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The QCD vacuum as a disordered medium: A simplified model for the QCD Dirac operator

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We model the QCD Dirac operator as a power-law random banded matrix (RBM) with the appropriate chiral symmetry. Our motivation is the form of the Dirac operator in a basis of instantonic zero modes with a corresponding gauge background of instantons. We compare the spectral correlations of this model to those of an instanton liquid model (ILM) and find agreement well beyond the Thouless energy. In the bulk of the spectrum the (dimensionless) Thouless energy of the RBM scales with the square root of system size in agreement with the ILM and chiral perturbation theory. Near the origin the scaling of the (dimensionless) Thouless energy in the RBM remains the same as in the bulk which agrees with chiral perturbation theory but not with the ILM. Finally we discuss how this RBM should be modified in order to describe the spectral correlations of the QCD Dirac operator at the finite temperature chiral restoration transition.

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THE CHIRAL RANDOM BANDED MODEL

Here we study the spectral properties of an ensemble of chiral random Hermitian $N \times N$ matrices given by

$$D_{RBM} = \begin{pmatrix} 0 & C \\ C^* & 0 \end{pmatrix}$$

where $C$ is a $N/2 \times N/2$ complex matrix with independently distributed Gaussian variables with zero mean. The variance of the matrix elements $C_{ij}$ are chosen to decay as a power of $r = |i - j|$ which measures the distance from the diagonal. Since the ILM uses periodic boundary conditions we use a periodic form of the power-law decay [17] given by

$$\langle |C_{ij}|^2 \rangle = \left\{ 1 + \left[ \sin(2\pi tr/N) \right]^{2a} \right\}^{-1}$$

where $a$ and $b$ are real parameters. The choice of complex matrix elements corresponds to a matrix model with a unitary symmetry which is appropriate for QCD with the phenomenologically relevant $SU(3)$ color group. Due to the chiral symmetry, the eigenvalues of (4) can be pairs of $\pm \epsilon_i$. This feature induces an additional level repulsion around zero which results in different spectral correlations for eigenvalues near zero (the origin) and away from zero (the bulk).

In the bulk the spectral correlations should not be affected by the block structure and should coincide with the non-chiral version of (4) which has been intensively studied in recent years [7, 17]. The use of the supersymmetry method [18] permits an analytical evaluation of both spectral properties and eigenfunction statistics [7] in a certain region of parameters. In the thermodynamic limit the eigenfunctions are multifractal for $\alpha = 1$ and localized (delocalized) for $\alpha > 1$ ($\alpha < 1$) respectively [7]. The spectral correlations in the $g = E_\epsilon/\Delta >> 1$ ($E_\epsilon$ is the Thouless energy and $\Delta$ is the mean level spacing) limit can be expressed through the spectral determinant of a classical diffusion operator [19]. The two point correlation function is defined as $R_2(s) = \Delta^2 (\rho(\epsilon)\rho(\epsilon + s\Delta)) - 1$ where $\rho(\epsilon)$ is the density of states at energy $\epsilon$ and the average is over an ensemble of RBM. For the unitary ensemble (our case)

$$R_2(s) = -\frac{1}{4\pi^2\delta s^2} \ln \frac{D(s, g)}{s^2} + \frac{\cos(2\pi s)}{2\pi^2s^2} D(s, g).$$

Due to the power-law decay, the spectral determinant $D(s, g) = \prod_{\epsilon_n < 0} (1 + s^2/\epsilon_n^2)^{-1}$ ($\epsilon_n = g/n^{2a-1}$) corresponds with a process of anomalous diffusion [7]. For $1/2 < \alpha < 1$ the dimensionless conductance increases with the system size as $g = C_0(b)N^{2-2a}$ with $C_0(b)$ a known constant. The scaling of $g$ thus resembles that of a weakly disordered conductor in $d = 2/(2a - 1)$ dimensions [7].
The observed behavior of the number of positive eigenvalues 
deviation within about 3/4 because the volume dependence of the dimensionless conductance in this case $g \approx 1.17\sqrt{\alpha N}$ [7] coincides with what is expected for QCD according to chiral perturbation theory [21]. As mentioned above, the spectral properties of the RBM at $\alpha = 3/4$ are similar to those of a disordered conductor in four dimensions.

Given the known results for the non-chiral RBM, one can now choose the parameter $\alpha$. Recall that our motivation to propose a random banded model is the decay of the ILM overlap matrix elements (3). It was shown in [5] that the spectral properties of systems with power-law hopping are similar in different dimensions provided that the decay exponent equaled the dimension. Since in the ILM the decay exponent (three) is less than the dimension (four) we expect this to map onto a 1D RBM model with $\alpha < 1$. We choose $\alpha = 3/4$ because the volume dependence of the dimensionless conductance in this case $g \approx 1.17\sqrt{\alpha N}$ [7] coincides with what is expected for QCD according to chiral perturbation theory [21]. As mentioned above, the spectral properties of the RBM at $\alpha = 3/4$ are similar to those of a disordered conductor in four dimensions.

In the ILM the decay exponent (three) is less than the theoretical expectation $g \approx 1.17\sqrt{N}$ within about 2%. In Figure 1 we show the spectral rigidity in the bulk of the chiral RBM obtained from numerical simulation at $b = 1$ along with the analytic formula of the non-chiral model for different values of $g$. The values of $g$ were chosen by eye to match the numerical results for the corresponding $N$. We find good agreement with the theoretical expectation $g \approx 1.17\sqrt{N}$ within about 2%.

In Figure 2 we compare the spectral rigidity in the bulk of the ILM with analytical predictions of the RBM. Again the values of $g$ for each volume were chosen by eye to closely match the ILM. The analytic results using these values of $g$ agree well with the ILM up to a scale of about 30% of the number of positive eigenvalues ($N/2$).

We generated sets of matrices for the RBM and the ILM and calculated their eigenvalues in order to compare the spectral correlations. The results of each simulation are averaged over $10^4$ configurations. For both models the simulations were done in the quenched approximation which allowed us to use values of $N$ ranging from 320 up to 1280. For the ILM we used the same model studied in [1] including the standard density of states $N/2 = 1 \text{fm}^{-4}$. Additional details can be found in [11].
model with power-law decay in order to describe the spectral correlations of the QCD Dirac operator beyond the Thouless energy. We have thus combined the asymptotic power-law tail observed in instanton liquid models with the random matrix approach valid for small spacings. We have provided some numerical evidence that the resulting chiral RBM does (at least for the two-point function) describe the spectral correlations of the QCD Dirac operator well beyond the Thouless energy. Finally we have mentioned that at the finite temperature chiral restoration transition the appropriately modified chiral RBM predicts a metal-insulator behavior including multifractal wavefunctions and the physics of the Anderson transition in the QCD vacuum.

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FIG. 3: Number variance $\Sigma^2(L)$ close to the origin. Points correspond to numerical simulation of the ILM. Lines correspond to numerical simulations of the chiral RBM for the given values of $b$.

RBM are in agreement with QCD theoretical predictions [1, 21] and lattice results [22].

We do not yet have analytical results for the spectral correlations of the RBM at the origin and therefore rely solely on numerical simulations. In Figure 3 we show the number variance near the origin for the ILM and chiral RBM. The corresponding values of $b$ for the RBM were also chosen by eye to provide a good fit. The agreement between both models is very good up to about 50 eigenvalues but we could not fit all volumes with a single $b$. The reason for that is that the chiral RBM has the same $g \sim \sqrt{N}$ scaling as in the bulk while the instanton liquid shows a weaker volume dependence. We find the scaling $b \approx 1.4N^{-0.13}$ which gives $g \approx 1.4N^{0.43}$ for the ILM. It would be interesting to compare these results with lattice simulations where a $g \sim \sqrt{N}$ scaling has also been reported close to the origin.

Finally we mention how the chiral RBM should be modified to describe QCD at finite temperature. As usual in field theory, temperature is introduced by compactifying one of the spatial dimensions. Thus the effect of temperature in Euclidean QCD is to reduce the effective dimensionality of the system to three. Now since the effective dimension of the space matches the power-law decay of the QCD Dirac operator ($\sim 1/R^3$) one expects, according to [5, 6], multifractal wavefunctions typical of a metal-insulator transition. The same chiral RBM proposed in this paper may be used in this situation but with $\alpha = 1$ (see [23] for a model with similar spectral properties). The above arguments suggest that if the restoration of chiral symmetry at finite temperature is dominated by instantons, the physical mechanism leading to the quark-gluon plasma state of matter would be similar to a metal-insulator transition. Clearly further work is needed to explore this exciting relation.

To conclude, we have proposed a chiral random banded

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