Numerical Implementation of Coulomb-Friction in Mechanical Systems

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1 Coulomb Friction

Coulomb friction is generally represented via the equation of dynamic Coulomb friction

$$F_{\text{fric}} = -\text{sign}(v)\mu F_{\text{normal}}, \tag{1}$$

where the friction force depends on the normal force, the velocity-independent coefficient of Coulomb friction μ and the sign of the velocity. For the static Coulomb friction, eqn. (1) holds only if the external force is equal to the Coulomb friction, e.g. for a block on an inclined plane at the critical angle. In all other cases, the static Coulomb friction is better described by the inequality

$$-\mu F_{\text{normal}} \leq F_{\text{fric}} \leq \mu F_{\text{normal}}. \tag{2}$$

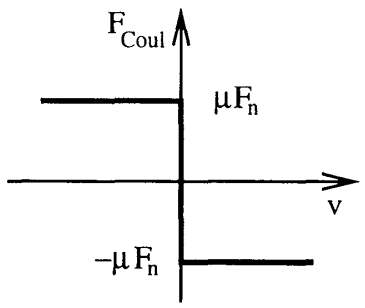

Figure 1: Graph for Coulomb-friction. At the singularity at $v = 0$, the value for the static Coulomb-friction is not unique.

Whereas the dynamic friction in Eqn. (1) describes and energy loss at the frictional contact, the static friction in Eqn. (2) describes a constraint of motion, where no energy loss takes place because the contacting bodies are at rest with respect to each other.
Generally, the importance of simulating Coulomb friction for macroscopic bodies lies in the fact that if results from friction experiments, no matter on which small time-scale, can be simulated via the macroscopic eqns. (1,2) one can be sure that the results still belong to the macroscopic world and offer no deeper understanding of the microscopic mechanisms of friction.

1.1 Choice of the friction Coefficient

Earlier tables of friction coefficients usually contained two coefficients of friction, one for the static, one for the dynamic friction, usually compiled from different sources, where the static coefficient of friction was usually larger than the dynamic coefficient of friction. In the newer literature, usually only a single coefficient for a pair of materials and a given boundary condition (lubrication, atmosphere) are predominant, except in the case of polymers, where the dynamic coefficient of friction often exceeds the static coefficient of friction, in contrast to the classical believe [2,3].
In the following we set the static coefficient of friction equal to the dynamic coefficient of friction due to the above experimental situation, the formalism can nevertheless be applied also in case of different coefficients of friction for the static and dynamic case, or also for time-dependent static coefficients of friction.

1.2 General Problem in numerical simulation of Coulomb friction

To compute particle trajectories from Newton's Equation of motion, we have to be able to compute unique values for the friction even in the case of static friction Eqn. (2). There are two fundamental problems:

1. When do we have to "switch" from "dynamic" to "static" friction ? Of course, one can always define an ad-hoc limit for which one forces the transition from the static to the dynamic case, but this will give spurious answers in the case of a velocity reversal where the velocity changes from positive to negative without an actual static situation.

2. The other problem is to decide the value for the friction, i.e. how to "regularize" Eqn. (2). The problem is simple for a single block on an inclined plane, but for a non-holonomic many-body problem with competition between sliding and rolling, like in the case of a sand-heap, a unique choice for the friction on grounds of the contacts alone is not possible.

One can, of course, implement Eqn. (1) instead of Eqn. (2) in a simulation and "hope" that everything will go well, except is doesn't, as the following section will show.

2 Why not use a straightforward implementation

For a block on an inclined plane like in Fig. 2, at the critical angle, the critical friction coefficient is given as

$$\mu = \tan(\vartheta).$$

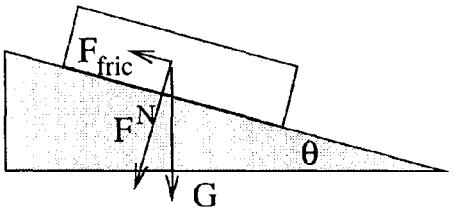

Figure 2: Block on an inclined plane

In Fig. 3, the simulation of a block on an inclined plate with with a 3rd-order Runge-Kutta method (Heun's method) is shown. For the friction coefficient below the critical $\tan(\vartheta)$, one obtains the expected accelerated motion, indicated by the parabolic graph. At the critical angle with friction coefficient $\mu = \tan(\vartheta)$, the force equilibrium leads to a uniform motion, also in agreement with the theoretical expectation, and the friction condition is still in the dynamic regime. For the friction coefficient $\mu = 3\tan(\vartheta)$, the static friction is clearly violated: The dynamic friction law in eq. (1) leads to an over-compensation of the downward motion, and the following upward motion is again over-compensated by the large friction term, so that the block slides downward in an up-down-down-sequence. The detailed motion depends slightly on the integration method chosen, but the overall effect is the same whether e.g. one-step methods or multi-step methods are used for the integration process.

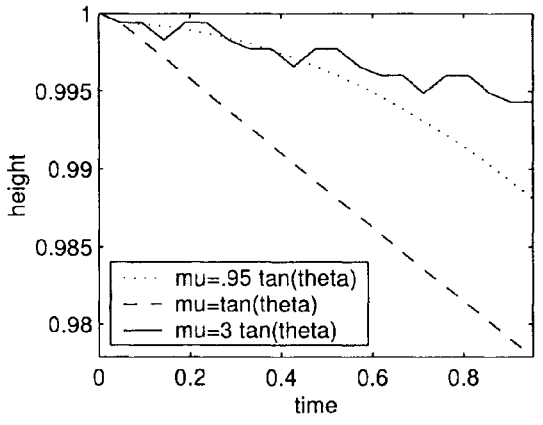

Figure 3: Time evolution of the movement of a block on an inclined slope for three different coefficients of friction.

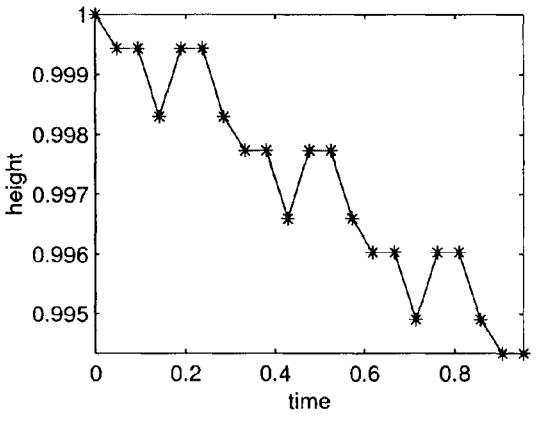

Figure 4: Closeup of the unphysical movement of a block with dynamic Coulomb friction and $\mu = 3\tan(\theta)$

3 Non-Smooth Mechanics and Constraint Motion

As seen in the previous section, static Coulomb friction cannot be modeled using the dynamic Coulomb friction. The deeper physical reason is, that static Coulomb friction acts as a constraint of motion, not as an energy dissipation mechanism as does dynamic Coulomb friction. Therefore, a more appropriate method is necessary to treat the static case of the Coulomb friction in such a way that the graph of the Coulomb friction in Fig. 1 is regularized so that the kinematic constraint holds the the two bodies in frictional in rest with respect to each other. The field which is concerned with such problems is usually called "non-smooth mechanics", as the arising problems lead to non-smooth terms in Newton's equation of motion.

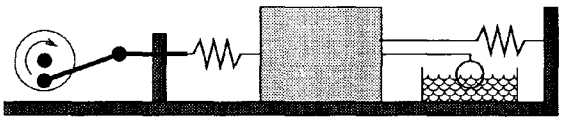

Figure 5: Stick-Slip-oscillator with Coulomb friction, viscous damping and periodic driving.

Because we cannot for static Coulomb friction, determine the movement from the force(-inequalities), we have to determine the forces from the movements (=constrains). The crucial problem is to find the condition for which the dynamic friction evolves into static friction. As an example, we use the equation of motion from p.198 in Ref. [1] with the coordinate y and the velocity $\dot{y}$

$$\ddot{y} + 2D\dot{y} + \mu \mathrm{sign}(\dot{y}) + y = A\cos(\omega t) \tag{3}$$

for which the physical model is shown in Fig. 5. The term with sign($\dot{y}$) must be regularized for static friction. Using the switching function $g(y) = \text{sign}(\dot{y})$, we can transform eqn.(3) so that we see the two "branches" of the flow of the differential equation with respect to the velocity $v = \dot{y}$

$$\dot{y} = \begin{cases} f_I(y) & = (A\cos(\omega t) - \ddot{y} - \mu - y)/2D \text{ if } g(y) > 0 \\ f_{II}(y) & = (A\cos(\omega t) - \ddot{y} + \mu - y)/2D \text{ if } g(y) < 0 \end{cases} \tag{4}$$

As f is discontinuous at $S = y; g(y) = 0$ and the difference is $f_{II}(y) - f_I(y) = 2\mu$, we look for the solution for $g(y) = 0$ in the "convex pencil"

$$f(y, \lambda) = (1 - \lambda)f_I(y) + \lambda f_{II}(y). \tag{5}$$

which interpolates smoothly between both branches of eqn.(4,4.) What is left is to determine the λ so that so that $g(y_0) = 0$). Differentiating $g(y) = 0$ versus time using the chain rule, we get

$$\nabla g(y)\dot{y} = 0.$$

As we have $\dot{y} = f(y, \lambda)$ from eqn.(5) , we have derived the left side of eqn.(4) to be 0 for $g(y) = 0$. Setting

$$a_I = \langle \nabla g(y_0), f_I(y_0) \rangle \tag{6}$$

$$a_{II} = -\langle \nabla g(y_0), f_{II}(y_0) \rangle \tag{7}$$

we obtain $(1 - \lambda)a_I(y) - \lambda a_{II}(y) = 0$,

or $\lambda = \dfrac{a_I}{a_I - a_{II}}$ as long as $v = 0$.

The last equations for λ show that the solution for static friction is a Lagrange-multiplier-problem. Lagrange-multipliers are usually encountered for problems of constraint motion in classical mechanics. As we have emphasized in the beginning that static friction is physically a problem of constraints of motion, it is satisfying that also the mathematical formalism shows the character of a constraint problem. Implementing the above equations for λ numerically with $D = 0.1$, $\mu = 4$, $A = 2$, $\omega = \pi$ and the initial conditions $v_0 = 4, y_0 = 3$ with constant time step and Heun's method, we get the result from Fig. 6.

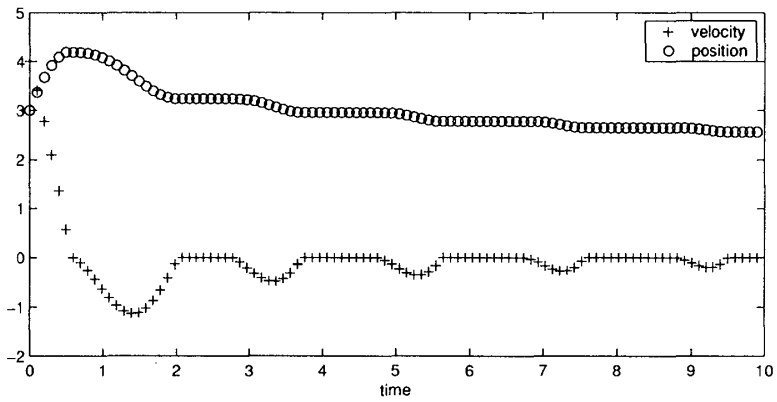

Figure 6: Numerically exact implementation of Coulomb friction using the Lagrange-multiplier-approach.

As can be seen, in the regime where the Coulomb friction exceeds the external forces, the block does not move at all, the static Coulomb friction is realized numerically exact. Due to the periodic driving of the system, we can also observe a "stick slip" motion, periods with motion alter with those of stillness. If we use the dynamic Coulomb friction $F_{\text{Coulomb}} = -\text{sign(v)}\mu F_{\text{normal}}$, the result looks like in Fig. 9. The block does no come to rest, but oscillates around the region with zero velocity in the case where the Coulomb friction exceeds the external forces, the stick-part of the stick-slip motion is not recognizable due to the noisiness of the simulation.

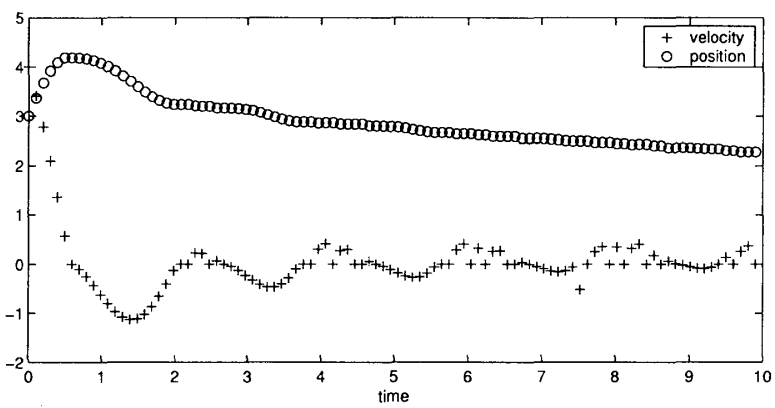

Figure 7: Numerically naive implementation using the dynamic coulomb friction also for the static case.

In principle it is possible to get "good" solution using adaptive stepsize control algorithms using the dynamic friction law only. In that case, in the "stick" regime, the stepsize is reduced considerably, so that the "over-compensation" of the external forces is limited to small timesteps and therefore the averaged result is still very similar to stick. Nevertheless, the result is far from satisfying, as the noise around the $v = 0$-regime is still considerable.

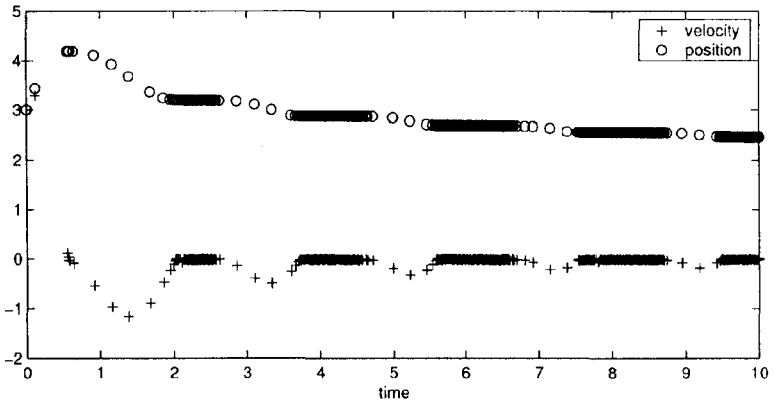

Figure 8: Numerically naive implementation using the dynamic coulomb friction with adaptive stepsize control.

Moreover, using adaptive stepsize control in some applications can be prohibitive as the timestep decreases so tremendously that simulations may become prohibitive for technically reasonable friction coefficients. We encountered such a problem for two coupled, driven sledges, where the computer time and accuracy necessary for $\mu = 0.01$ was comparable for the DAE-solution and for the solution with dynamic friction only, but the computer time necessary at $\mu = 0.3$ increased hundredfold for the dynamic friction case and for $\mu = 0.6$ we terminated the simulation without result after using tenthousandfold the computer time of the Lagrange-multiplier-solution had been consumed by the dynamic-friction-only simulation. Moreover, the Lagrance-multiplyer-method can also be used with the adaptive-stepsize method and leads to numerically superior solutions.

On a side-note, we would like to emphasize that the stick-slip in the above case has not been tuned by the choice of the friction coefficient, but is a result of a periodic driving of the model. Fig. ??

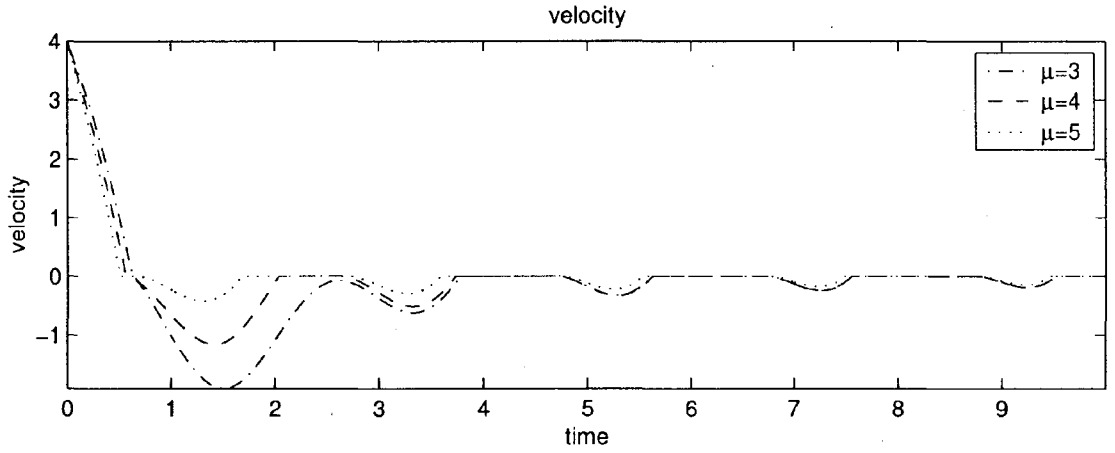

Figure 9: Stick-slip for various choices for μ.

References:

[1] E. Hairer, S.P. Nœrsett, G. Wanner, Solving Ordinary Differential Equations I, 2nd Edition, Springer 1993, page 1999

[2] Engineering Material, Ashby, Jones, Pergamon 1980

[3] CRC Handbook of Lubrication and Tribology, CRC 1994