Title	Quantum Chaos in Generic Systems
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Citation	物性研究 (2004), 82(5): 662-665
Issue Date	2004-08-20
URL	http://hdl.handle.net/2433/97885
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

Quantum Chaos in Generic Systems

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Abstract

In this paper I offer an overview of the problem of energy level statistics in generic Hamiltonian systems, where classically we have regular motion on invariant tori for some initial conditions, and chaotic motion for some other complementary initial conditions. We look at this problem in the semiclassical limit of sufficiently small effective Planck's constant $\hbar_{eff}$, assuming that we do have enough objects (asymptotically infinitely many) to define the statistics in this limit.

1 Introduction

We consider Hamiltonian systems with only bounded, finite, classical motion, and ask what are the statistical properties of the eigenstates (eigenfunctions in configuration space, Wigner functions, the energy spectra, the matrix elements of other observables) of the corresponding quantum systems. In order to have a clear definition of the spectral statistics (energy level statistics) we ought to consider the semiclassical limit of sufficiently small effective Planck's constant $\hbar_{eff}$, and in order to have a clear relationship or correspondence between the quantal and classical aspects the classical dynamics must be well defined (i.e. the phase portrait of the classical phase space must be the same for all eigenstates considered in the statistics), which in the general case forces us to consider only the levels in a small energy interval, which in the semiclassical limit $\hbar_{eff} \to 0$ nevertheless contains arbitrarily large number of levels, so that the energy level statistics are well defined, asymptotically. In the rare case of a scaling system, like in all billiard systems, the classical dynamics is the same for all energies and the advantage of such a situation is obvious. Sometimes also continuous and smooth potentials have a scaling property, like hydrogen atom in strong magnetic field or helium atom and others. The main references of this paper are refs.[1,2].

2 The main assertion of stationary quantum chaos

The main assertion of stationary quantum chaos is the following answer to *the main problem of quantum chaos* in the semiclassical limit of sufficiently small $\hbar$:

2.1 Classical integrability

The case (**I**) of **classically integrable quantal systems** $\hat{H}$:
If H is classically integrable, then the Wigner function $W(\mathbf{q},\mathbf{p})$ of the eigenstate is a delta function on the invariant torus. The eigenvalues of $\hat{H}$, i.e. the eigenenergies, in a small interval,

after unfolding, obey (typically) the Poissonian statistics: The probability $\mathrm{E}(k, L)$ of observing k levels inside an interval of length L is given by

$$\mathrm{E}_{integrable} = \mathrm{E}_{Poissonian}(k, L) = \frac{L^k}{k!}\exp(-L) \tag{1}$$

The untypical cases have measure zero, and are characterized by some number theoretic special properties like e.g. the rectangle billiards with rational squared sides ratio. An extensive study was published by Robnik and Veble (1998).

2.2 Classical ergodicity

The case (**E**) of **classically ergodic quantal systems** $\hat{H}$:
If H is classically ergodic system, then to the leading approximation we have the microcanonical Wigner function for *almost all* eigenstates.

The eigenenergies of $\hat{H}$ in a small interval, after unfolding, obey the predictions of classical *Random Matrix Theories* (RMT), namely the statistics of the eigenvalues of the ensemble of orthogonal Gaussian matrices (GOE) or of unitary Gaussian ensembles (GUE) (depending on the existence or nonexistence of an antiunitary symmetry).

This assertion has been proposed originally by Giulio Casati and coworkers (1980) and later probably independently by Bohigas, Giannoni and Schmit (1984). It implies that $\mathrm{E(k, L)}$ statistics must obey the RMT laws, the so-called Casati-Valz-Gris-Guarneri- Bohigas-Giannoni-Schmit-Conjecture:

$$\mathrm{E}_{ergodic}(k, L) = \mathrm{E}_{RMT}(k, L) \tag{2}$$

2.3 Classically mixed systems

The case (**M**) of **classically mixed (generic) quantal systems** $\hat{H}$:
If H is classically mixed system, then we can distinguish between regular and irregular states. Percival (1973) was the first to propose such a qualitative characterization of eigenstates. The regular states are associated with classical invariant tori (semiclassically EBK/Maslov quantized tori, to the leading semiclassical approximation), and the chaotic states are associated with chaotic components. This view has been made more quantitative in the work of Berry and Robnik (1984). The Berry-Robnik picture rests upon the *The Principle of Uniform Semiclassical Condensation* (PUSC), which states that the Wigner functions of quantal states in the limit $\hbar \to 0$ become positive definite, and since they are mutually orthogonal, they must "live" on disjoint supports, and the phase space volume (Liouville measure) of each of them is of the order of $(2\pi\hbar)^N$ (the so-called elementary quantal or Planck cell). The question is, what is the geometry of the object on which they "condense", and the answer - as a conjecture - is: uniformly on a classical invariant object. Therefore we have regular and irregular states. The assumption is that there is no correlation between the spectral sequences (the regular one and a series of irregular states). If $N \geq 3$ we have only one chaotic component (the Arnold web of chaotic motion pervades the entire phase space - energy surface - and is dense, i.e. its closure is the energy surface) and one associated irregular sequence of eigenstates, whereas in $N = 2$ we have many, even infinite number of sequences of irregular states, of smaller and smaller invariant measure, each sequence being associated with one chaotic component. It is thus assumed that the Wigner function of a regular state is of type (**I**), whilst for irregular states, and generally, it is

$$W(\mathbf{q},\mathbf{p}) = \frac{\delta_f(\mathbf{F}(\mathbf{q},\mathbf{p}))\chi_\omega(\mathbf{q},\mathbf{p})}{\int d^N\mathbf{q}d^N\mathbf{p}\delta_f(\mathbf{F}(\mathbf{q},\mathbf{p}))\chi_\omega(\mathbf{q},\mathbf{p})} \tag{3}$$

where $\chi_\omega(\mathbf{q},\mathbf{p})$ is the characteristic function of the invariant component, labeled by ω, being a (either smooth or nonsmooth, generally possibly also fractal) subset of the smooth $(2N-f)$-dimensional invariant surface defined by the f implicit equations (global integrals of motion), namely $\mathbf{F}(\mathbf{q},\mathbf{p}) = 0$, where $\mathbf{F} = (F_1, F_2, \ldots, F_f)$. The characteristic function $\chi_\omega(\mathbf{q},\mathbf{p})$ is defined to have value unity on ω and zero elsewhere. The integer number f can be anything between 1 (ergodic system) and N (integrable system).

Obviously, the formula (3) is the most general expression for a condensed Wigner function of a (pure) eigenstate. It generalizes the cases (**I**) and (**E**). Namely, if we have ergodicity, then $f = 1$, we put $F_1(\mathbf{q},\mathbf{p}) = E - H(\mathbf{q},\mathbf{p})$, and $\omega =$ entire energy surface. In the other extreme (**I**), we have N global integrals of motion in involution, and so $\mathbf{F}(\mathbf{q},\mathbf{p}) = \mathbf{I_n} - \mathbf{I}(\mathbf{q},\mathbf{p})$, and $\omega =$ the invariant torus labeled by $\mathbf{I_n}$, and we recover the case (**I**). In the most general case, therefore, formula (3) applies. Obviously, W is normalized

$$\int d^N\mathbf{q}d^N\mathbf{p}W(\mathbf{q},\mathbf{p}) = 1 \tag{4}$$

For generic (mixed) systems the most typical case is $f = 1$, $F_1(\mathbf{q},\mathbf{p}) = E - H(\mathbf{q},\mathbf{p})$ and ω is a (nonsmooth, typically fractal, chaotic) subset of the energy surface F_1. We write down this most important case explicitly:

$$W(\mathbf{q},\mathbf{p}) = \frac{\delta_1(E - H(\mathbf{q},\mathbf{p}))\chi_\omega(\mathbf{q},\mathbf{p})}{\int d^N\mathbf{q}d^N\mathbf{p}\delta_1(E - H(\mathbf{q},\mathbf{p}))\chi_\omega(\mathbf{q},\mathbf{p})} \tag{5}$$

It is important to know the relative invariant (Liouville) measure of chaotic and regular eigenstates because the Hilbert space of a mixed Hamiltonian system is split into regular and irregular eigenstates, in the strict semiclassical limit, precisely in proportion to the classical invariant measure of the integrable component (invariant tori) and of the irregular components. It is quite obvious by looking at the equation (5) that the invariant Liouville measure of a subset ω of the energy surface is equal to

$$\rho(\omega) = \frac{\int d^N\mathbf{q}d^N\mathbf{p}\delta_1(E - H(\mathbf{q},\mathbf{p}))\chi_\omega(\mathbf{q},\mathbf{p})}{\int d^N\mathbf{q}d^N\mathbf{p}\delta_1(E - H(\mathbf{q},\mathbf{p}))} \tag{6}$$

The relative invariant Liouville measure of the regular components will be denoted by ρ_1, and the measures of chaotic components (ordered in sequence of decreasing measure) by $\rho_2, \rho_3, \ldots, \rho_m$, where $m = \infty$ for $N = 2$ and $m = 2$ for $N \geq 3$, as already explained.

Assuming the above mentioned absence of correlations pairwise between m spectral sequences, due to the fact that they have disjoint supports and thus do not interact, where m is infinite for $N = 2$ and 2 for $N \geq 3$, the spectral statistics can be written as

$$\mathrm{E}_{mixed}(k,L) = \sum_{k_1+k_2+\ldots+k_m=k} \prod_{j=1}^{m} \mathrm{E}_j(k_j, \rho_j L) \tag{7}$$

which is a manifestation of Berry-Robnik (1984) picture. Here $\mathrm{E}_j(k,L)$ is $\mathrm{E}_{Poisson}(k,L)$ for $j = 1$, and $\mathrm{E}_{RMT}(k,L)$ for $j = 2, 3, \ldots, m$. See cases (I) and (**E**), equations (1) and (2). The picture is based on the reasonable assumption that (after unfolding) the mean density of levels in the j-th sequence of levels is ρ_j, simply applying the Thomas-Fermi rule of filling the phase space volume with elementary cells of size $(2\pi\hbar)^N$ in the thin energy shell embedding the corresponding subset ω. Therefore, please note that the second argument of $\mathrm{E}_j(k, L)$ is weighted precisely by the classical relative invariant measure of the underlying invariant component.

Also, if there were several regular (Poissonian) sequences they can be lumped together into a single Poissonian sequence, because we have some kind of a central limit theorem, saying that the statistically independent superposition of Poisson sequences results in a Poisson sequence.

The case (**M**) is the most general one, and as the limiting extreme cases includes cases (**I**) and (**E**).

2.4 Limitations of the universality

There are two important limitations of the above stated asymptotic behaviour as $\hbar \to 0$, when $\hbar$ is not yet small enough: One is the existence of the outer energy scale, and the other one is the localization phenomena. Due to the lack of space the reader is referred to refs.[1,2].

3 Conclusions and discussion

PUSC is a good basis for treating the semiclassical behaviour of eigenfunctions and their Wigner functions in general bound quantal Hamiltonian systems with discrete energy spectrum and infinitely many levels. It has great predictive power to describe qualitatively and quantitatively the properties of eigenfunctions as well as of the energy spectra. The numerical analysis in the billiard systems[3] has brilliantly confirmed this picture and the theory.

If the regular and irregular eigenstates start to interact, due to the tunneling phenomena, deviations from Berry-Robnik statistics become visible, which happens at lower energies, where $\hbar_{eff}$ is not sufficiently small. This has been pointed out, qualitatively, already in the original paper by Berry and Robnik (1984), but has been worked out quantitatively by Podolskiy and Narimanov[4] very recently (December 2003). In addition, in general we shall see also dynamical localization of chaotic eigenstates on classically chaotic regions in the phase space (seen in the Wigner functions), which gives rise to the phenomenon of fractional power law level repulsion first observed by Prosen and Robnik (1993 and 1994). Here the exponent β in the power law behaviour of the level spacing distribution $P(S)$ at small spacings, $P(S) \propto S^{\beta}$, is intermediate between 0 (Poisson) and 1 (in case of GOE of RMT), hence the name "fractional power law level repulsion". This phenomenon is not captured by the formula of Podolskiy and Narimanov[4] (2003), and has no quantitative dynamical derivation yet.

Acknowledgements

This work was supported by the Ministry of Educatioon, Science and Sports of the Republic of Slovenia and by the Nova Kreditna Banka Maribor. The author thanks the organizers of Kyoto Conference "Quantum Mechanics and Chaos: From Fundamental Problems through Nanoscience", Professors K. Nakamura, K. Takatsuka and I. Ohba, held at the Yukawa Institute in Kyoto, on 11 - 13 November 2003, for the kind invitation and great hospitality.

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