

Geometric Phase and Classical-Quantum Correspondence

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We study the geometric phase factors underlying the classical and the corresponding quantum dynamics of a driven nonlinear oscillator exhibiting chaotic dynamics. For the classical problem, we compute the geometric phase factors associated with the phase space trajectories using Frenet-Serret formulation. For the corresponding quantum problem, the geometric phase associated with the time evolution of the wave function is computed. Our studies suggest that the classical geometric phase may be related to the the difference in the quantum geometric phases between two neighboring eigenstates.

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Since the discovery of Berry phase [1], the study of geometric phases based on the common mathematical theme of anholonomy, has emerged in a variety of fields. [2] The Berry phase is a path-dependent geometric phase associated with the adiabatic time evolution of the wave function, associated with circuits in parameter space. This concept has been extended [3–5] to non-adiabatic cases and also to non-cyclic circuits, since the phase acquired by the wave function in *any* type of time evolution, may have a component that is of purely geometric origin. This phase is a gauge invariant quantity and is equal to the difference between the total phase and the dynamical phase acquired by the wave function.

One of the open questions has been the classical limit of the Berry phase or its generalization describing the geometric part of the phase of the quantum wave function. For the special case of integrable Hamiltonian systems, described in terms of action-angle variables, the so-called Hannay angle θ_h [6] represents the semiclassical limit of the Berry phase. Berry gave an explicit formula relating the classical angle, called the Hannay angle and the n^{th} quantum eigenstate ϕ_n as, $\theta_h = -\partial_n \phi_n$. There have been some attempts to describe the classical limit of Berry phase for chaotic systems where the effort has been focused on finding a generalization of the *2-form* within Wigner-Weyl formalism. [7]

Viewing the Berry phase as an *anholonomy effect* underlying dynamical evolution described by Schrödinger equation, we seek a classical analog of anholonomy, underlying the corresponding classical evolution described by the Newton's equations of motion. Here, we are concerned with the geometric phases associated with periodic, quasiperiodic and chaotic dynamics of a driven nonlinear oscillator. We compute the geometric component of the phase of the wave function using a kinematic formulation [4] of the Berry phase [4] as given by Mukunda and Simon. In the corresponding classical problem, we find the anholonomy underlying nonplanar

periodic phase space trajectories and then extend this formulation to quasiperiodic and chaotic trajectories. It should be emphasized that unlike in Berry phase, our circuits are not in parameter space but are in phase space. By treating a classical phase space trajectory as a space curve, we show that a geometric phase can be associated with every trajectory, by using a Frenet-Serret (FS) formulation [8]. The classical geometric phase is the integrated torsion of the phase trajectory, and it bears a strong analogy to the geometric phase factor associated with the wave function.

To illustrate this correspondence, we begin with the Frenet-Serret equations, describing the time evolution of the orthonormal FS triad, consisting of the tangent \mathbf{T} , the normal \mathbf{N} , and the binormal \mathbf{B} to the space curve $\mathbf{r}(t)$,

$$\dot{\mathbf{T}} = v\kappa\mathbf{N}, \quad \dot{\mathbf{N}} = -v\kappa\mathbf{T} + v\tau\mathbf{B}, \quad \dot{\mathbf{B}} = -v\tau\mathbf{N}, \quad (1)$$

where κ and τ are respectively the curvature and torsion of the curve and $v = |\dot{\mathbf{r}}|$. Here, the overdot denotes derivative with respect to time. The above equations can be written as, $\dot{\mathbf{F}} = \xi \times \mathbf{F}$, where $\mathbf{F} = \mathbf{T}, \mathbf{N}$, or \mathbf{B} , and $\xi = -v\kappa\mathbf{B} + v\tau\mathbf{T}$. That is, the FS triad rotates around \mathbf{T} and \mathbf{B} . One way to quantify this rotation is to work in a frame in which \mathbf{T} is parallel transported, and measure the angle of rotation around the tangent \mathbf{T} . This will be given by the angle $\phi_c(t) = \int_0^t v\tau dt'$. Thus $\phi_c(t)$ is the geometric phase characterizing the anholonomy associated with the corresponding phase space orbit.

If we define a complex vector $\mathbf{M} = (\mathbf{N} + i\mathbf{B})/\sqrt{2}$, the *classical* geometric phase can be written as $\phi_c(t) = i \int_0^t \mathbf{M}^* \cdot \dot{\mathbf{M}} v dt'$. This expression has exactly the same form as the *quantum* geometric phase found by Berry [1], when the classical vector \mathbf{M} is replaced by a quantum eigenstate ψ_n . Additionally, by mapping the closed phase space trajectory to a circuit on a unit sphere traced by the tip of the

tangent vector (tangent indicatrix), ϕ_c can be shown to be the solid angle subtended by this tangent indicatrix at the center of the sphere. [10] These results are valid also for a non-periodic trajectory, since it can always be closed using a geodesic on the sphere. [4]

Motivated by this close analogy between the geometric phase in a quantum system and the FS geometric phase, we investigate any possible relationship between the two. The underlying key question is whether the classical anholonomy is related to the corresponding quantum one. Here we calculate the classical and the quantum geometric phases for a periodically driven nonlinear oscillator exhibiting complex dynamics. The system under investigation is an "impact oscillator" [9], the oscillator that rebounds elastically whenever its displacement x drops to zero. For $x > 0$, the system is described by,

$$H = \frac{1}{2}p^2 + \frac{1}{2}\omega_0^2 x^2 - f \cos(\omega t)x \quad (2)$$

The system is piecewise linear, and the analytic solutions can be obtained for $x > 0$. The discontinuity at the origin makes it essentially nonlinear. Without loss of generality, we choose the units of t and x such that $\omega = 1$ and $f = 1$. The phase space trajectory of the dynamical system can be viewed as a space curve generated by the three-dimensional vector $\mathbf{r}(t) = (x, \dot{x}, \ddot{x})$ parameterized by the time t .

As we discuss below, the impact oscillator exhibits very rich and complex dynamics, where periodic, quasiperiodic and chaotic dynamics coexist. Our choice of this example was motivated by the fact that in addition to the simplicity underlying the classical analysis of the oscillator, the quantum wave functions for the driven oscillator are known in terms of the classical solution as, [11],

$$\psi(x, t) = \chi(x', t) \exp \frac{1}{\hbar} [\dot{x}_c(t)x' + \int_0^t L(t')dt'] \quad (3)$$

Here $x' = x - x_c(t)$, with $x_c(t)$ being the solution of the classical equation of motion and L is the Lagrangian of the driven system. $\chi(x, t)$ is the wave function of the oscillator in the absence of driving. Note that the wave functions of the driven oscillator are centered on the position of the classical forced oscillator.

If we take χ to be an eigenstate of the undriven oscillator, $\chi_n(x, t) = u_n(x) \exp -i(E_n t/\hbar)$, the eigenfunction can be written in terms of Hermite polynomials as $u_n(x) = \exp -[\omega_0/(2\hbar)x^2] H_n(\sqrt{\omega_0/\hbar}x)$. It should be noted that corresponding to the eigenstate, $|\psi(x, t)|^2 = |\chi_n(x', t)|^2$. Therefore, the center of the wave packet $x_c(t)$ obeys classical equation of motion, and the shape of the wave packet (the density distribution with respect to the center x_c) is unaffected by the driving force.

In view of the fact that we have the analytic solution for the classical system for $x > 0$, and *also* a closed form

solution for the quantum wave function, we can compute the classical and the quantum geometric phases with extreme precision. These will be presented below.

Figure 1 shows richness and complexity underlying the classical dynamics of the oscillator as we vary the initial energy (initial conditions) of the oscillator. We see periodic orbits and quasiperiodic tori, in addition to chaotic trajectories.

All results are for a fixed $\omega_0 = 1.6$ which corresponds to the oscillator frequency of 3.2, in units of ω , the frequency of the driver, putting our analysis is close to the adiabatic regime.

Furthermore, we believe that the probability of inducing a transition (due to driving) is rather small because the classical energies of the particle under consideration here are far below the threshold for the transition between two quantum states n_1 and n_2 in units of $\hbar\omega_0$.

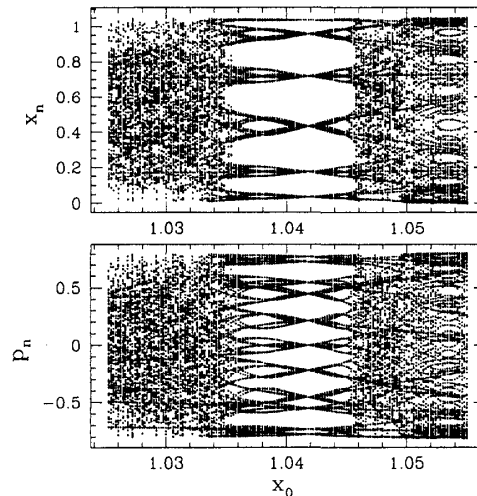


FIG. 1. For $p_0 = 0$, the figure shows the possible x and p values (once every period of the driver), as a function of x_0 , the initial position of the oscillator. For $x_0 \approx 1.0416$, there is a period-11 orbit. For $x_0 > 1.046$, we get invariant quasiperiodic tori or cantori as seen in the lower figure. For $x_0 < 1.035$, almost all initial conditions result in chaotic dynamics.

Using the classical solution, the FS geometric phase ϕ_c can be computed as the integral of the torsion, given by $\tau = \dot{\mathbf{r}} \cdot (\ddot{\mathbf{r}} \times \dot{\mathbf{r}}) / |\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|^2$. For the corresponding quantum problem, we can obtain the geometric phase, using kinematic formulation by Mukunda and Simon [4]. For a given wave function $\psi(x, t)$, the quantum geometric phase ϕ_q is a gauge invariant quantity is given by $\phi_q(t) = \arg \int [\psi^*(x, 0)\psi(x, t)]dx - \text{Im} \int_0^t \psi^*(x, t')\partial_t \psi(x, t')dt'$. It is clear that the first term describes total phase while the second term describes the dynamical phase accumulated by the wave function in time t . Note that this formulation does not require any circuits to define geometric

phase.

For the driven impact oscillator, if we substitute the solution for the wave function, given by equation (3), the geometric phase can be written in terms of the expectation values of $\langle x \rangle$ and the classical solution x_c

$$-\hbar\phi_{q,n} = \int_0^t [L(t') - \langle x \rangle \dot{x}_c(t')] dt' + G(t) \quad (4)$$

where $G(t)$ is given by, $G(t) = (\dot{x}_c(t)x_c(t) - \dot{x}_c(0)x_c(0)) + \text{arg}[\int (u_n(x'(0))u_n(x'(t)) \exp \frac{i\pi}{\hbar} (x_c(t) - x_c(0))) dx$. For periodic evolution, $G(t) = 0$. Also, $G(t) = 0$ if we consider t values where the classical particle is at the turning points. In view of this, we will confine our calculations of geometric phases to only such values of t .

Since the classical impact oscillator is confined to $x \geq 0$, the eigenstates of the corresponding quantum system are restricted to *odd* n values only. Substituting the explicit expression for the Hermite polynomials, $\langle x \rangle$ can be expressed in terms of parabolic cylindrical functions which are functions of x_c .

$$\langle x \rangle_{2n-1} = \frac{\sum_{l=1}^{4n} A_{4n-l}(y_c) D_{-l}}{\sum_{l=1}^{4n-1} B_{4n-l}(y_c) D_{-l}} \quad (5)$$

where $D_{-m}(y_c)$ is a parabolic cylinder functions of order m . Here $y_c = \alpha x_c$ where $\alpha = \sqrt{2\omega_0/\hbar}$. $A_l(y_c)$ and $B_l(y_c)$ are polynomials of y_c of degree l that are uniquely determined by n . For example, for $n = 1$, we have

$$\langle x \rangle_1 = \frac{1}{\alpha} \frac{6D_{-4}(y_c) - 4y_c D_{-3}(y_c) + y_c^2 D_{-2}(y_c)}{2D_{-3}(y_c) - 2y_c D_{-2}(y_c) + y_c^2 D_{-1}(y_c)} \quad (6)$$

For higher values of n , the expressions are very complicated. Our analysis has been confined to the ground state $n = 1$, and the first excited state $n = 3$. For all values of n , $\hbar\phi_{q,n}$ reduces to the classical action as $\hbar \rightarrow 0$. We factor out this part of the phase factor ϕ_0 and write $\phi_{q,n} = \phi_0 + \phi_n$. As we show below, it is the difference in the phases between the two neighboring eigenstates that exhibit quantum fingerprints of classical dynamics.

Figures 2 and 3 show geometric phases for a fixed time interval T , equal to the driving period, for a periodic and a chaotic trajectory. These results as well as other similar analysis suggest that ϕ_c may be related to $\phi_{n-1} - \phi_n$.

For dynamical evolution involving arbitrary time t , our detailed analysis shows that the integrated quantum phase factors can be expressed as a sum of linear and oscillatory functions of t and hence can be written as, $\phi_n(t) = v_n t + f_n(t)$. Here v_n are constants, independent of t and $f_n(t)$ are oscillatory functions which are found to retain the fingerprints of the corresponding classical dynamics for all times.

In contrast to quantum phases, the classical $\phi_c(t)$ is found to be an oscillatory function for periodic, quasiperiodic as well as for the chaotic trajectories. This

should be contrasted with the the driven and damped oscillator phases [12] where the classical phase averaged over the period of the driver was finite.

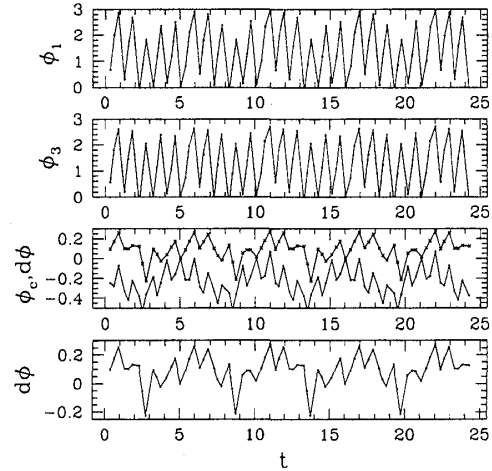


FIG. 2. For a period-11 trajectory, the four figures (from top to bottom) show the quantum geometric phases for the ground state ($n = 1$), the first excited state ($n = 3$), classical geometric phase ϕ_c and $d\phi = \phi_1 - \phi_3$, respectively.

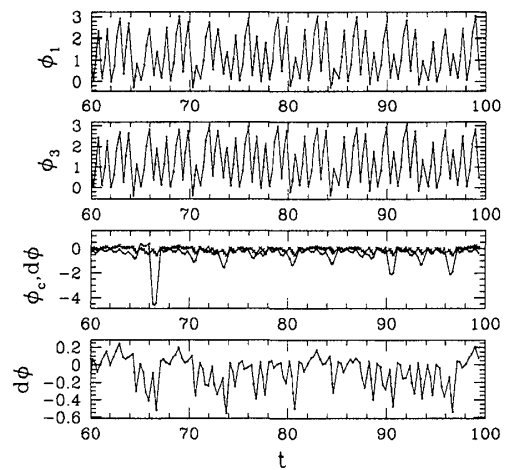


FIG. 3. For a chaotic trajectory, same plots as in Fig. 2. In the third figure, we superimpose $d\phi$ with ϕ_c .

The results for arbitrary time evolution are shown in figures 4 and 5. Here we factor out the linearly increasing parts of the phases and show the fluctuating parts along with the classical phase. For the parameter values corresponding to the figures 2 to 5, $v_1/(2\pi) = .4218$ and $v_3/(2\pi) = .3956$ for the periodic orbit and $v_1/(2\pi) =$

.3723 and $v_3/(2\pi) = .4021$ for the chaotic orbits. We notice that ϕ_c is of the same order of magnitude as $f_1 - f_3$ with ϕ_c exhibiting intermittent fluctuations. In view of the fact that $v_1 \approx v_3$, it is conceivable that v_n approaches a constant, independent of n as $n \rightarrow \infty$ and therefore ϕ_c may be related to the $d\phi$ for arbitrary time t . This is analogous to Berry's relation $\theta_n = -\partial_n \psi_n$ which was true for integrable systems in semiclassical limit.

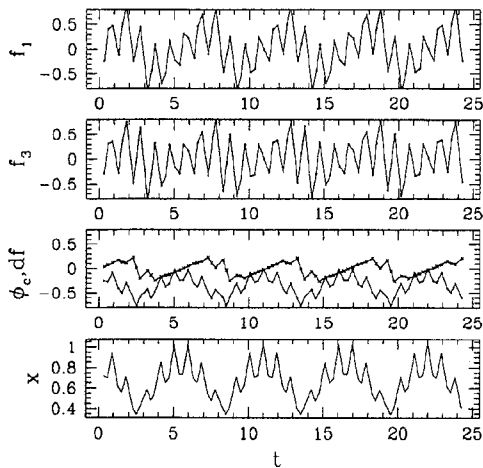


FIG. 4. For a period-11 trajectory, the top two figures show the oscillatory components f_1 and f_3 of the net accumulated phases for the ground state and the excited state. The third figure shows the classical phase and the difference $df = f_1 - f_3$. The lowest figure shows the position of the oscillator.

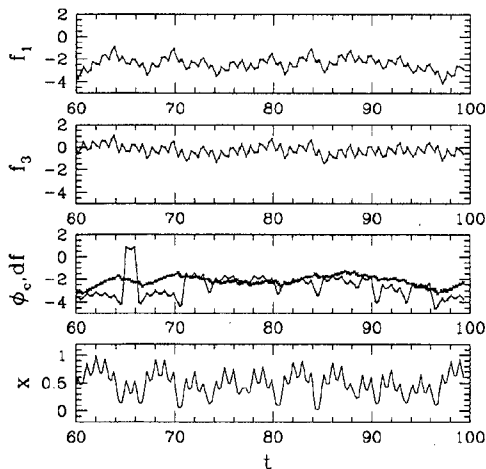


FIG. 5. For a chaotic trajectory, the same plots as in Fig. 4.

In summary, our results as depicted above describe preliminary studies of classical and quantum phases underlying a driven nonlinear oscillator. An interesting result is the possible relationship between the classical phases and the difference in the quantum phases between the two neighboring states, reminiscent of the Berry relation $\theta_h = -\partial_n \phi_n$. As a consequence, the smallest of the classical phase will have its origin in the relative stability of the quantum phase with respect to the quantum number n . It is rather intriguing that the fluctuations in the quantum geometric phases retain the finger prints of the corresponding classical dynamics. Further detailed studies involving higher excited states are needed to confirm these speculative views.

In physical applications such as atom optics, Hamiltonians of the form, $H(x, t) = H_0 + \lambda x \sin(\omega t)$ are relevant where H_0 is the time-independent Hamiltonian and the time-dependent term describes the interaction with a single mode radiation field in dipole approximation. For nonlinear H_0 , nontrivial dynamics may lead to many surprises. The impact oscillators exhibit dynamics with many features that are typical of a nonlinear oscillator. This suggests that the driven impact oscillator may provide an interesting theoretical model to explore various issues relevant to quantum chaos.

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