数理解析研究所講究録
510 巻 1984 年 169-176



# Exact Solutions of Navier-Stokes Equation and Vorticity Dynamics

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## 1. INTRODUCTION

Motion of vorticity in a viscous incompressible fluid in three-dimensional space may be decomposed into three fundamental elements: convecton, stretching and viscous diffusion. The governing equation of the vorticity $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\boldsymbol{v} + \nu\nabla^2\boldsymbol{\omega} \tag{1}$$

where $\boldsymbol{\omega} = \mathrm{rot}\,\mathbf{v}$, $\mathbf{v}(\mathbf{x},t)$ is the incompressible velocity field ( $\mathrm{div}\,\mathbf{v}=0$) and $\nu$ the kinematic viscosity. The three elements of motion mentioned first correspond to the three terms except the first $\boldsymbol{\omega}_t$, respectively.

Here we consider motion of shear vorticity superimposed on a straining motion whose velocity $\mathbf{v}_s$ is represented by (ax, by, cz) in the Cartesian coordinate system (x,y,z). The form of the shear layer is assumed to be rectilinear (§2) or axisymmetric (§3). Although their geometrical forms are simple, it is remarkable that exact solutions of their motion are given for arbitrary initial profiles (Kambe 1983 a, b, c). These solutions for the initial value problem show cascade of the Fourier components to higher wave numbers.

In a particular case of uniform strain in which the parameters a, b and c are assumed to be constant and satisfy a particular relation, a steady state is approached from an arbitrary initial state. In the axisymmetric case this steady state corresponds to the steady solution found by Burgers (1948). The solution of a diffusing axisymmetric vortex given by Oseen (1911) without the external strain field is also included in the general solution given below.

Bellamy-Knights (1970) presented exact solutions of viscous vortex motion with the method of similarity solution which include also both of the Burgers' and Oseen's vortex as particular cases.

## 2. SOLUTION IN CARTESIAN COORDINATES

### 2.1 General expression of the solution

Consider a shear layer in a straining field (ax, by, cz) in Cartesian coordinate system, and suppose that the velocity $\mathbf{v} = (u,v,w)$ is given by $u = a(t)x$, $v = b(t)y + V(x,t)$ and $w = c(t)z$ where a, b and c are given functions of time t only. The continuity condition demands the relation

$$a(t) + b(t) + c(t) = 0.$$

The vorticity $\boldsymbol{\omega}$ has only z-component:

$$\boldsymbol{\omega} = (0,\ 0,\ \omega), \qquad \omega(x,t) = \partial V/\partial x.$$

The vorticity equation (1) reduces to

$$\omega_t + a x \omega_x = c\omega + \nu \omega_{xx} . \tag{2}$$

This represents the motion of the shear vorticity $\omega(x,t)$ under the combined action of viscous diffusion ($\nu\omega_{xx}$), stretching ($c\omega$) and convective straining ($ax\omega_x$). Using the new variables defined by

$$\left.\begin{aligned} &\xi = A(t)x, \qquad \tau(t) = \int_0^t A^2(t')dt', \qquad W = \frac{\omega}{C(t)}, \\ &A(t) = \exp\left[-\int_0^t a(t')dt'\right], \quad C(t) = \exp\left[+\int_0^t c(t')dt'\right], \end{aligned}\right\} \tag{3}$$

instead of x, t and $\omega$, we can transform the equation (2) into the diffusion equation (Kambe 1983 a,b),

$$W_\tau = \nu W_{\xi\xi} . \tag{4}$$

(Axisymmetric case is considered by Lundgren (1982)).

For a general initial condition of the form $\omega(t=0)=\omega_0(x)$, the solution of (4) is written as, using $\omega = C(t)W$,

$$\omega(x,t) = \frac{C(t)}{\sqrt{4\pi\nu\tau(t)}} \int_{-\infty}^{\infty} \omega_0(x') \exp\left[-\frac{(Ax-x')^2}{4\nu\tau}\right] dx', \tag{5}$$

where C(t) represents the effect of vortex stretching.

## 2.2 Steady straining flow

Suppose that the parameters a, b and c are constant. We assume further that a is negative and write

$$a = -\alpha = const\ (<0), \quad c = \gamma = const, \quad \alpha > 0.$$

Then we have

$$A = e^{\alpha t}, \quad C = e^{\gamma t}, \quad \tau = \frac{1}{2\alpha}(e^{2\alpha t} - 1). \tag{6}$$

Substituting these expressions into (5) and taking an asymptotic limit as t tends to infinity, we find the asymptotic expression,

$$\omega(x,t) = \frac{1}{\sqrt{2\pi}\,\ell} \exp\left((\gamma-\alpha)t - \frac{x^2}{2\ell^2}\right) \left[\int_{-\infty}^{\infty} \omega_0(x')dx' + \frac{x}{\ell^2} e^{-\alpha t} \int_{-\infty}^{\infty} x'\omega_0(x')dx' + O(e^{-2\alpha t})\right], \tag{7}$$

where

$$\ell = (\nu/\alpha)^{\frac{1}{2}} \tag{8}$$

has dimension of length.

Let us consider the special case of $\alpha = \gamma$ and $b = 0$. This is the case where the straining flow is in the plane perpendicular to the direction of the shear velocity (0,V,0). Then the leading term of the asymptotic expression (7) becomes independent of time t and the second correction term decays like $\exp[-\alpha t]$. Thus it is found that a single steady shear layer develops from an arbitrary initial distribution of vorticity $\omega_o(x)$. By using the initial net amount of vorticity

$$\Gamma = \int_{-\infty}^{\infty} \omega_o(x')\, dx' ,$$

the steady shear layer is represented by

$$\frac{\Gamma}{\sqrt{2\pi}\, l} \exp[-x^2/2l^2] . \tag{9}$$

However if the initial distribution is composed of same amount of opposite vorticities, the integral $\Gamma$ vanishes and the expression (7) takes the form

$$\frac{1}{2\sqrt{2\pi}\, l^3}\, e^{-\alpha t}\, x \exp[-x^2/2l^2] \int_{-\infty}^{\infty} x' \omega_o(x') dx' . \tag{10}$$

This shear layer disappears in due course of time like $\exp[-\alpha t]$. This is interpreted as cancellation of vorticities. The non-cancelling case (9) represents amalgamation of vorticity fluctuations. The solution of the form $\omega \propto \exp[-x^2/2l^2]$ like (9) is known as a steady solution satisfying the equaiton (2) without first term $\omega_t$ and with $-a = c = \text{const}$ ( $b = 0$ ) (Townsend (1951) and Batchelor (1967)).

## 2.3 Cascade of Fourier components

The Fourier spectrum of the vorticity (5) is given by

$$\hat{\omega}(k,t) = \frac{C(t)}{A(t)}\, \hat{\omega}_o\left(\frac{k}{A(t)}\right) \exp\left[-\frac{\nu k^2 \tau(t)}{A^2(t)}\right] , \tag{11}$$

where $\hat{\omega}_o(k_o)$ is the initial spectrum. This expression states that the straining field $v_s$ produces transfer of the initial spectrum component $\hat{\omega}_o(k_o)$ to a higher wave number $k = k_o A(t)$ with the magnitude of the component being diminished by the viscosity. The viscous cutoff wave number $k_d = A/(2\nu\tau)^{\frac{1}{2}}$, defined from the argument of the exponential function, is given by the asymptotic relation $k_d = 1/l$ as $t \to \infty$.

### 2.4 Two-dimensional problem

If c is set equal to zero (hence $C = 1$ and $a + b = 0$), the above expressions except (9) and (10) reduce to those of two-dimensional problem. (Kraichnan (1974) shows a solution of the same equation (2) with $c = 0$ governing the convection of a passive scalar.) Assuming further that a is a negative constant $-\alpha$ as above and using the relation $k/A(t) = k_o$ in (11), we find that the argument in the exponential function takes the form,

$$\nu \tau k_o^2 \approx \frac{\nu k_o^2}{2\alpha} e^{2\alpha t} = \frac{1}{2} \exp\left[2\alpha\left(t - \frac{1}{2\alpha} \ln R_k\right)\right], \tag{12}$$

where $R_k = \alpha/\nu k_o^2 = (k_d/k_o)^2$ is the Reynolds number of the eddy of length scale $1/k_o$. Thus it is found that the component of the scale $1/k_o$ has a critical time of $(\ln R_k)/2\alpha$ for the viscous cutoff, i.e. the critical time is proportional to $\ln R_k = 2 \ln (k_d/k_o)$.

## 3. SOLUTION IN AXISYMMETRY

### 3.1 General expression of the solution

We find a similar solution in an axisymmetric distribution. Suppose that the velocity is represented by

$$\boldsymbol{v} = (-\alpha r,\ v_\theta,\ 2\alpha z), \qquad v_\theta = v_\theta(r, t), \quad \alpha = \alpha(t), \tag{13}$$

in the cylindrical coordinate system $(r, \theta, z)$. This corresponds to assuming $a = b = -\alpha$ for the straining field in the previous section. The vorticity is given by

$$\boldsymbol{\omega} = \operatorname{rot} \boldsymbol{v} = (0, 0, \omega), \quad \omega = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta).$$

Then the vorticity equation (1) reduces to

$$\omega_t - \alpha r \omega_r = 2\alpha\omega + \nu \frac{1}{r}(r\omega_r)_r . \tag{14}$$

Defining the new variables,

$$\rho = A(t)\, r, \qquad \tau = \int_0^t A^2(t')\,dt' ,$$
$$W = \omega / A^2(t), \qquad A(t) = \exp\left[\int_0^t \alpha(t')\,dt'\right],$$

as before, we can transform the equation (14) into

$$W_\tau = \nu \frac{1}{\rho}(\rho W_\rho)_\rho \ , = \nu (W_{\xi\xi} + W_{\eta\eta}), \tag{15}$$

where $\xi = Ax$ and $\eta = Ay$ are the Cartesian coordinates. This is the diffusion equation of axisymmetry in two-dimensional space.

For an arbitrary axisymmetric initial condition of the form

$$W\Big|_{\tau=0} = \omega\Big|_{t=0} = \omega_0(r),$$

the solution is written as

$$\omega(r,t) = \frac{A^2(t)}{4\pi\nu\tau}\iint \omega_0\left(\sqrt{\xi_1^2+\eta_1^2}\right)\exp\left[-\frac{(\xi-\xi_1)^2+(\eta-\eta_1)^2}{4\nu\tau}\right]d\xi_1 d\eta_1 . \tag{16}$$

3.2 Initial condition of the form of δ-function

For the initial condition of a vortex filament,

$$\omega_0 = \Gamma\, \delta(x)\, \delta(y) ,$$

we find the solution:

$$\omega(r,t) = \frac{\Gamma A^2(t)}{4\pi\nu\tau} \exp\left[-\frac{\rho^2}{4\nu\tau}\right]. \tag{17}$$

(i) $\alpha = 0$ ( no external straining). In this case we have $A = 1$, $\tau = t$ and $\rho = r$. The solution (17) is reduced to the expression of the diffusing vortex filament,

$$\omega(r,t)=\frac{\Gamma}{4\pi\nu t}\exp\left[-\frac{r^2}{4\nu t}\right]. \qquad (18)$$

This was first given by Oseen (1911).

(ii) $\alpha$ = const ( > 0, constant straining). Substituting the relations,

$$A=e^{\alpha t},\quad \rho=re^{\alpha t},\quad \tau=\frac{1}{2\alpha}\left(e^{2\alpha t}-1\right), \qquad (19)$$

we find the solution

$$\omega(r,t)=\frac{\alpha\Gamma}{2\pi\nu(1-e^{-2\alpha t})}\exp\left[-\frac{\alpha}{2\nu}\frac{r^2}{(1-e^{-2\alpha t})}\right]$$

$$\rightarrow \omega_{st}=\frac{\Gamma}{\pi l_\alpha^2}\exp\left[-r^2/l_\alpha^2\right],\quad (t\rightarrow\infty)$$

where

$$l_\alpha=(2\nu/\alpha)^{\frac{1}{2}}.$$

It is found that a steady state is approached in the limit $t\rightarrow\infty$. The steady solution $\omega_{st}$ can be found directly from the equation (14) with putting $\partial\omega/\partial t=0$. For the equation (14) can then be integrated once as

$$\frac{\alpha}{\nu}r^2\omega+r\frac{d\omega}{dr}=\mathrm{const}$$

This leads to the same expression as $\omega_{st}$, which was first shown by Burgers (1948) (see also Batchelor (1967) ) with putting the const to zero.

### 3.3 Arbitrary initial profile $\omega_0(x)$

Assuming $\alpha$= const ( > 0) and substituting the expressions (19) into (16), we find

$$\omega(r,t)=\frac{\alpha}{2\pi\nu(1-e^{-2\alpha t})}\iint\omega_0\left(\sqrt{\xi_1^2+\eta_1^2}\right)\exp\left[-\frac{(x-\xi_1e^{-\alpha t})^2+(y-\eta_1e^{-\alpha t})^2}{\frac{2\nu}{\alpha}(1-e^{-2\alpha t})^2}\right]d\xi_1\,d\eta_1.$$

This tends to the asymptotic expression, as $t\rightarrow\infty$,

$$\omega(r,t)=\frac{1}{\pi l_\alpha^2}e^{-r^2/l_\alpha^2}\left[\int_0^\infty\omega_0(\zeta)\,2\pi\zeta\,d\zeta\right.$$

$$+e^{-2\alpha t}\frac{2\pi}{l_a^4}\int_0^\infty \omega_0(\zeta)\left[(r^2-l_a^2)\zeta^2-r^2l_a^2\right]\zeta\,d\zeta+O(e^{-4\alpha t})\Big].$$

Therefore if

$$\Gamma=\int_0^\infty \omega_0(\zeta)\,2\pi\zeta\,d\zeta \neq 0,$$

then we have

$$\lim_{t\to\infty}\omega(r,t)=\frac{1}{\pi l_a^2}\Gamma\, e^{-r^2/l_a^2}.$$

Thus it is found that a steady vortex of an effective core size $l_a$ is formed in the final state. This is the Burgers' vortex $\omega_{st}$. However if $\Gamma = 0$, the vortex disappears exponentially like $\exp(-2\alpha t)$.

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