

Uni-directional transport in billiards

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Abstract

We outline an analysis of a *classical billiard chain* – a *channel* that can serve as a model for bended optical fibers. The phase space is split into two dis-joined parts corresponding to left and right *unidirectional motion*. The dynamics is analyzed in terms of a jump model defined by a *jump map* and a *time function*. The jump map has a mixed phase space with dominant chaotic component. We numerically examine the stability of motion and diffusion along the chain. As a result of singularity of the time function we find marginally-normal diffusion after subtracting the average drift.

We analyze dynamical properties of a *billiard chain* constructed as a periodic array of cells of semi-circular walls as shown on the fig. 1. The walls are parallel to each other in the sense that a normal vector to one wall is perpendicular to the other wall. Geometry of the billiard implies that the particle traveling in one direction can not change the direction so the phase space of the dynamics is split into two disjoint parts corresponding to left and right unidirectional motion.

In our work we concentrate only on the motion of the particles to the right. The cells of the chain are labeled with n , starting with zero and increasing to the right. We examine dynamics of particles, with unit velocity, starting at the left side of zeroth ($n = 0$) cell and moving forward to the right. Dynamics is described in terms of a *Poincaré map* (*jump map*) $\mathbf{F} : \mathcal{S} \rightarrow \mathcal{S}$ with the surface of section (SOS) $\mathcal{S} = \{(x, v_x) : x \in [-1, -q], v_x \in [-1, 1]\}$, where x is the position and v_x is the horizontal velocity of the entry point into a cell. In addition we measure the time in a given cell entering at the point \mathbf{x} and denote it as $T(\mathbf{x})$. The pair (\mathbf{F}, T) now represents the *jump model* [2] corresponding to our billiard channel. We should note that numerical routines for computing the map \mathbf{F} and the function T are very elementary and efficient. The jump map \mathbf{F} is non-integrable possessing mixed phase space symmetric around the mean radius $x = (1 + q)/2$ of walls, see fig. 2.

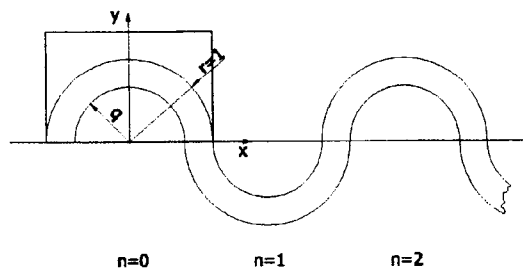


Figure 1: A schema of the serpent billiard model with one cell put in a rectangular frame and index n labeling the cells.

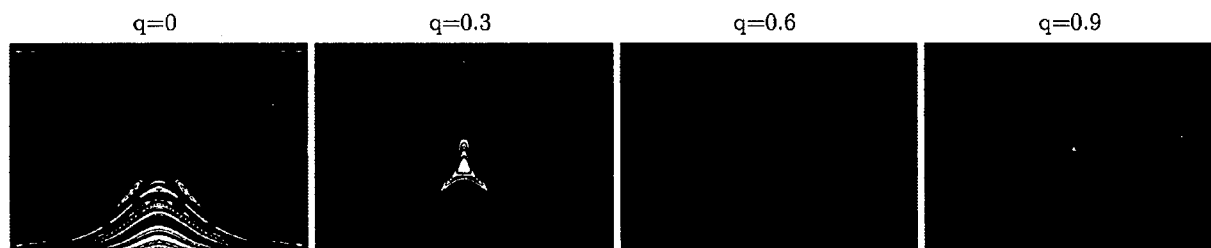


Figure 2: Phase space portraits of the jump map \mathbf{F} for different value of q . Horizontal axis: x , vertical axis: v_x . Each diagram shows 10^4 successive iterations of 400 random initial points.

The chaoticity of the map F is measured by the *Lyapunov exponent* λ [1]. We calculated it numerically and is shown on the fig. 3. It is always positive and monotonically increasing when narrowing the channel (increasing q).

The *time function* $T(\mathbf{x})$ is expected to have a square-root singularity for $v_x \rightarrow \pm 1$ as it may take arbitrary long time to traverse a cell for sufficiently small value of angular momentum,

$$T(x, v_x) \sim (1 - |v_x|)^{-1/2}, \quad |v_x| \sim 1.$$

Another quantity which can illustrate the dynamical behavior of observable T is the probability distribution $P(T)$ of times $T(\mathbf{x})$ for a very long chaotic trajectory. By assuming ergodicity on the SOS $P(t) = \int \delta(t - T(\mathbf{x})) d^2\mathbf{x}$, with the asymptotics $P(t \rightarrow \infty) \sim t^{-3}$ that essentially does not depend on the ergodicity. The only important condition is that the chaotic component is extending to the lines of singularity $v_x = \pm 1$.

We have also studied transport properties of the chain in the context of the jump model. Length is measured in the number of traversed cells. In SOS of zeroth cell we prepare initial ensemble of particles. The transport on the regular components (islands) of SOS is obviously ballistic. Therefore we only focus on the transport on the chaotic component, on which the particles are initially uniformly distributed. We find that in the asymptotics $n \sim vt \gg 1$ the probability distribution $P(n, t)$ of particles over the chain can be expressed as

$$P(n, t) = \frac{1}{\sigma_n} g\left(\frac{n - vt}{\sigma_n}\right), \quad \sigma_n^2 = \sigma_0^2 v^3 t \log(vt),$$

where $v^{-1} = \int tP(t)dt$, $g(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$ is Gaussian function and σ_0 is some system dependent parameter. From this expression we see that the particles exhibits *marginally normal diffusion* $\sigma_n^2 \sim t \log(t)$ with a drift $\langle n \rangle = vt$, which is consistent with our numerical result, see figure 4.

We have analyzed a periodic billiard chain, the serpent billiard, with the property of unidirectional motion, using the jump model [2]. We have shown that the jump map is chaotic with mixed phase space. The size the chaotic component as well as the maximal Lyapunov exponent are increasing by narrowing of the channel. The transport of particles exhibits marginally-normal diffusion $\sigma_n^2 \sim t \log t$ when the drift term is subtracted. The model and its generalizations may be relevant for real world problems of transport in optical fibers or electromagnetic wave-guides.

References

- [1] M. Horvat, T. Prosen, article submitted to *J. Phys. A: Math. Gen.*
- [2] G. Zumofen, J. Klafter Scale-invariant motion in the intermittent chaotic systems, *Phys. Rev. E* **47** 851-863 (1993)

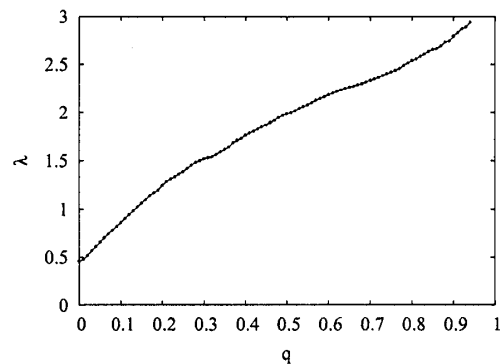


Figure 3: Lyapunov exponent λ of the jump map \mathbf{F} as a function of q .

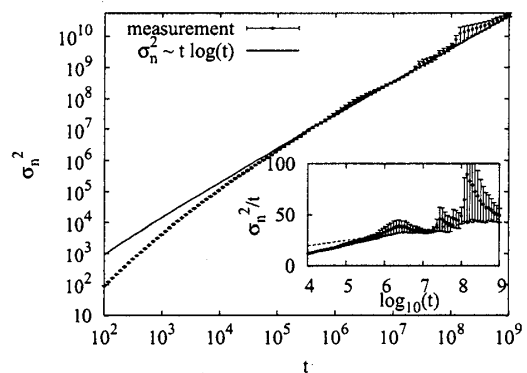


Figure 4: The spread of particles σ_n^2 with time t at $q = 0.6$. The statistics is performed over 25000 initial points.