Link between Financial and Nonlinear Systems

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Abstract

株価や為替の収益率は正規分布よりも裾野が厚いことが知られている。この現象を説明するモデルとして Ehrenfest モデルを一般化したモデル (ELJ) [2] を紹介した。また、市場でディーラーが多数派を志向して行動することとディーラーの行動が価格から影響を受けていることに着目したモデル [3] を紹介した。さらに、そのような系が非線形方程式で知られる van der Pol 方程式に帰着することを示した。

1 Introduction

Changes (returns) in stock index prices and exchange rates for currencies are argued, based on empirical data, to obey a stable distribution with characteristic exponent $\alpha < 2$ for short sampling intervals and a Gaussian distribution for long sampling intervals. In order to explain this phenomenon, an Ehrenfest model with large jumps (ELJ) is introduced to explain the empirical density function of price changes for both short and long sampling intervals.

We also introduced another urn model that is a majority orienting model with a feedback from price. It explains the oscillation of the price in a stock market. Using a deterministic approximation of the proposed model, we showed that the model is connected with the van der Pol equation which is known as a nonlinear equation.

2 Ehrenfest model with large jumps

The Ehrenfest model [1] envisages two boxes + and $-$, with $2R$ particles distributed in these boxes. A particle is chosen at random and moved from one box to the other and the same procedure is repeated.

We consider a generalized Ehrenfest model in which $a\,b^j$ steps of the Ehrenfest model take place with probability C/Q^j at each step, where $a,\ b > 0,\ Q > 1,\ j = 0,\ 1,\ 2, \ldots$ and C is a normalization constant defined by $C = 1 - 1/Q$. For simplicity, a is fixed to be 1 in the following discussion. This generalized model becomes identical to the original Ehrenfest model in the limit $Q \to \infty$. The time series of the ELJ is interpreted as a time series chosen randomly from that of the original Ehrenfest model.

The proposed model well describes the changes in the stock indices such as the S&P500, TOPIX and currency exchange rates. Let each particle in the box + represent a buy stance dealer and each particle in the box $-$ represent a sell stance dealer. These dealers will get a lot of information and their view will be changed from buy to sell or from sell to buy on the basis of such information. These changes will generate a change in the stock price or the exchange rate. We assume that a change in the dealer's mind directly causes a change in the price so that the particle number in the box + is interpreted as the price.

As the information is spatially and temporally distributed, and any piece of information may have a different impact on the dealers, the price will sometimes change drastically within minutes, or sometimes only slightly over hours. The difference between the progress of time in the ELJ and in the original Ehrenfest model describes this phenomenon. The parameters b and Q might thus represent the dealers' (price change) sensitivity and frequency of information.

We denote the particle number in the box + after s steps as $M(s)$, and define the changes as $Z_{\Delta s}(s) = M(s) - M(s - \Delta s)$, where Δs denotes the step interval. Using this $Z_{\Delta s}(s)$, we define

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the empirical density $P_{\Delta s}(z)$ as:

$$\sum_{z_1<z<z_2} P_{\Delta s}(z) = \sum_{z_1<z<z_2} \sum_{s} \delta_{z, Z_{\Delta s}(s)}/s_{\text{last}}, \tag{1}$$

where $s = 1, 2, 3, ..., s_{\text{last}}$ and $\delta_{l,k}$ denotes the Kronecker delta.

Using the diffusion approximation, the probability density function of the ELJ is approximated as:

$$\begin{aligned} P^{\text{ELJ}}_{\tau\Delta s}(z) \;=\; & \sum_{i_1,\ldots,i_{\Delta s}} \frac{C^{\Delta s}}{Q^{i_1+\cdots+i_{\Delta s}}} \sqrt{\frac{\gamma}{4\pi D(1-\exp\{-\gamma\tau(b^{i_1}+\cdots+b^{i_{\Delta s}})\})}} \\ & \exp\left(-\frac{\gamma}{4D}\frac{z^2}{1-\exp\{-\gamma\tau(b^{i_1}+\cdots+b^{i_{\Delta s}})\}}\right). \end{aligned} \tag{2}$$

Consider a density function given by a combination of an infinite number of Gaussian density functions:

$$P^{\text{G}}(x) = \frac{Q-1}{Q} \sum_j \frac{1}{Q^j} \sqrt{\frac{1}{2\,\pi\,\Delta^2\,b^j}} \exp\left(-\frac{x^2}{2\,\Delta^2\,b^j}\right). \tag{3}$$

It is easy to confirm from the knowledge of domains of attraction that the density function of the above equation belongs to the domain of attraction of the symmetrical Lévy stable density with characteristic exponent given by:

$$\alpha = 2\frac{\log Q}{\log b}. \tag{4}$$

The density functions given by Eqs. (2) and (3) are identical under a first order approximation in the limit $\gamma \to 0$ and $\Delta s = 1$. Therefore, the symmetrical Lévy stable density with characteristic exponent given by Eq. (4) fits the density function (2) well as far as the central part of the density function of the ELJ is concerned.

The results of a numerical simulation of the trajectory of $M(s)$ is shown in Fig. 1 (a), where $N = 10000$, $b = 3$, and $Q = 2$. Big jumps that do not exist in the original Ehrenfest model are observed in the ELJ. Because of these big jumps, the empirical density $P_{\Delta s}(z)$ of the new model is a better fit to a stable distribution ($\alpha < 2$), whereas the empirical density $P_{\Delta s}(z)$ is given by a Gaussian distribution in the original Ehrenfest model.

In Fig. 1 (b), the empirical density $P_{\Delta s}(z)$ is shown for different time intervals $\Delta s = 16, 64, 512$. These results were obtained for a sequence of about one million steps of the ELJ. The density obtained by simulation using the proposed model is well described by a symmetrical Lévy stable density function. In the model, α increase to 2.0 with increasing Δs.

The empirical density of the exchange rate between the US dollar and the Japanese yen over one year (Jan. '94-Dec. '94) [4] is shown in Fig. 2 (a), along with the estimated stable density functions. This figure shows that the empirical density is close to the stable distribution with a characteristic exponent $\alpha < 2$ for short sampling intervals and a Gaussian distribution for long sampling intervals. When the sampling interval is small, the return is distributed according to a stable law with characteristic exponent $\alpha < 2$. In addition, α becomes larger with expansion of the sampling interval and approaches 2.

3 Majority orienting model with feedback process

Consider an urn model. Assume that there are two boxes, the box plus (+) and the box minus (−) and the number of particles contained in each box is N_+ and N_- respectively. The total

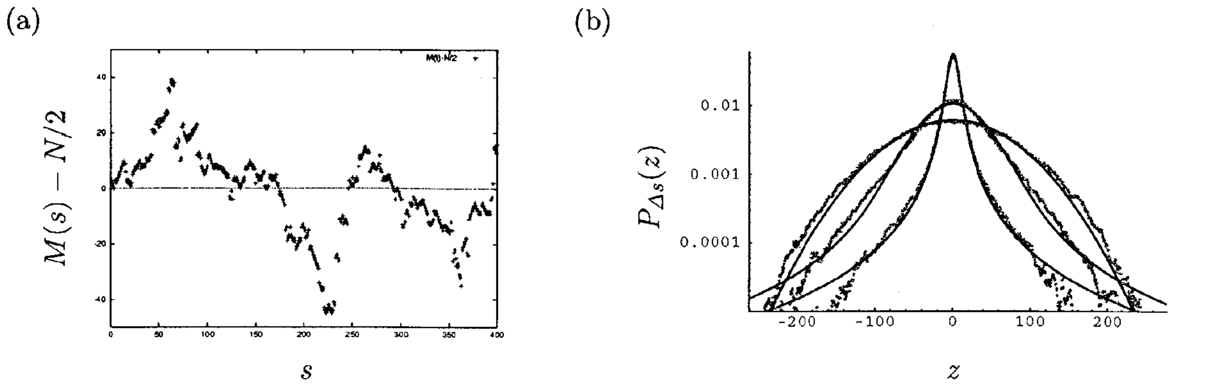

Figure 1: (a) Trajectory of the number of particles in box + ($M(s)$) for $N = 10000$, $b = 3$, $Q = 2$. (b) Empirical densities $P_{\Delta s}(z)$ for $\Delta s = 8$, 64, 512 (dotted line). $N = 10000$, and $b = 3$, $Q = 2$. The Lv́y stable density estimated using the maximum log likelihood method is also shown (solid line). $\alpha = 1.310$, 1.763, 2.0 and $\gamma \Delta s = 9.223$, 335.0, 2418.

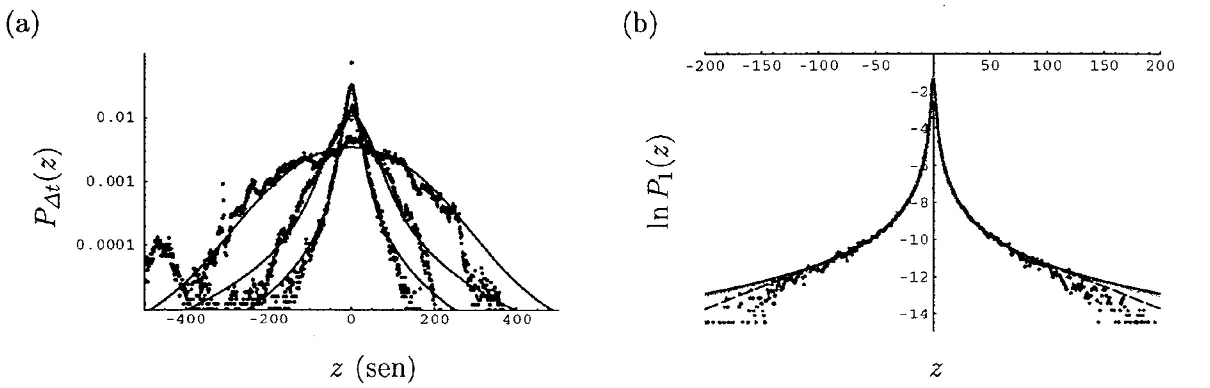

Figure 2: (a) Empirical density functions $P_{\Delta t}(z)$ for sampling intervals of $\Delta t = 2^{15}$, 2^{18}, 2^{21} seconds (dots) and the corresponding stable density functions (lines, $\alpha = 1.451$, 1.588, 1.900 and $\gamma\Delta t = 23.86$, 184.5, 4276) estimated by the maximum log likelihood. (b) Comparison the shapes of $P_1^{ELJ}(z)$, $P_1^{G}(z)$, the stable density and the simulation's empirical density of ELJ when $\Delta s = 1$. $b = 3$, $Q = 2$, $N = 10000$.

number of the particles is $N = N_+ + N_-$, and every particle is numbered from 1 to N. The following step of three substeps i), ii) and iii) are successively applied to particles in the two boxes in a step.

i). One particle is chosen at random and it moves from its box into the other box with probability r and does not move with probability $1 - r$, $(0 \le r \le 1)$.

ii). *Majority rule*: We take three particles at random at each step. If two of the particles taken are in the box plus and one is in the box minus, the one in the box minus is moved to the box plus and the price S is increased by 1, while, if two of the particles are in the box minus and one is in the box plus, the one in the plus is moved to the minus and the price S is decreased by 1. If the three particles taken are in the box plus, no change occurs for the particles and the price S is increased by 3, while, if the three particles taken are in the box minus, no change occurs for the particles and the price S is decreased by 3.

iii). *Feedback*: We change the number of the particles in the box plus N_+ (so in the box minus N_-) with the probability which is proportional to the price S. That is, if S is a positive number, N_+ is decreased by 1 with probability S/N, while, if S is negative, N_+ is increased by 1 with probability $-S/N$.

Let each particle in the box + represent a buy position dealer, and each particle in the box − represent a sell position dealer. The i) models that the position of a dealer changes randomly and it correspond to mutation in population genetics. The ii) models the change of a dealers' position from sell to buy or buy to sell considering the behaviour of the other dealers. The iii) models that the change of dealers' position under the influence of the price in the market. If the price goes to high, the buy-position dealers usually change their position to sell, whereas if the price goes to low, the sell-position dealers maybe change their position to buy.

Let us represent N_+ and S at s step as $N_+(s)$ and $S(s)$ respectively. We assume the step interval as τ. Then, on condition that the values of $N_+(s)$, $N_-(s)$ and $S(s)$ are given, our model is described as:

$$
\begin{aligned}
\mathrm{E}\left[\frac{N_+(s+1)-N_+(s)}{\tau N}\right] &= r\left\{-\frac{N_+(s)}{N}+\frac{N_-(s)}{N}\right\}-\lambda\frac{S(s)}{N} \\
&\quad +3\frac{N_+(s)(N_+(s)-1)N_-(s)}{N(N-1)(N-2)}-3\frac{N_+(s)N_-(s)(N_-(s)-1)}{N(N-1)(N-2)}, \\
\mathrm{E}\left[\frac{S(s+1)-S(s)}{\tau N}\right] &= 3\frac{N_+(s)(N_+(s)-1)(N_+(s)-2)}{N(N-1)(N-2)}+3\frac{N_+(s)(N_+(s)-1)N_-(s)}{N(N-1)(N-2)} \\
&\quad -3\frac{N_-(s)(N_-(s)-1)N_+(s)}{N(N-1)(N-2)}-3\frac{N_-(s)(N_-(s)-1)(N_-(s)-2)}{N(N-1)(N-2)},
\end{aligned}
$$

where E represent the expectation. λ is a parameter which we introduce for convenience of analysis of our model and $\lambda = 1$ in our case.

For sufficiently small τ and large N, approximating $\frac{N_+(s)-N/2}{N}$ by $x(t)$ and $\frac{S(s)}{N}$ by $y(t)$, we have the following system of ordinary differential equation as a deterministic approximation:

$$
\begin{aligned}
\frac{d}{dt}x &= -2\,r\,x+6\,x\left(\frac{1}{2}+x\right)\left(\frac{1}{2}-x\right)-\lambda\,y \qquad (5)\\
\frac{d}{dt}y &= 6\,x,
\end{aligned}
$$

where $-1/2 \le x \le 1/2$.

The following van der Pol equation is obtained from Eq. (5)

$$
\frac{d^2}{dt^2}x-6\left(\frac{3-4\,r}{12}-3\,x^2\right)\frac{d}{dt}x+6\,\lambda\,x=0, \qquad (6)
$$

when $r < 3/4$ ($\simeq 0.75$). The van der Pol equation has the limit cycle, therefore the price oscillates.

References

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