2.3) holds if $\beta < \sqrt{\ln(1/\pi_d)}$.

Remark 2.1.1 The first rigorous proof of Theorem 2.1.1 was obtained by J. Z. Imbrie and T. Spencer [ImSp88] in the case of Bernoulli environment. Soon afterwards, a more transparent proof based on the martingale analysis was given by E. Bolthausen [Bol89]. The martingale proof was then extended to general environment under condition (2.3) by R. Song and X. Y. Zhou [SoZh96]. By the argument in [Bol89, SoZh96], it is possible to get a much more precise

statement than (2.4). In fact, under the same assumption in Theorem 2.1.1, the following central limit theorem holds; for all $f \in C(\mathbb{R}^d)$ with at most polynomial growth at infinity,

$$\lim_{n \nearrow \infty} \mu_n \left[f\left(\omega_n/\sqrt{n}\right) \right] = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f\left(x/\sqrt{d}\right) \exp(-|x|^2/2) dx, \quad Q\text{-a.s.} \qquad (2.5)$$

The diffusive behavior (2.4) follows from (2.5) by choosing $f(x) = |x|^2$. In [Bol89], (2.5) is obtained for the Bernoulli environment only. However, with the help of the observation made in [SoZh96], it is not difficult to extend the central limit theorem to general environment under the assumption in Theorem 2.1.1.

We now recall the following well known fact for the simple random walk, i.e., the case of $\beta = 0$:

$$\max_{x \in \mathbb{Z}^d} P\{\omega_n = x\} = \mathcal{O}(n^{-d/2}), \quad \text{as } n \nearrow \infty. \qquad (2.6)$$

The decay rate $n^{-d/2}$ in (2.6) can be understood as a manifestation of the fact that the possible position of ω_n is spread over a ball in $\mathbb{Z}^d$ with radius const. $\times \sqrt{n}$.

For $\beta \neq 0$, we can still prove (2.6) in a weaker form as follows.

Theorem 2.1.2 *(Delocalization;[CaHu02, CSY03]) Suppose that $d \geq 3$ and that β is small enough so that (2.3) holds. Then,*

$$\sum_{n \geq 1} \max_{x \in \mathbb{Z}^d} \mu_{n-1}\{\omega_n = x\}^2 < \infty, \quad Q\text{-a.s.} \qquad (2.7)$$

and thus,

$$\lim_{n \nearrow \infty} \max_{x \in \mathbb{Z}^d} \mu_{n-1}\{\omega_n = x\} = 0, \quad Q\text{-a.s.} \qquad (2.8)$$

Remark 2.1.2 Theorem 2.1.2 was obtained for Gaussian environment by P. Carmona and Y. Hu [CaHu02] and then for general environment by F. Comets, T. Shiga and N. Yoshida [CSY03].

2.2 The strong disorder phase

The result we present in this subsection shows that the behavior of the polymer is quite different from the usual random walk if either (i) $d = 1, 2$ and $\beta \neq 0$ or (ii) $d \geq 3$ and β is large[2]. For this model, it is rather recent that the phenomena of this kind began to be studied rigorously.

We now present a result which is in sharp contrast with (2.6) and (2.8).

Theorem 2.2.1 *(Localization to the favorite sites [CaHu02, CSY03]) Suppose either that*

(i) $d = 1, 2$ *and* $\beta \neq 0$ *or*

(ii) $d \geq 1$ *and*

$$\gamma_2(\beta) \overset{\text{def.}}{=} \beta\lambda'(\beta) - \lambda(\beta) > \ln(2d). \qquad (2.9)$$

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[2]This is again, up to some particular environmental distributions, like the one discussed in Example 2.1.1; to be on the safe side for this statement, one can consider unbounded environments, or bounded ones taking positive values with positive probability and without mass on the top point of its support

Then, there exsists a constant $c = c(d, \beta) > 0$ such that

$$\varlimsup_{n \nearrow \infty} \max_{x \in \mathbb{Z}^d} \mu_{n-1}\{\omega_n = x\} \geq c, \quad Q\text{-}a.s. \tag{2.10}$$

The bound (2.10) suggests that the position of ω_n, viewed under the polymer measure μ_{n-1}, is concentrated at a small region (the "favorite sites") with the size $\mathcal{O}(1)$ as $n \nearrow \infty$.

Note that γ_2 is increasing on $[0, \infty)$ and therefore that (2.9) holds for large enough β if

$$\lim_{\beta \nearrow \infty} \gamma_2(\beta) > \ln(2d). \tag{2.11}$$

We see from Theorem 2.1.2 and Theorem 2.2.1 that, if $d \geq 3$ and (2.11), then a phase transition occurs as β increases from the weak disorder phase to the strong disorder phase.

Example 2.2.1 Consider the Bernoulli environment (Example 1.2.1). Then, it is not difficult to see from direct computations that $\lim_{\beta \nearrow \infty} \gamma_2(\beta) = \ln(1/(1-p))$. This shows that (2.9) holds for large enough β if $p > 1 - \frac{1}{2d}$.

Example 2.2.2 Consider the Gaussian environment (Example 1.2.1). Then, $\gamma_2(\beta) = \beta^2/2$ and hence (2.9) holds if $\beta > \sqrt{2\ln(2d)}$.

Remark 2.2.1 Theorem 2.2.1 was obtained for Gaussian environment by P. Carmona and Y. Hu [CaHu02] and then for general environment by F. Comets, T. Shiga and N. Yoshida [CSY03].

2.3 The normalized partition function and its positivity in the limit

We now introduce an important process on $(H, \mathcal{G}, Q)$ ((2.12) below), which is a martingale in fact. The large time behavior of this process characterizes the phase diagram of this model and for this reason, many of results on the model can be best understood from the viewpoint of this process.

Define the *normalized partition function* by

$$W_n = \exp(-\lambda(\beta)n)Z_n, \quad n \geq 1. \tag{2.12}$$

We then have

Lemma 2.3.1 *The limit*

$$W_\infty = \lim_{n \nearrow \infty} W_n \tag{2.13}$$

exists Q-a.s. Moreover, there are only two possibilities for the positivity of the limit;

$$Q\{W_\infty > 0\} = 1 \,, \tag{2.14}$$

or

$$Q\{W_\infty = 0\} = 1 \,. \tag{2.15}$$

The above contrasting situations (2.14) and (2.15) can be considered as the characterization of the weak disorder phase and the strong disorder phase, respectively. In fact, as are shown in [CaHu02, CSY03], (2.14) implies (2.7), while (2.15) implies a weaker form of (2.10) that $\sum_{n \geq 1} \max_{x \in \mathbb{Z}^d} \mu_{n-1}\{\omega_n = x\} = \infty$, Q-a.s. It is even expected that (2.14) implies (2.5) and that (2.15) implies (2.10).

We close this subsection with the following result, which, in consistency with what we discuss above, describes the basic phase diagram of the model.

Theorem 2.3.2 (a) *For $d \geq 3$, (2.3) implies (2.14).*

(b) *Either (i) or (ii) in Theorem 2.2.1 implies (2.15).*

Remark 2.3.1 For $d \geq 3$, it is an interesting question to find a characterization of (2.14) (or (2.15)) in terms of the distribution of $\eta(n, x)$. This question has somewhat similar flavor to some other topics in the probability theory such as Kakutani's dichotomy for infinite product measure (e.g.,[Dur95, page 244]), nontriviality of the limit of the normalized Galton-Watson process [AtNe72] and of multiplicative chaos [KaPe76].

3 Some related models

The simple random walk model which we have discussed so far has a number of close relatives in the literature. We now mention some of them.

3.1 Gaussian random walk model

This model considered in M. Petermann [Pet00] and by O. Mejane [Mej02]. The polymer measure for this model is defined by the same expression (1.4). Here, however, the random walk $(\omega_n)_{n \geq 1}$ is the summation of independent Gaussian random variables in $\mathbb{R}^d$, i.e., Ω is replaced by $\Omega = \{\omega = (\omega_n)_{n \geq 0}; \omega_n \in \mathbb{R}^d, n \geq 0\}$ and P is the unique measure on $(\Omega, \mathcal{F})$ such that $\omega_1 - \omega_0, \ldots, \omega_n - \omega_{n-1}$ are independent and

$$P\{\omega_0 = 0\} = 1, \quad P\{\omega_n - \omega_{n-1} \in dx\} = (2\pi)^{-d/2} \exp(-|x|^2/2)dx.$$

Moreover, as the random environment, one takes a random field $\{\eta(n, x) \; ; \; (n, x) \in \mathbb{N} \times \mathbb{R}^d\}$ with a certain mild correlation in x variables. A major technical advantage in working with the Gaussian random walk rather than the simple random walk is the applicability of a Girsanov-type path transformation, which plays a key role in analyzing this model.

3.2 Brownian directed polymer

This model is introduced in [CY03] as a continuous model of directed polymers in random environment, defined in terms of Brownian motion and of a Poisson random measure. We first fix notation we use for the Brownian motion and Poisson random measure. Then, we introduce the polymer measure. We write $\mathbb{R}_+ = [0, \infty)$.

- *The Brownian motion:* Let $(\{\omega_t\}_{t \geq 0}, P)$ denote a d-dimensional standard Brownian motion. To be more specific, we let the measurable space $(\Omega, \mathcal{F})$ be $C(\mathbb{R}_+ \to \mathbb{R}^d)$ with the cylindrical σ-field, and P be the Wiener measure on $(\Omega, \mathcal{F})$ such that $P\{\omega_0 = 0\} = 1$.

- *The space-time Poisson random measure:* We let η denote the Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ with the unit intensity, defined on a probability space $(\mathcal{M}, \mathcal{G}, Q)$. To make the

definitions more precisely, we let $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ denote the class of Borel sets in $\mathbb{R}_+ \times \mathbb{R}^d$. Then, η is an integer valued random measure characterized by the following property: If $A_1, ..., A_n \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ are disjoint and bounded, then

$$Q\left(\bigcap_{j=1}^{n} \{\eta(A_j) = k_j\}\right) = \prod_{j=1}^{n} \exp(-|A_j|)\frac{|A_j|^{k_j}}{k_j!} \quad \text{for } k_1, ..., k_n \in \mathbb{N}. \tag{3.1}$$

Here, $|A|$ denotes the Lebesgue measure of $\mathbb{R}^{d+1}$.

• *The polymer measure:* We let V_t denote a "tube" around the graph $\{(s, \omega_s)\}_{0 < s \leq t}$ of the Brownian path,

$$V_t = V_t(\omega) = \{(s, x) \ ; \ s \in (0, t], \ x \in U(\omega_s)\}, \tag{3.2}$$

where $U(x) \subset \mathbb{R}^d$ is the closed ball with the unit volume, centered at $x \in \mathbb{R}^d$. For any $t > 0$, define a probability measure μ_t on the path space $(\Omega, \mathcal{F})$ by

$$\mu_t(d\omega) = \frac{\exp\left(\beta\eta(V_t)\right)}{Z_t} P(d\omega), \tag{3.3}$$

where $\beta \in \mathbb{R}$ is a parameter and

$$Z_t = P[\exp\left(\beta\eta(V_t)\right)]. \tag{3.4}$$

The Brownian motion model defined above can be thought of as a natural transposition of the simple random walk model into continuum setting.

Analogous results of Theorem 2.1.2, Theorem 2.2.1, Theorem 2.3.2, as well as an almost sure large deviation principle for the polymer measure are obtained for this model in [CY03]. The model allows application of stochastic calculus, with respect to both Brownian motion and Poisson process, leading to qualitative properties of the quenched Lyapunov exponent and handy formulas for the fluctuation of the free energy.

Another strong motivation for the present model is its relation to some *stochastic partial differential equations*. To describe the connection, it is necessary to relativize the partition function, by specifying the ending point of the Brownian motion at time t. Let $P[\cdot|\omega_t = y]$ be the distribution of the Brownian bridge starting at the origin at time 0 and ending at y at time t. Define

$$Z_t(y) = P[\exp\left(\beta\eta(V_t)\right)|\omega_t = y](2\pi t)^{-d/2}\exp\{-|y|^2/2t\}. \tag{3.5}$$

Then, by definition of the Brownian bridge,

$$Z_t = \int_{\mathbb{R}^d} Z_t(y)dy.$$

Similar to the Feynman-Kac formula, one obtains [CY03] the following stochastic heat equation (SHE) with multiplicative noise in a certain weak sense,

$$dZ_t(y) = \tfrac{1}{2}\Delta Z_t(y)dt + (e^\beta - 1)Z_{t^-}(y)\eta(dt \times U(y)), \quad t \geq 0, y \in \mathbb{R}^d, \tag{3.6}$$

where $dZ_t(y)$ denotes the time differential and $\Delta = (\frac{\partial}{\partial y^1})^2 + ... + (\frac{\partial}{\partial y^d})^2$.

In the literature, this equation has been extensively considered in the case of a Gaussian driving noise, instead of the Poisson process η here. Although we are able to prove (3.6) only in the weak sense, let us now pretend that (3.6) is true for all $y \in \mathbb{R}^d$. We would then see

from Itô's formula that the function $h_t(y) = \ln Z_t(y)$ solves the Kardar-Parisi-Zhang equation (KPZ):

$$dh_t(y) = \tfrac{1}{2}\left(\Delta h_t(y) + |\nabla h_t(y)|^2\right) dt + \beta\eta(dt \times U(y)) \, .$$

We observe that, since h has jumps in the space variable y, the non-linearity makes the precise meaning of this equation somewhat knotty. This equation was introduced in [KaPaZh86] to describe the long scale behavior of growing interfaces. More precisely, the fluctuations in the KPZ equation –driven by a δ-correlated, gaussian noise–, are believed to be non standard, and universal, i.e., the same as in a large class of microscopic models. See [KrSp91] for a detailed review of kinetic roughening of growth models within the physics literature, in particular to Section 5 for the status of this equation.

3.3 Crossing Brownian motion in a soft Poissonian potential

This model is studied by M. Wüthrich [Wut98a, Wut98b, Wut98c], see also [Szn98]. The model investigated there is described in terms of Brownian motion and of Poisson points. However, the Brownian motion there is "undirected", in other words, the d-dimensional Brownian motion travels through the Poisson points distributed in space $\mathbb{R}^d$, not in space-time as in the Brownian directed polymer.

3.4 First and last passage percolation

The first (resp. last) passage percolation can be thought of as an analogue of directed polymers at $\beta = -\infty$ (resp. $\beta = +\infty$). In fact, it is expected and even partly vindicated that the properties of the path with minimal/maximal passage time has similar feature to the typical paths under the polymer measure [Kes86, NePi95, LiNePi96].

3.5 Other models

Directed polymers in random environment, at positive or zero temperature, relate – even better, can sometimes be exactly mapped – to a number of interesting models of growing random surfaces (directed invasion percolation, ballistic deposition, polynuclear growth, low temperature Ising models), and non equilibrium dynamics (totally asymmetric simple exclusion, population dynamics in random environment); We refer to the survey paper [KrSp91] by Krug and Spohn for detailed account on these models and their relations.

4 Critical exponents

We write $\xi(d)$ for the "wandering exponent",i.e., the critical exponent for the transversal fluctuation of the path, and $\chi(d)$ for the the critical exponent for the longitudinal fluctuation of the free energy. Their definitions are roughly

$$\sup_{0 \le j \le n} |\omega_j| \approx n^{\xi(d)} \quad \text{and} \quad \ln Z_n - Q[\ln Z_n] \approx n^{\chi(d)} \quad \text{as } n \nearrow \infty. \tag{4.1}$$

There are various ways to define rigorously these exponents, e.g. (0.6) and (0.10-11) in [Wut98a], (2.4) and (2.6-7-8) in [Piz97], and the equivalence between these specific definitions are often non trivial. Here, we do not go into such subtlity and take (4.1) as "definitions". The polymer is said to be *diffusive* if $\xi(d) = 1/2$ and *super-diffusive* if $\xi(d) > 1/2$.

These exponents are investigated in the context of various other models and in a large number of papers. In particular, it is conjectured in physics literature that the *scaling identity* holds in any dimension,

$$\chi(d) = 2\xi(d) - 1, \quad d \geq 1, \tag{4.2}$$

and that the polymer is super diffusive in dimension one;

$$\chi(1) = 1/3, \quad \xi(1) = 2/3. \tag{4.3}$$

See, e.g., [HuHe85],[FiHu91, (3.4),(5.11),(5.12)], [KrSp91, (5.19),(5.28)].

On the other hand, rigorous results prove (or suggest) for example that

$$\chi(d) \geq 2\xi(d) - 1 \text{ for all } d \geq 1, \tag{4.4}$$

$$\xi(d) \leq 3/4 \text{ for all } d \geq 1, \tag{4.5}$$

$$\xi(1) > 1/2 \tag{4.6}$$

M. Piza [Piz97] discusses (4.4) –(4.6) for the simple random walk model. For the Gaussian random walk model, M. Petermann [Pet00] proves (4.6), while O. Mejane [Mej02] shows (4.5). F. Comets and N. Yoshida [CY03] discuss (4.4)–(4.6) in the framework of Brownian directed polymer. Critical exponents similar to the above are also discussed for the crossing Brownian motion in a soft Poissonian potential by M. Wüthrich [Wut98a, Wut98b, Wut98c] and for the first passage percolation by C. Licea, M. Piza and C. Newman [NePi95, LiNePi96].

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