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Sound Radiation from Interaction of a Vortex Ring with a Fixed Sphere

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This paper attempts to find the sound pressure generated by the interaction of an axisymmetric vortex ring with a fixed rigid sphere. For this purpose, the above problem is replaced with the problem of sound emission from the vortex system which consists of the actual vortex ring and its image in the sphere. Then the results of Kambe and Minota for the sound generated by the vortex systems are directly applied to the present case and it is found that the radiated sound is a dipole-like sound related to the time derivative of the stream function of a potential flow around the fixed sphere in the direction of the vortex motion.

1. Introduction

Since the appearance of Lighthill's aerodynamic sound theory (1952), it has been made clear that sound is generated by unsteady

motion of fluid regardless of rigid body vibrations. The well--known Lighthill's equation, which is an inhomogeneous linear wave equation, is derived from the fundamental equations of fluid mechanics (the equations of continuity and momentum conservation), and the inhomogeneous term is interpreted as the source of sound generated. For low Mach number flows, the above term has been shown to be equivalent to Powell's source term $\rho_o \mathrm{div}(\boldsymbol{\omega}\times\mathbf{u})$, due to Powell(1964) and Howe(1975), where ρ_o is the density of the fluid at rest and $\boldsymbol{\omega}=\nabla\times\mathbf{u}$ is the vorticity, and hence it confirms that the source region is the vorticity non-vanishing region. For this reason the radiated sound, which is called as aerodynamic sound after Lighthill, is also called as "vortex sound" after Powell.

Sound generation by interaction of unsteady vortex motion has been studied in our laboratory both theoretically and experimentally. Sound radiation from the region with locally distributed vorticities to the outer region has been obtained by the matched asymptotic expansion method and the results have been applied to the problem of sound radiation from vortex systems (Kambe & Minota 1981). Recently, acoustic wave emitted by head-on collision of two vortex rings was observed and compared with theoretical predictions by Kambe & Minota(1983). It has been also investigated by numerical simulation of viscous vortex motion by Kambe & U Mya Oo(1983) to get insight into viscous effect on sound emission from collision interaction of two vortex rings.

The foregoing cases are concerned only with sound generated by fluid flows in free space. The purpose of the present paper

is to find sound pressure from the flow region which contains solid boundaries. For simplicity of configuration, we consider here the sound generated by the interaction of an axisymmetric vortex ring with a fixed rigid sphere. The fluid is assumed to be inviscid, slightly compressible with low Mach number and irrotational except locally concentrated vorticity. The motion of the vortex ring can be regarded as steady when it is far from the sphere, but as it passes the sphere sound is emitted.

The influence of solid boundaries upon the generation of sound by fluid flow was considered by Curle (1955) as an extension of Lighthill's general theory, but his equation is an intractable integral equation. Radiation of sound from a line vortex interacting with a rigid half-plane was considered by Crighton (1972) and Howe (1975) as a two dimensional problem, using the Green function tailored to the geometry of the flow field. For the three dimensional case Obermeier (1980) generalized the results obtained by Möhring (1978) and Obermeier (1979) to include the effects of solid boundaries upon aerodynamic sound generation. Here also we have to know vector Green functions which are to be adjusted to the geometry of the flow field in question.

In our case, we only have to replace the original problem with the interaction problem of the actual vortex ring and its image in the sphere, and then this enables us to apply directly the results of Kambe & Minota (1981) for sound radiation from vortex systems to the present problem.

2. Sound emission from vortex systems

2.1 General theory

Let l and u be the charecteristic length and velocity of the flow field, then time scale is l/u and scale of the radiated sound field is thus $\lambda = (l/u)c_o = l\ M^{-1}$, where c_o is sound velocity in a uniform flow at rest and M is Mach number u/c_o of the flow. For low Mach number flow, $l/\lambda = u/c_o = M \ll 1 \longrightarrow l \ll \lambda$, which means that the flow is acoustically compact. For such flows, the whole region can be divided into the inner region scaled by the length l and the outer region scaled by the wave length $\lambda = lM^{-1}$. (See figure 1).

In the inner region hydrodynamic flow processes dominate and it can be described by the equations of an incompressible flow as

$$\frac{\partial v_i}{\partial x_i} = 0 \quad , \qquad \nabla^2 p = -\frac{\partial^2 v_i v_j}{\partial x_i \partial x_j} + \frac{\partial f_i}{\partial x_i} \quad . \tag{2.1}$$

In the outer region, where sound field processes dominate, the equations of motion are reduced to a homogeneous wave equation for velocity potential Φ :

$$\frac{\partial^2 \Phi}{\partial \hat{x}_i^2} - \frac{\partial^2 \Phi}{\partial t^2} = 0 \quad , \tag{2.2}$$

where $\hat{x}_i = M\ x_i$ (i=1,2,3).

In the above equations the velocity v_i, the coordinates x_i, the pressure p, the external force f_i and the potential Φ are scaled by u, l , $\rho_o u^2$, $\rho_o u^2/l$ and lu respectively. To determine the radiated sound field the matching of the inner solution and the outer solution in both regions is carried out as follows in an intermediate region.

The inner region, in which the fluid behaves like an incompressible inviscid flow, can again be divided into a finite region D where vorticity $\omega(x,t)$ has non-zero values and the rest where it vanishes. Then the solenoidal velocity $\mathbf{v}(x,t)$ driven by the vorticity field can be expressed by means of the Biot-Savart law as follows :

$$\mathbf{v} = \nabla \times \mathbf{B} \quad , \qquad \mathbf{B} = \frac{1}{4\pi} \int \frac{\boldsymbol{\omega}(y,t)}{|x - y|} dy \quad . \tag{2.3}$$

Outside the region D the velocity is also represented by a potential in an irrotational flow as $\mathbf{v} = \nabla\Phi$. When $r = |x|$ is large we can write

$$\frac{1}{|x - y|} = \frac{1}{r} - y_i \frac{\partial}{\partial x_i} \frac{1}{r} + \frac{1}{2} y_i y_j \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r} + \dots \ , \tag{2.4}$$

where $\mathbf{x} = (x_i)$ and $\mathbf{y} = (y_i)$, and hence the asymptotic expression of the potential is found to be

$$\Phi = \frac{1}{4\pi} P_i \frac{\partial}{\partial x_i} \frac{1}{r} + Q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{r} + \dots\dots \ , \tag{2.5}$$

where

$$P_i(t) = \frac{1}{2} \int (y \times \omega)_i dy \quad , \tag{2.6}$$

and

$$Q_{ij}(t) = -\frac{1}{12\pi} \int y_i \ (y \times \omega)_j dy \ . \tag{2.7}$$

Here use has been made the fact that $\int \omega dy$ vanishes since the vortex lines are all closed curves lying in D and the vorticity ω is zero outside D.

For the outer region, the solution must satisfy both the wave equation (2.2) and the radiation condition. Therefore, the solution can be expressed as a multipole expansion

$$\Phi = \frac{A_o(t-\hat{r})}{\hat{r}} + \frac{\partial}{\partial \hat{x}_i} \frac{A_i(t-\hat{r})}{\hat{r}} + \frac{\partial^2}{\partial \hat{x}_i \partial \hat{x}_j} \frac{A_{ij}(t-\hat{r})}{\hat{r}} + \dots \ , \tag{2.8}$$

where $\hat{r} = |\hat{x}| = M\ r$.

The matching is carried out in an intermediate region where both expressions (2.5) and (2.8) are asymptotically valid as $M \to 0$ and it is found that $A_o(t) = 0$, $A_i(t) = M^2 P_i(t)/4$ and $A_{ij}(t) = M^3 (Q_{ij}(t) + C(t)\ \delta_{ij})$, where $C(t)$ is an arbitrary function to be fixed by comparing the pressure expression (2.9) with the alternative expression for the pressure obtained from the dynamical equation (Crow 1970). Thus the pressure for the outer region, given by $p = -\frac{\partial \Phi}{\partial t}$, takes the form :

$$p = -\frac{M^2}{4\pi} \frac{\partial}{\partial \hat{x}_i} \frac{\dot{P}_i(t-\hat{r})}{\hat{r}} - M^3 \frac{\partial^2}{\partial \hat{x}_i \partial \hat{x}_j} \frac{\dot{Q}_{ij}(t-\hat{r}) + \dot{C}(t-\hat{r})\delta_{ij}}{\hat{r}} + O(M^4) \quad . \tag{2.9}$$

By comparing this equation with Crow's equation for the pressure as stated above we obtain the following relations:

$$\dot{P}_i(t) = \int f_i dy \quad , \tag{2.10}$$

$$\dot{Q}_{ij}(t) + \dot{C}(t)\delta_{ij} = \frac{1}{4\pi}\int (v_i v_j + f_i y_j)\, dy \tag{2.11}$$

$$\dot{C}(t) = -\frac{1}{12}\int (v_k^2 + y_k f_k)\, dy \quad . \tag{2.12}$$

2.2 Sound radiation from the system of N co-axial vortex rings

Consider N co-axial vortex rings with common axis on the z-axis of the cylindrical coordinate system (Z, R, ϕ). Let the axial position be $Z_i(t)$, the strength Γ_i and the radius $R_i(t)$ for ith vortex ring. Suppose that θ denotes the angle between the direction of observation point and the positive direction of z-axis (Fig. 2) .

Then, applying the asymptotic form of (2.9) in the far

field, in dimensional variables,

$$p = \frac{\rho_o}{4\pi c_o} \frac{x_i}{r^2} \frac{\partial^2}{\partial t^2} P_i(t-\frac{r}{c_o}) + \frac{\rho_o}{c_o^2} \frac{x_i x_j}{r^3} \frac{\partial^3}{\partial t^3}[Q_{ij}+C(t-\frac{r}{c_o})\delta_{ij}]+.. \quad (2.13)$$

to the sound pressure radiated from the system of N co-axial axisymmetric vortex rings we obtain the sound pressure as

$p = p_d + p_q + p_m$, where p_d, p_q and p_m are as follows.

The term p_d is a dipole and its value is

$$p_d = \frac{\rho_o}{4\, c_o r} \cos\theta \quad \ddot{P}(t-\frac{r}{c_o}) \quad , \quad (2.14)$$

where P(t) denotes the total impulse of the system and it can be written as $P(t) = \pi \iint R^2 \omega dR\, dZ = \pi \sum_{i=1}^{N} \Gamma_i R_i^2$ (2.15)

from equation (2.6).

The term p_q is a quadrupole with the value

$$p_q = \frac{\rho_o}{4\, c_o^2 r} \dddot{Q}(t-\frac{r}{c_o}) \; (\cos^2\theta - \frac{1}{3}) \quad (2.16)$$

where $Q(t) = \iint R^2 Z\omega\, dR\, dZ = \sum_{i=1}^{N} \Gamma_i R_i^2 Z_i$ (2.17)

is related to the mean axial position of the vortex system.

The last term p_m is a monopole and its value is as follows:

$$p_m = \frac{\rho_o}{6\pi c_o^2 r} \; (\ddot{K} + \frac{d^2}{dt^2} \int \frac{1}{2} y_k f_k dy) \quad (2.18)$$

where $K(t) = \frac{1}{2} \int v_i^2 dy$ is the total kinetic energy of the system.

Thus the sound radiated from the system is found to be the sum of a dipole related to the total impulse of the system, which is also related to the external force, a quadrupole related to the mean axial position of the vortex system and a monopole related to the total kinetic energy of the system.

2.3 Relative efficiency of the three terms p_d, p_q and p_m

It is advantageous to know the relative effeciency of the three components p_d ,p_q and p_m for later part of this paper. For this purpose we can do as follows. By dimensional consideration we can estimate the intensity I_d of sound generated by the dipole component as

$$I_d = p_d^2 / \rho_o c_o \sim \rho_o u^6 c_o^{-3} l^2 r^{-2}$$

at a distance r from the origin in the flow field using equation (2.14) and (2.15).

Similarly, from equations (2.16) - (2.18), we can see that the intensities I_q and I_m of sound generated by the quadrupole and the monopole are respectively as

$$I_q = p_q^2 / \rho_o c_o \sim \rho_o u^8 c_o^{-5} l^2 r^{-2}$$

$$I_m = p_m^2 / \rho_o c_o \sim \rho_o u^8 c_o^{-5} l^2 r^{-2} .$$

Thus we can see that the relative order estimates of the ratios I_q/I_d and I_m/I_d are

$$I_q/I_d \sim (u/c_o)^2 << 1 , \quad \text{and}$$

$$I_m/I_d \sim (u/c_o)^2 << 1 \quad \text{for low Mach number flows.}$$

It follows that the dipole component is more efficient than the other terms for low Mach number flow when it does not vanish. For this reason we should note that the sound generated by the vortex system with an external force is dominated by dipole-like sound .

3. Vortex ring interacting with a sphere

3.1 Equation of motion

Before proceeding to investigate the sound pressure, let

us consider the motion of a vortex ring with strength Γ, radius R and axial coordinate Z near a sphere of radius a with its centre at the origin. As is well known the flow field is equivalent to that of the actual vortex ring and its image in the sphere in free space. For the image vortex ring the strength Γ_2 radius R_2 and axial-coordinate Z_2 are related to those of the actual vortex as follows:

$$\Gamma_2 = -\Gamma r/a \ , \quad R_2 = (a^2/r^2)R \quad \text{and} \quad Z_2 = (a^2/r^2)Z \ , \tag{3.1}$$

where $r^2 = Z^2 + R^2$.

If we denote the radius of core cross-section of the vortex by δ and the values of R and δ for the vortex ring at the initial position far from the sphere by R_o and δ_o we have

$$\delta^2 R = \delta_o^2 R_o \quad . \tag{3.2}$$

Using the above relations (3.1) and (3.2) we can write the equations of motion (Dyson, 1893) after some simplification as

$$\frac{dZ}{dt} = -\frac{\Gamma}{4\pi R}\left[\left(\log\frac{8R_o}{\delta_o}+\frac{3}{2}\log\frac{R}{R_o}-\frac{1}{4}\right)-f-\frac{df}{d\kappa}\kappa\frac{(Z^2+R^2-a^2)(Z^2-R^2-a^2)}{(Z^2+R^2-a^2)^2+4a^2R^2}\right] \tag{3.3}$$

$$\frac{dR}{dt} = \frac{\Gamma}{4\pi R}\frac{df}{d\kappa}\kappa\frac{2ZR(Z^2+R^2-a^2)}{(Z^2+R^2-a^2)^2+4a^2R^2} \tag{3.4}$$

where

$$\kappa^2 = \frac{4a^2R^2}{(Z^2+R^2-a^2)^2+4a^2R^2} \tag{3.5}$$

and $f(\kappa) = (2/\kappa)[\,F(\kappa)-E(\kappa)\,] - F(\kappa)$. (3.6)

Here $F(\kappa)$ and $E(\kappa)$ are the first and the second kinds of complete elliptic integral.

Using Runge-Kutta method we can integrate the equations (3.3) and (3.4) to get the values of Z(t) and R(t) and their time derivatives dZ/dt and dR/dt. For later use the second time derivatives d^2Z/dt^2 and d^2R/dt^2 can be obtained by

differentiating the equations (3.3) and (3.4) with respect to t and by using the relations

$$df/d\kappa = \frac{2-\kappa^2}{\kappa^2(1-\kappa^2)} E(\kappa) - \frac{2}{\kappa^2} F(\kappa)$$

$$d^2f/d\kappa^2 = \frac{(-\kappa^4+7\kappa^2-4)}{\kappa^3(1-\kappa^2)^2} E(\kappa) + \frac{(4-5\kappa^2)(1-\kappa^2)}{\kappa^3(1-\kappa^2)^2} F(\kappa)$$

together with the numerical values of Z , R , dZ/dt and dR/dt.

3.2 Radiated sound

To find the radiated sound from the interaction of the vortex ring with the fixed sphere we consider the sound radiated from the interaction of two vortex rings in free space, the actual vortex ring and its image vortex ring in the sphere, since the motions in the two cases are equivalent as mentioned in the previous section . By doing this we can apply the theory for sound emission from the vortex system in free space stated in section 2.1 to the computation of the sound pressure . But it is important to note that to keep the strength of the actual vortex constant and to satisfy the boundary condition on the surface of the sphere the strength of the image must be changed and hence that there must be an external force to maintain the above situation. Thus as remarked at the end of section 2.3 the sound radiated in this case must be dipole-like sound , and so we only have to consider this component.

Applying the theory for N co-axial vortex rings to the present case with N=2 , vortex strengths Γ, Γ_2 , ring radii R, R_2, and axial coordinates Z , Z_2 , we obtain

$$p = p_d = \frac{\rho_o}{4\pi c_o r} \cos\theta \frac{d^2}{dt^2} P(t - \frac{r}{c_o}) \quad , \tag{3.7}$$

where $$P(t) = \sum_{i=1}^{N} \Gamma_i R_i^2 = (\Gamma R^2 + \Gamma_2 R_2^2) \quad . \tag{3.8}$$

Substituting the values from equation (3.1) into (3.8), we have

$$P(t) = 2\pi\Gamma \ (1/2) R^2 [1 - \frac{a^3}{r^3}] \ . \tag{3.9}$$

Here it is interesting to find accidentally that the function $\psi = (1/2)R^2[1 - a^3/r^3)$ in the right hand side of (3.9) is the stream function for a potential flow around the fixed sphere and the sound radiated by the interaction of the vortex ring with the sphere is proportional to time derivative of that function as

$$p = p_d = \frac{\rho_o}{4\pi c_o r} \cos\theta \ 2\pi\Gamma \frac{d^2\psi}{dt^2} \tag{3.10}$$

By writing p_d in the form :

$$p = p_d = \frac{\rho_o \Gamma}{2c_o r} \cos\theta \ (\dot{Z}^2\psi_{ZZ} + \dot{R}^2\psi_{RR} + 2\dot{Z}\dot{R}\psi_{ZR} + \ddot{Z}\psi_Z + \ddot{R}\psi_R)$$

and by using the results of the previous section we can compute the radiated sound numerically.

4. Results and discussion

Numerical integration of the equations (3.3) and (3.4) was performed by setting the values of the initial parameters as $Z_o/a = 4.$, $\delta_o/R_o = 0.1, 0.3$ and $R_o/a = 0.8, 1.0, 1.2, 1.4$ and 1.6 .

Figure 4 shows the path of the vortex core centre $(R(t), Z(t))$ for various initial values of R_o: $R_o/a = 0.8, 1.0$ and 1.2. Figure 5 represents the streamline pattern of the imaginary potential flow

around the fixed sphere. As can be seen from equation (3.10) the radiated sound would be zero if the path of the core centre coincide with one of the streamlines because the time derivative of the function ψ along the streamlines is zero. Thus we can see that sound is generated only when the path of the vortex core crosses the streamlines of the imaginary potential flow around the sphere. The phenomenon that the emitted sound pressure is in some way related to time derivatives of the stream function of a certain potential flow around the body can be seen also in other cases such as sound radiation from a vortex filament negotiating the edge of a half plane (Howe 1975) .

Figures 6(a), (b) and (c) show time variations of the functions ψ, $d\psi/dt$ and $d^2\psi/dt^2$ for various values of R_o/a: 0.8, 1.0 and 1.2 respectively. Here the stream function ψ and the time t are normalized by a^2 and $4\pi a^2/\Gamma$. The last curve of each figure also represents the radiated sound normalized with $\rho_o\Gamma^3/(32\pi^2a^2c_or)$.

In figure 7, the radiated sound pressure is plotted against the axial position of the vortex for various values of R_o/a (= 0.8,1.0,1.2,1.4 and 1.6) with δ_o=0.3 R_o. With the same values of the parameters except δ_o(=0.1 R_o), the pressure curves are shown in figure 8. From these figures we can observe that the peaks of the pressures occur when the vortex reaches the end points (Z/a=-1, 1) and the centre of the sphere (Z/a=0). It can also be seen in these figures that with the same strengths the larger the vortex ring is, the weaker the radiated sound becomes. Further, by comparing these figures we can see that the vortex with the smaller core cross-section produces the weaker sound for the same radius of the ring and the same strength, when it passes over the centre of the sphere.

5. Conclusion

The generation of sound from an unsteady flow containing a rigid body has been considered by using Kambe and Minota formula for free space sound radiation with the image method. It has been found that the dipole component predominates in the sound field generated by a vortex ring passing a fixed sphere. Further, it is interesting to find that the dipole-like sound is related to the second time derivative of the stream function for a potential flow around the sphere. Such kind of phenomenon that the radiated sound is in some way related to the stream function of a hypothetical potential flow around the body can be found also in other cases.

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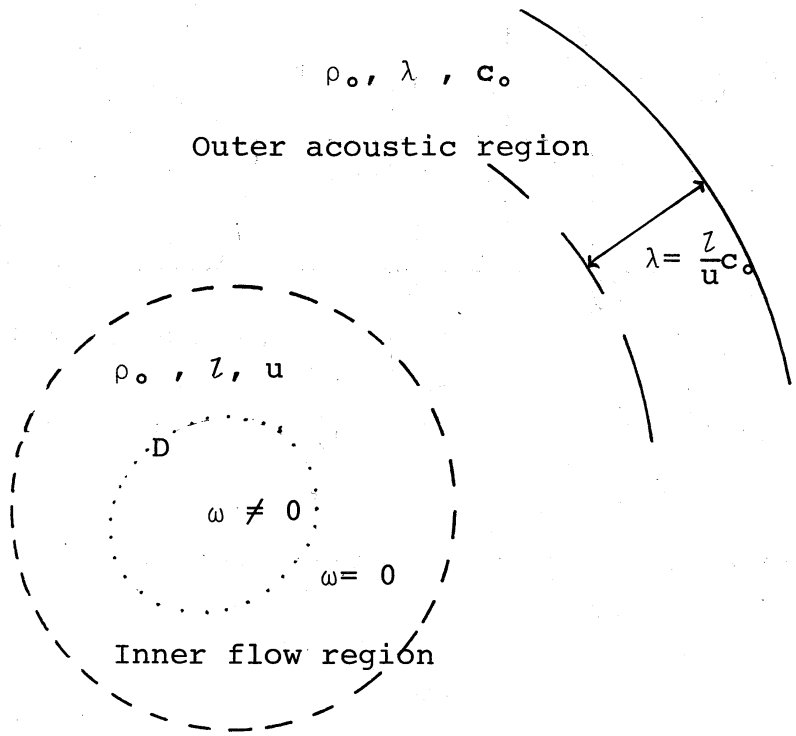


Fig. 1 Geometry of the flow and the radiated sound field.

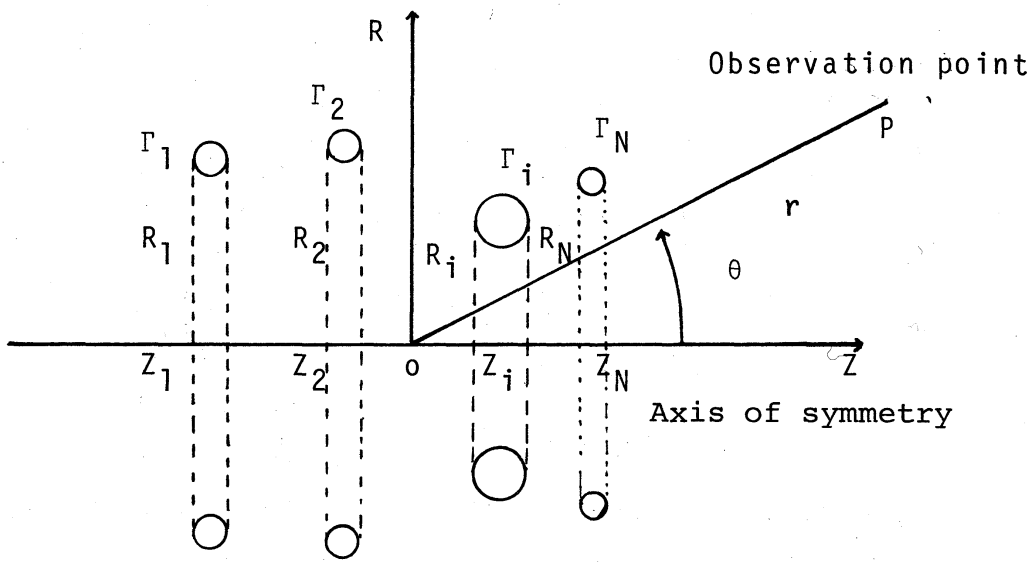


Fig.2 System of N vortex rings.

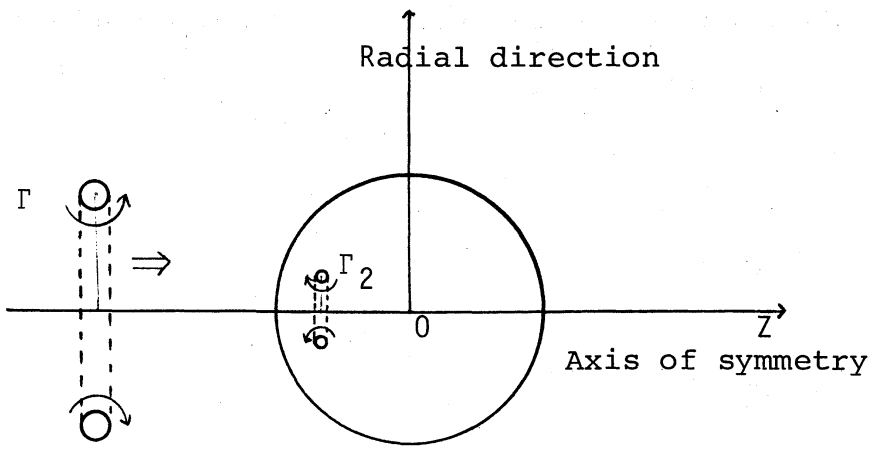


Fig.3 A vortex ring and its image in a fixed sphere .

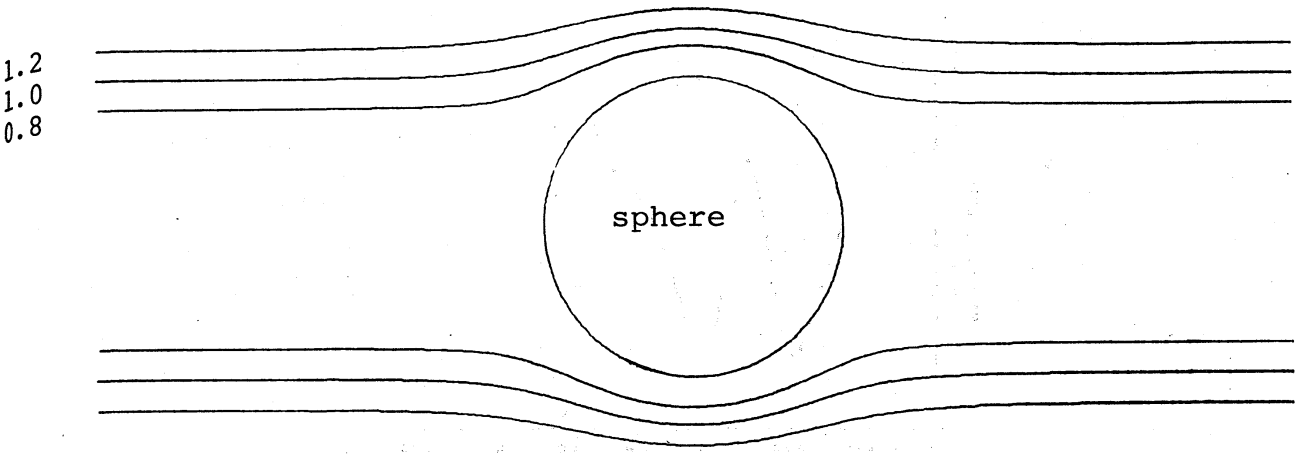


Fig.4 Path of core centre of the vortex ring for R_o/a =0.8, 1.0 and 1.2.

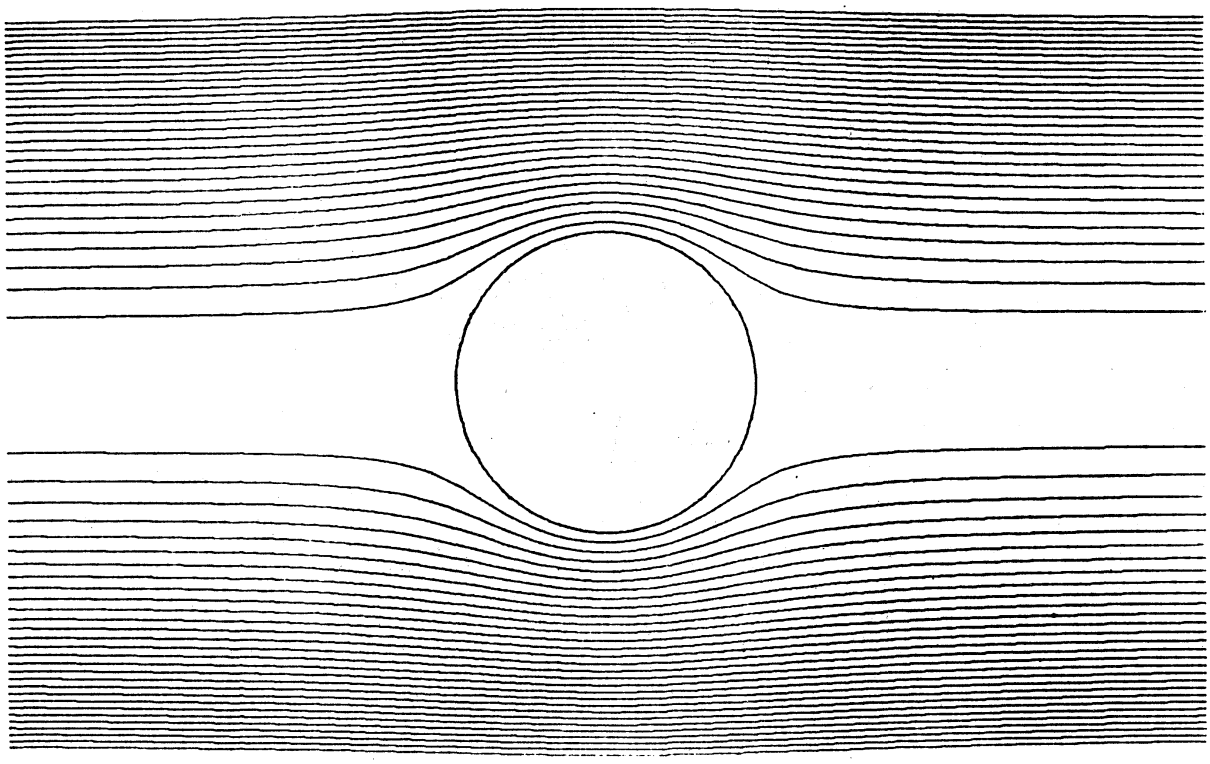
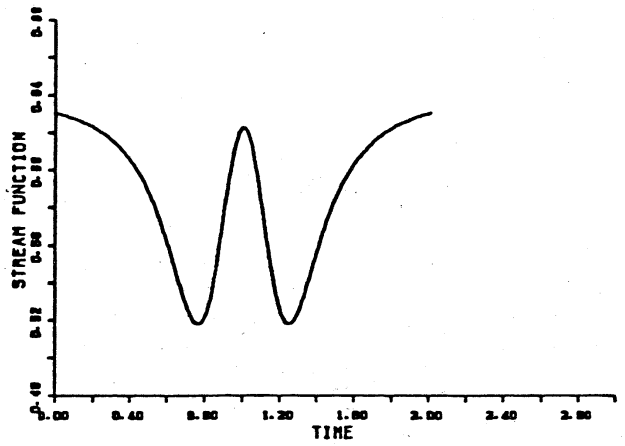


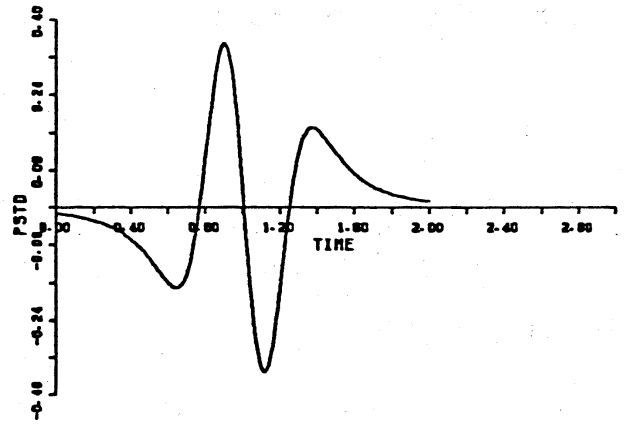
Fig.5 Streamlines of a potential flow around the sphere ;

$$\psi = (1/2)\ R^2\ [\ 1 - a^3/(z^2+R^2)^{3/2}] = \text{const.}$$

ψ

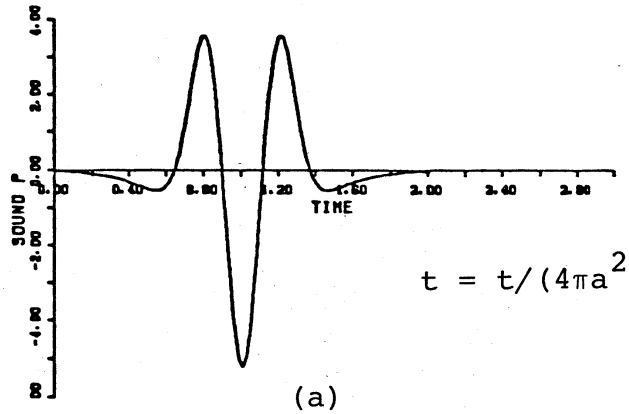


$\frac{d\psi}{dt}$



$$\tilde{p} = p \ / \ \{ \frac{\rho_0 \Gamma^3}{32\pi^2 a^2 c_0 r} \} .$$

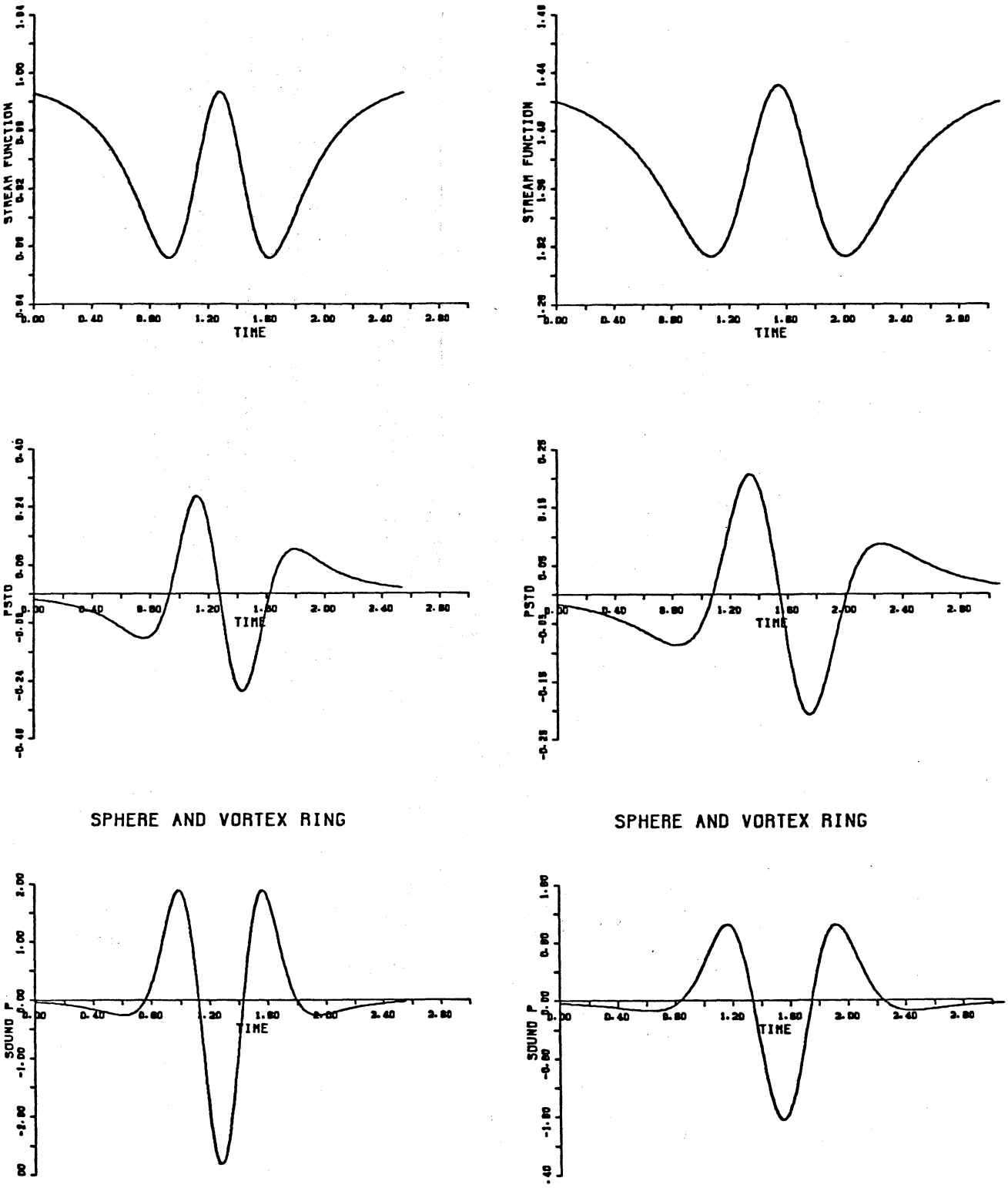
$\frac{d^2\psi}{dt^2}$



$t = t/(4\pi a^2/\Gamma)$

(a)

Fig.6 Time variations of ψ, $\frac{d\psi}{dt}$ and $\frac{d^2\psi}{dt^2}$ for (a) $R_o/a = 0.8$,(b) $R_o/a = 1.0$ and (c) $R_o/a = 1.2$.


STREAM FUNCTION
TIME
PSTD
TIME
SPHERE AND VORTEX RING
SOUND P
TIME
STREAM FUNCTION
TIME
PSTD
TIME
SPHERE AND VORTEX RING
SOUND P
TIME

(b) (c)

Fig.6

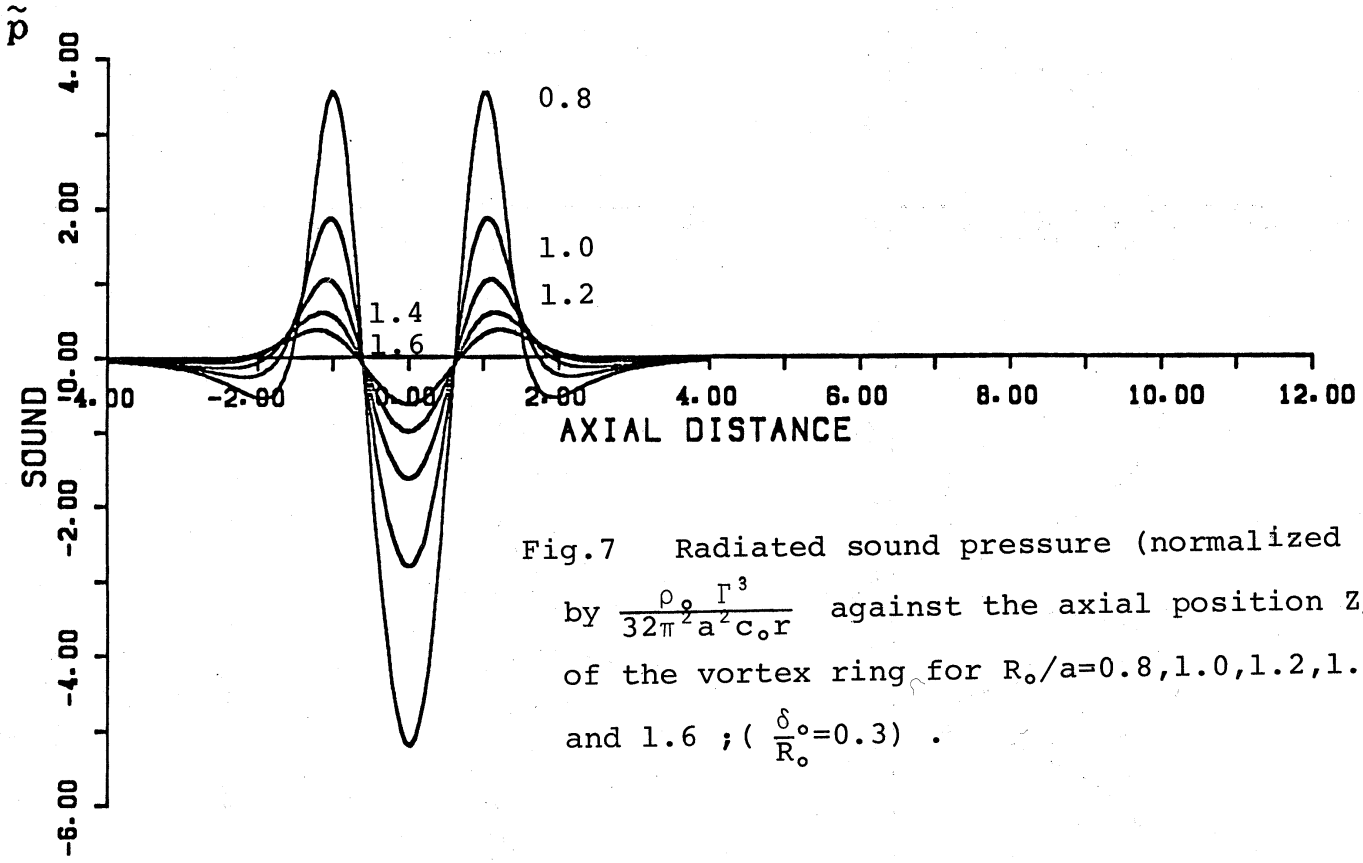


Fig.7 Radiated sound pressure (normalized by $\frac{\rho_o \Gamma^3}{32\pi^2 a^2 c_o r}$ against the axial position Z/a of the vortex ring for R_o/a=0.8,1.0,1.2,1.4 and 1.6 ;($\frac{\delta_o}{R_o}$=0.3) .

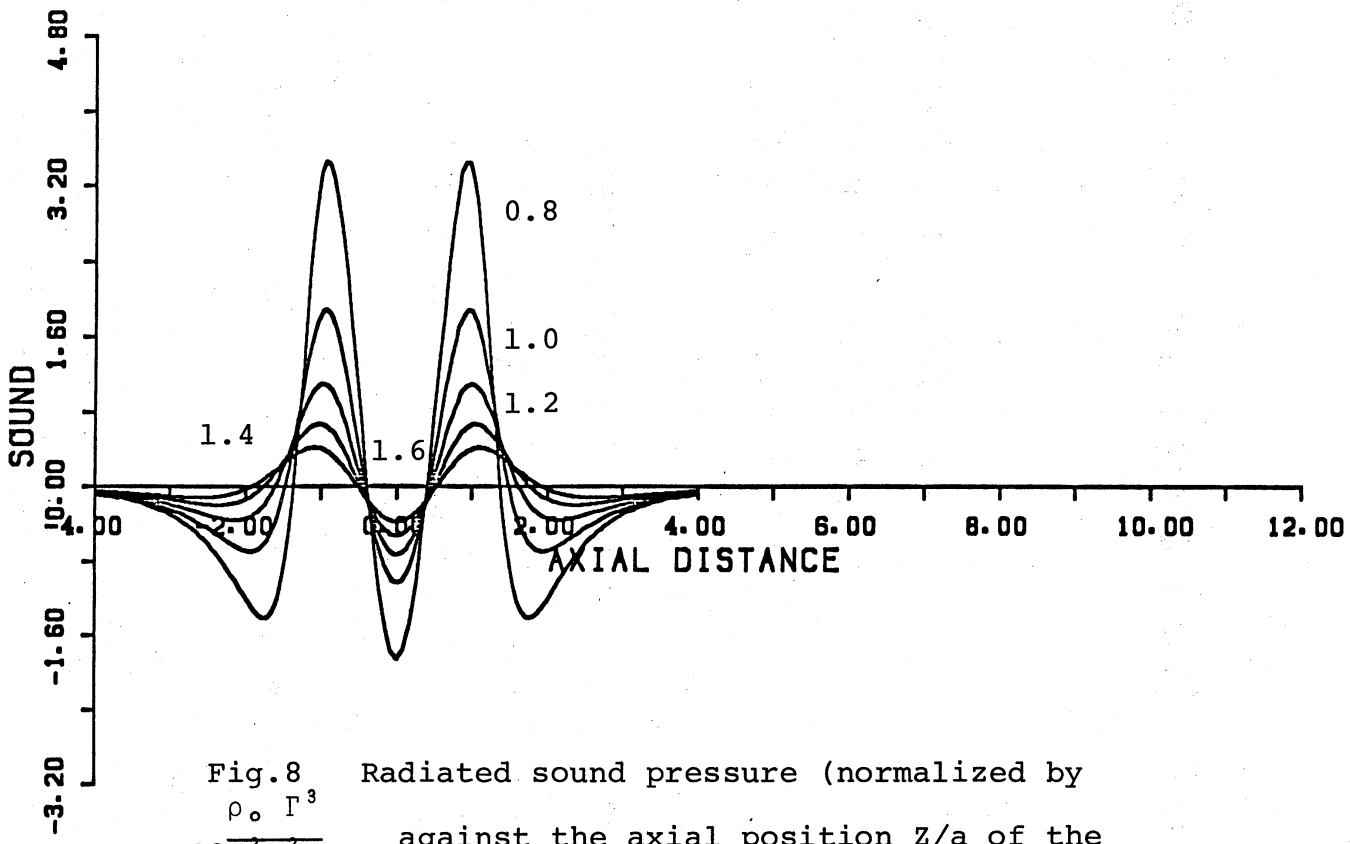


Fig.8 Radiated sound pressure (normalized by $\frac{\rho_o \Gamma^3}{32\pi^2 a^2 c_o r}$ against the axial position Z/a of the vortex ring for R_o/a=0.8,1.,1.2,1.4 and 1.6;($\frac{\delta_o}{R_o}$=o.1) .