A New Interpretation on Scars of Short Periodic Orbits

— An Interference Effect in Semiclassical Regime —

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We give a new interpretation of scars of short periodic orbits observed in quantum systems with chaotic classical limit. The enhancement of density is shown to be caused by the mutual interference between two specific paths on the periodic orbit.

我々は古典的にカオスを示すような量子系での周期軌道のスカーの新しい解釈を与えます。確率密度の増加は周期軌道に乗っている二つの特別な軌道の干渉から来ています。

1 Introduction

Since the discovery by McDonald and Kaufman[1] in their numerical work, scars, phenomena of enhancement of density probability along short unstable periodic orbits of classically chaotic systems, has gained a growing attention both theoretically and numerically. It provides a clear example to the inaccuracy of random matrix theory (RMT), according to which, the wave function of classically ergodic systems must be evenly distributed over phase-space, up to quantum fluctuations. Apparently, the semiclassical approach is indispensable to understand this phenomena. The first theoretical explanation was given by Heller[2] based on semiclassical evolution of a Gaussian wave packet near a periodic orbit. Later works by Bogomolny[3] and Berry[4] involved semiclassical calculations in coordinate representation and Wigner representation, respectively. We should note that, in last two works[3, 4], besides its regorousness, scars appears as a mathematical consequence and thereby covers its physical origin.

In this present work, scars will be extracted from the dynamic of the system. In this sense, our approach is in the same spirit as the work by Heller[2]. The role of the initial wave packet will be considered from the beginning. However, in contrast to his, we shall give an explicit periodic orbit formula for the enhancement of density probality which shares similar essential properties as the one derived by Bogomolny in ref. [3]. During the derivation, we shall emphasize the physical origin of this beautifull phenomena. It will be shown later that, in semiclassical regime, the

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enhancement is a natural consequence of interference phenomena between two specific classical trajectories on the periodic orbit under consideration.

2 Extracting an Eigenstate

In this section, a brief review on how to extract an eigenstate from the time developed wave packet is discussed. Let us suppose a bounded system with a Hamiltonian $\hat{H}$ which satisfies the following eigenvalue equation $\hat{H}|\phi_n\rangle = E_n|\phi_n\rangle$, where $|\phi_n\rangle$ and E_n denote the energy eigenfunction and eigenvalue respectively. The time developed wave packet can then be written as

$$|\varphi(t)\rangle = e^{-i\hat{H}t/\hbar}|\varphi(0)\rangle = \sum_n c_n e^{-iE_nt/\hbar}|\phi_n\rangle, \qquad c_n = \langle\phi_n|\varphi(0)\rangle, \tag{1}$$

where $|\varphi(0)\rangle$ is the initial state. Applying Fourier transform to both side, one obtains the following identity

$$-\frac{1}{2\pi i}(\hat{G}^+(E) - \hat{G}^-(E))|\varphi(0)\rangle = \sum_n c_n|\phi_n\rangle\delta(E - E_n), \tag{2}$$

where the Green's function representation for a delta function has been applied and G^+ and G^- denote the retarded and the advanced Green's function respectively. In position representation, a single eigenstate $\phi_n(\mathbf{q})$ weighted by c_n can therefore be obtained as

$$c_n\phi_n(\mathbf{q}) = -\frac{1}{2\pi i}\frac{1}{\varsigma}\int_{E_n-\varsigma/2}^{E_n+\varsigma/2} dE \int d\mathbf{q}' \ \left(\hat{G}^+(\mathbf{q},\mathbf{q}';E) - \hat{G}^-(\mathbf{q},\mathbf{q}';E) \right) \varphi(\mathbf{q}';0), \tag{3}$$

where ς is taken to be sufficiently small but finite. It will later be shown that the presence of ς will guarantee the convergence of the formula we are going to derive.

The probability density of an eigenstate of mode "n" in position space, weighted by the probability that the initial state is to be found in mode "n" is then obtained by taking the absolute square of the above equation, $c_n^2|\phi_n(\mathbf{q})|^2$. One then finds that it is given by combinations of the retarded and the advanced Green's functions. It turns out that, semiclassically, only terms which involve the products of advanced Green's function with its complex conjugate, and the product of the retarded Green's with its complex conjugate will give non-vanishing contributions. Using the relation $G^-(\mathbf{q}',\mathbf{q};E) = [G^+(\mathbf{q},\mathbf{q}';E)]^*$ and writing the retarded Green's function G^+ as G, one finally obtains

$$|c_n|^2|\phi_n(\mathbf{q})|^2 \approx \frac{1}{2\pi^2}\int d\mathbf{q}_1' \int d\mathbf{q}_2' \, \varphi(\mathbf{q}_1';0)\varphi^*(\mathbf{q}_2';0) \left\langle \ G(\mathbf{q},\mathbf{q}_1';E_1) \ \right\rangle \left\langle \ G^*(\mathbf{q},\mathbf{q}_2';E_2) \ \right\rangle, \tag{4}$$

where we have defined $\left\langle \ G(\mathbf{q},\mathbf{q}';E) \ \right\rangle = \frac{1}{\varsigma}\int_{E_n-\varsigma/2}^{E_n+\varsigma/2} G(\mathbf{q},\mathbf{q}';E)dE$. Evaluating this averaging procedure one finds straightforwardly that as $\hbar \to 0$, only trajectories that satisfy the following inequality, contribute to the Green's function: $T_s < \hbar/\varsigma$, where T_s is the time needed by the trajectory s starting from $\mathbf{q}'$ to reach $\mathbf{q}$.

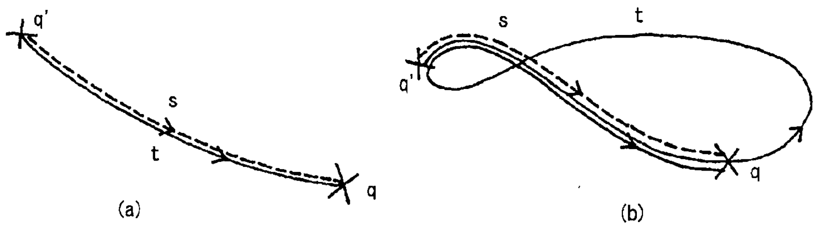

図 1: (a) A pair of identical trajectories and (b) a pair of non-identical trajectories that satisfy the stationary phase condition.

3 Scars, An Interference Phenomena

Our semiclassical consideration begin, firstly, by choosing a sufficiently narrow Gaussian as the initial wave packet

$$\varphi(\mathbf{q};0)=\frac{1}{(\gamma\sqrt{\pi\hbar})^{d/2}}e^{-\mathbf{q}^2/2\gamma\hbar}, \tag{5}$$

with width $\Delta\equiv\gamma\sqrt{\hbar}$ in the configuration space. This choice of initial wave packet is plausible since a quantum-classical correspondence can only be done consistently using coherent state, and that any wave packet can be expanded in a superposition of coherent states. To be able to discuss the quantum-classical correspondence, Δ, should be taken to be sufficiently small compared to the typical length scale of the system under consideration. Yet, to preserve the wave nature of the Gaussian, it should be much larger than the Planck's length. In the semiclassical limit, these two conditions are naturally satisfied by our choice of Gaussian initial state of Eq. (5). This can easily be verified since, as $\Delta\sim\sqrt{\hbar}$, by taking the limit $\hbar\to 0$, Δ will also decrease smaller and smaller, yet, we still have $\Delta\gg\hbar$. In this regime, the following approximate identity holds

$$\varphi^*(\mathbf{q};0;\gamma)\varphi(\mathbf{q}';0;\gamma)=\frac{1}{(\gamma\sqrt{\pi\hbar})^d}e^{(\mathbf{q}-\mathbf{q}')^2/(2\gamma^2\hbar)}e^{-\mathbf{q}\mathbf{q}'/(\gamma^2\hbar)} \tag{6}$$

$$\approx 2^{d/2}\delta(\mathbf{q}-\mathbf{q}')e^{-\mathbf{q}\cdot\mathbf{q}'/(\gamma^2\hbar)}. \tag{7}$$

Applying this to the Eq. (4) one obtains

$$P_n(\mathbf{q})\equiv|c_n|^2|\phi_n(\mathbf{q})|^2\approx\frac{2^{d/2}}{2\pi^2}\int d\mathbf{q}'\;e^{-\mathbf{q}^2/\gamma^2\hbar}\langle G_{sc}(\mathbf{q},\mathbf{q}';E_1)\rangle\langle G^*_{sc}(\mathbf{q},\mathbf{q}';E_2)\rangle. \tag{8}$$

The main contribution to the integral, thus, clearly comes from pairs of paths that start from the same point at $\mathbf{q}'$ and end at the same point at $\mathbf{q}$. The semiclassical approximation can then proceed by replacing the Green's function by its semiclassical version and evaluating the above integral using the stationary phase approximation. The non trivial part of the semiclassical Green's function is given as follows

$$G^s_{sc}(\mathbf{q},\mathbf{q}';E)=\frac{1}{i\hbar(2i\pi\hbar)^{(d-1)/2}}\frac{1}{|\dot{q}\dot{q}'|^{1/2}}\left|\|\;D^s_\perp\;\|\right|^{1/2}e^{\frac{i}{\hbar}S_s-i\frac{\pi}{2}m_s}, \tag{9}$$

where $D^s_\perp(\mathbf{q},\mathbf{q}';E) = -\frac{\partial^2 S_s(\mathbf{q},\mathbf{q}';E)}{\partial q_\perp \partial q'_\perp} = \frac{1}{m_s^{12}}$, $M_s = (m)^{ij}_s$ is the stability matrix of the trajectory s. Performing the stationary phase approximation, one finds that the main contribution is given by pairs of trajectories s and t that satisfy the following stationary phase conditions

$$\frac{\partial S_s}{\partial \mathbf{q}'}\Big|_{\mathbf{q}'=\mathbf{q}'_*} - \frac{\partial S_t}{\partial \mathbf{q}'}\Big|_{\mathbf{q}'=\mathbf{q}'_*} = -\mathbf{p}'_s + \mathbf{p}'_t = 0. \tag{10}$$

Namely, both trajectories have the same initial momentum at the same initial point $\mathbf{q}' = \mathbf{q}'_*$.

Two immediate implications can easily be drawn. First, if $\mathbf{q}$ is semiclassically quite far from any periodic orbits, then the above conditions can only be satisfied by two identicdal trajectories: $s = t$ (See Fig. 1 (a)), for which the phase will cancel out each other. The contribution from this diagonal term therefore gives the non-interesting part which is independent of $\mathbf{q}$ for sufficiently chaotic system. Second, if $\mathbf{q}$ is lying on a periodic orbits po, then besides pairs of identical trajectories, the stationary phase conditions are also satisfied by pairs of different trajectories $s \neq t$ that are both part of a periodic orbit that passes through the point under consideration $\mathbf{q}$, as illustrated in Fig. 1 (b). Then s and t can differ by an integer number $n = 1, 2, \ldots$ of period of the primitive periodic orbit po. The superposition of both of the trajectories results in a total action that is equal to the integer n multiplication of the action of the periodic orbit po. The density probability at point on a periodic orbit can therefore be written as $P_n(\mathbf{q}) = P_n^{diag}(\mathbf{q}) + P_n^{scar}(\mathbf{q})$, where the first and the second term denote the contribution from the diagonal and the off diagonal term respectively. Evaluating the integral, the enhancement can be written as

$$P_n^{scar}(\mathbf{q}) = \frac{2^{d/2}}{2\pi^2} \sum_{po} \sum_r \frac{\sqrt{2}}{\pi^{1/2}\hbar^{5/2}} \frac{1}{|\dot{q}_{po}|\sqrt{(m_{po}^{21})^r}} \frac{\Gamma_{po}}{\sqrt{Tr\tilde{M}_{po}^r - 2}} \cos\Big(r\,\Big(\frac{S_{po}}{\hbar} - \sigma_{po}\frac{\pi}{2} \Big)\Big), \tag{11}$$

which shares similar properties as Bogomolny's[3], except that ours is weighted by $\Gamma_{po} = \int_0^{T_{po}} dt\, e^{-\frac{1}{\gamma^2\hbar}|q(t)|^2}$, which contains the information on the initial wave packet.

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