Quantization of Open System Based on Quantum State Diffusion II

Department of Physics, Waseda University, Tokyo 169–8555, Japan
Ichiro Ohba[1] and Yukihiro Ota[2]

開いた系を量子化する1つの方法，quantum state diffusion 法 (確率的シュレーディンガー方程式) の応用例を示す．この方法では，系の量子–古典対応を論じる際，古典論で位相空間内を時々刻々に変化していく事象が見て取れるのに対応するように，ヒルベルト空間内で，ハミルトニアンによる時間発展はもちろん，散逸効果も確率過程的取り込んで，状態ベクトルを時間発展させる．ここでは古典的に散逸系でカオスが現れているダフィン振動子の量子化を取り上げ，有効プランク定数を変えて擬リヤプノフ数を求め，有効プランク定数のある臨界値でこの系が古典系から量子系へのクロスオーバしていることを示す．

1 Introduction

The quantum mechanics is a very fundamental theory in microscopic and closed systems. The system evolves unitarily; there is any loss of information. On the other hand, there are various irreversible phenomena, the transition from a pure state to a mixed state in macroscopic or nano scale systems. The method of open quantum systems have been researched in order to explain such phenomena by the use of a theory consistent with the quantum mechanics. The dynamics of open quantum systems is phenomenologically described by the Lindblad master equation[1] under the Markovian assumption. The positivity of the density matrix ρ and $\mathrm{Tr}\{\rho\} = 1$ are preserved as long as the system evolves according to this equation. The quantum state diffusion (QSD)[2] is equivalent to this formalism and a very effective tool for numerical simulation of complex problems, compared with the description depending on the master equation. In this paper, we study an application of the QSD for a dissipative quantum chaos.

So far, many researchers have studied the quantum–classical correspondence (QCC) for Hamiltonian systems. Such researches and results are very fruitful and interesting[3]. However, there is an another type of chaotic systems; the chaotic behavior can occur in a dissipative system. Thus, it is necessary to investigate a dissipative quantum chaos in order to discuss the QCC in more detail. Moreover, this subject is also related to a measurement theory in the quantum mechanics[4]. We examine a quantity sensitive to the initial condition and define pseudo–Lyapunov exponent as its candidate. Then, we discuss a crossover in the quantum version of Duffing oscillator. The effective Planck constant β and the effective Planck cell play an important role in this analysis. The more detail discussion is given in the Ref. [5].

[1]E-mail: ohba@waseda.jp
[2]E-mail: ota@suou.waseda.jp

2 Quantum version of Duffing oscillator

The classical equation of Duffing oscillator is $m\ddot{x} - 2\gamma m\dot{x} + m\omega_0^2 x^3/l^2 - m\omega_0^2 x = m\omega_0^2 l^2 g\cos(\omega t)$. If we choose a set of dimensionless parameters $(\Gamma,\ g,\ \Omega) \equiv (\gamma/\omega_0,\ g,\ \omega/\omega_0) = (0.125,\ 0.3,\ 1.00)$, we find the chaotic motion in the Poincaré surface[6]; strange attractor.

In order to treat the quantum version of Duffing oscillator, we determine the *Hamiltonian* $\hat{H}$ $(\hat{H}^\dagger = \hat{H})$ and the Lindblad operator $\hat{L}$ phenomenologically[5]: $\hat{H} = \hat{P}^2/2 + \beta^2\hat{Q}^4/4 - \hat{Q}^2/2 + \Gamma(\hat{Q}\hat{P} + \hat{P}\hat{Q})/2 - g\hat{Q}\cos(\Omega t)/\beta$ and $\hat{L} = \sqrt{\Gamma}(\hat{Q} + i\ \hat{P})$, where $\hat{Q}$ and $\hat{P}$ are the dimensionless position and momentum operator, respectively, and $[\hat{Q}, \hat{P}] = i$. The dimensionless parameters Γ, g and Ω are the same as ones in the classical equation of motion. Notice that β is introduced naturally when the operators are transformed to the dimensionless expressions. We show stro-

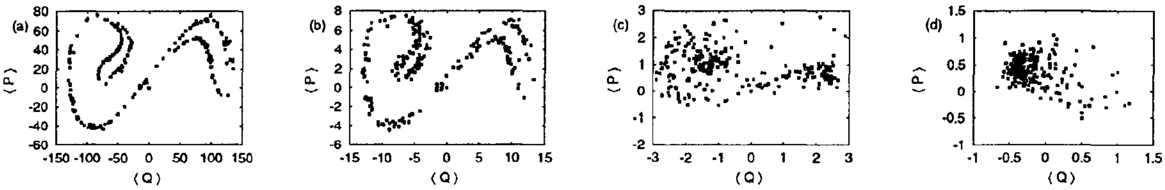

Figure 1: These are the stroboscopic maps for $(\langle\hat{Q}\rangle, \langle\hat{P}\rangle)$. The each point in these figures represents the data at every $2\pi/\Omega$ for a single realization of complex Wiener process. Figures (a), (b), (c) and (d) are for $\beta = 0.01$, 0.10, 0.40 and 1.00, respectively.

boscopic maps for $(\langle\hat{Q}\rangle, \langle\hat{P}\rangle)$ in the Fig. 1 for a certain realization of complex Wiener process $\zeta(t)$ which is related to the QSD[2, 5]. The each point in Fig. 1 represents the data at every $2\pi/\Omega$ for a single realization of $\zeta(t)$. The initial state is a pure coherent state $|\alpha\rangle\langle\alpha|$, where $\mathrm{Re}\{\alpha\} = \sqrt{2}\langle\hat{Q}\rangle$ and $\mathrm{Im}\{\alpha\} = \sqrt{2}\langle\hat{P}\rangle$. These show that a strange attractor appears certainly and the system behaves chaotically in $\beta = 0.01$, while it has been lost in $\beta \sim \mathcal{O}(1)$. For intermediate case, there remains the remnant of strange attractor. We find that the scale of system gets large as β goes to zero. These observations are successful to show the loss of chaotic behavior except for $\beta = 0.01$ at least. Therefore, let us call that the system is in the classical region for $\beta = 0.01$ and in the quantum region for $\beta = 1.00$, respectively. These analyses without averaging over the ensemble for $\zeta(t)$ have been already studied in Ref. [2]. In particular, Fig. 1 agrees with the results in it.

3 Sensitivity to the initial conditions

The above results are inadequate to understand fully the QCC in this model due to the following points. First, it is doubtful whether in $\beta = 0.01$ the chaotic dynamics survives or not, since we only obtain a figure like strange attractor in the stroboscopic maps. The problem remains even if one claims on the basis of this assertion that the chaotic behavior may occurs in $\beta = 0.01$. It is not clear at what region of intermediate $\beta = 0.01$ and $\beta = 1.00$ the classical behavior survives. The definition of the classical region or the quantum region is obscure. Finally, we do not consider the proper quantity related to the behavior of system, as $\beta \to 1.00$.

We examine the above three points, based on an analysis of a quantity sensitive to initial conditions: $\Delta(\tau) = N^{-1}\sum_{\{1,2\}}(\delta\overline{Q}_{12}(\tau)^2 + \delta\overline{P}_{12}(\tau)^2)^{1/2}$, where $\delta\overline{Q}_{12}(\tau) = \mathrm{Tr}\{\hat{Q}\rho_1(\tau)\} - \mathrm{Tr}\{\hat{Q}\rho_2(\tau)\}$ and $\delta\overline{P}_{12}(\tau) = \mathrm{Tr}\{\hat{P}\rho_1(\tau)\} - \mathrm{Tr}\{\hat{P}\rho_2(\tau)\}$. $\rho_1(\tau)$ and $\rho_2(\tau)$ denote two density matrices for different initial states $\rho_1(0)$ and $\rho_2(0)$, respectively. τ is the dimensionless time, $\tau = \omega_0 t$. The subscript of $\{1, 2\}$ represents the summation over the sets of chosen initial conditions and N is the number of those sets. We have randomly chosen initial states, using the data of constant phase maps calculated.

The calculation of $\Delta(\tau)$ is similar to the derivation of Lyapunov exponent in the classical mechanics. We call a resultant value got from the simulation as pseudo–Lyapunov exponent. Before numerical simulations, we have to determine a suitable value of $\epsilon \equiv \Delta(\tau = 0)$. Notice that two points in the phase space are not *distinguishable* in the view of quantum mechanics, if they coexist inside the same Planck cell. The Planck cell is limited by the Heisenberg's uncertainty relation. In this model, the commutator $[\hat{Q}, \hat{P}] = [\hat{x}, \hat{p}]/\beta^2 S_0 = i$ is fulfilled, where $\hat{x}$ and $\hat{p}$ are original position and momentum operators, respectively. Then, the Planck cell has a constant volume of $\Delta Q \Delta P = 1/2$ in the scaled phase space, whereas it has $\Delta x \Delta p = \hbar/2 = \beta^2 S_0/2$ in the original phase space. With the fixed value of typical action S_0 for the system, the smaller β^2 corresponds to the smaller $\hbar$ and the system exhibits the more classical behavior. Thus we define an effective Planck cell as $\beta^2 S_0/2$; its linear size is almost equivalent to β in the unit of $\sqrt{S_0/2}$. We present the results in the case of $\epsilon = 0.01$ (fixed), where two points in the phase space are *distinguishable* only for the classical region ($\beta = 0.01$). Another case is given by Ref. [5]. In Fig. 2

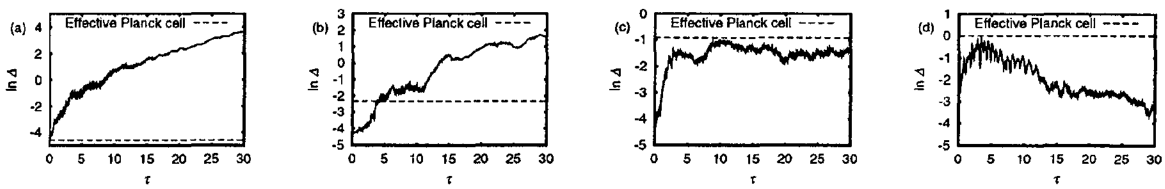

Figure 2: These figures are the time evolution of $\Delta(\tau)$ with ϵ fixed as 0.01. The complex Wiener process is used in QSD. Figures (a) and (b) are obtained with single realization of complex Wiener process for each initial condition (20 samples). Figures (c)–(d) are obtained with averaging over 100 realizations of complex Wiener process for each initial condition (10 samples). Figures (a), (b), (c) and (d) are for $\beta = 0.01$, 0.10, 0.40 and 1.00, respectively.

(a), we find an exponential increase of $\Delta(\tau)$, a characteristic behavior of chaos. This corresponds to the fact that maximal Lyapunov exponent is positive in classical mechanics. This behavior is also consistent with the existence of the strange attractor, and verifies that the quantum version of Duffing oscillator keeps still a chaotic behavior for $\beta = 0.01$. In Fig. 2, we see very different behaviors between (b) and (c)–(d). For these values of β, all initial two points coexist inside of the effective Planck cell and are *indistinguishable* from each other. Nevertheless, starting from the inside of the effective Planck cell, $\Delta(\tau)$ for $\beta = 0.10$ increases gradually and crosses the size of effective Planck cell after some duration and then increases simply. This suggests that the remnant of chaotic dynamics still survives for $\beta = 0.10$. However, $\Delta(\tau)$s for $\beta = 0.40$ and 1.00 always stay within the effective Planck cell. The chaotic dynamics has been completely lost in these cases. This observation suggests that there exists some critical stage as β goes from 0.10 to 1.00. In Ref. [5], we have discussed an effect of dissipation in this system, using a degrees of localization introduced in the QSD formalism. we find that it suppresses the occurrence of chaos in the quantum region, while it, combined with the periodic external force, plays a crucial role in the chaotic behaviors of classical system.

4 Summary

We apply the QSD to the quantum version of Duffing oscillator and discuss the QCC and dissipative quantum chaos in this system. We calculate the constant phase map and the pseudo–Lyapunov exponent, varying Planck constant effectively and show there is a certain clear critical stage in which a crossover from classical to quantum behavior occurs. Furthermore, we discuss the effect of dissipation in the dissipative quantum chaos by means of a degrees of localization introduced in the QSD formalism[5]. Unfortunately, the pseudo–Lyapunov exponent is not a quantity characterized a dissipative system. Therefore, it will be necessary to discuss the connection between the classical trajectory (strange attractor) and the quantum observable which characterize a dissipative quantum system.

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