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Specialized Characters and Power Series Identities

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1. Introduction —— Weyl-Kac character formula:

Let me start my talk with a skip review on representation theory of Kac-Moody Lie algebras. Details would be available on many good references [3][6][7][10][19][20] and so on.

A realization of an $n \times n$-generalized Cartan matrix (= GCM) $A = (a_{ij})_{i,j=1,\ldots,n}$ is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ such that

i) $\mathfrak{h}$ is an (n + corank A)-dimensional complex vector space,

ii) $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ (resp. $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$) is a linearly independent subset in $\mathfrak{h}^*$ (resp. $\mathfrak{h}$),

iii) $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ for every i and j.

The corresponding Lie algebra $\mathfrak{g}(A)$ is generated by $\mathfrak{h}$, $e_1, \ldots, e_n$, $f_1, \ldots, f_n$ and satisfies

i) $[e_i, f_j] = \delta_{ij} \alpha_i^\vee$

ii) $[h, e_i] = \alpha_i(h)e_i$ and $[h, f_i] = -\alpha_i(h)f_i$

for every $h \in \mathfrak{h}$ and $i = 1,\ldots,n$,

iii) $[\mathfrak{h}, \mathfrak{h}] = \{0\}$,

iv) $\mathfrak{g}(A)$ contains no proper ideals which intersect trivially with $\mathfrak{h}$.

Let Z, N or N_0 denote the set of all, all positive or all non-negative integers respectively, and we set

$W = W(A) =$ the Weyl group of $(\mathfrak{g}(A), \mathfrak{h})$,

$\Delta = \Delta(A) = \{\text{non-zero roots}\}$,

$\Delta_+ = \Delta_+(A) = \{\text{positive roots}\}$,

$P = P(A) = \{\text{integral forms on } \mathfrak{h}\}$

$= \{\lambda \in \mathfrak{h}^* \,;\, \langle \alpha_i^\vee, \lambda \rangle \in Z \text{ for every } i\}$,

$P_+ = P_+(A) = \{\text{dominant integral forms on } \mathfrak{h}\}$

$= \{\lambda \in \mathfrak{h}^* \,;\, \langle \alpha_i^\vee, \lambda \rangle \in N_0 \text{ for every } i\}$.

It is known that for each $\Lambda \in P_+$ there exists a unique irreducible integrable highest weight module $L(\Lambda)$ with highest weight Λ and that the character ch $L(\Lambda)$ is given by the celebrated Weyl-Kac character formula:

Proposition 1.1. Assume that A is symmetrizable and $\Lambda \in P_+$, then

$$\mathrm{ch}\, L(\Lambda) = \frac{\sum_{w \in W} (\mathrm{sgn}\, w)\, e^{w(\Lambda+\rho)-\rho}}{\prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\mathrm{mult}\,\alpha}}$$

where $\rho \in \mathfrak{h}^*$ is chosen such that $\langle \alpha_i^\vee, \rho \rangle = 1$ for every i.

The denominator formula

$$\sum_{w \in W} (\mathrm{sgn}\, w)\, e^{w\rho - \rho} = \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha})^{\mathrm{mult}\,\alpha}$$

follows immediately with the special choice of $\Lambda = 0$.

For an n-tuple $s = (s_1, \ldots, s_n)$ of positive integers, the specialization of type s is the ring homomorphism

$$\begin{array}{ccc} F_s : \mathbb{C}[[e^{-\alpha_1}, \ldots, e^{-\alpha_n}]] & \longrightarrow & \mathbb{C}[[q]] \quad , \\ \cup\!\!\!\!| & & \cup\!\!\!\!| \\ e^{-\alpha_i} & \longmapsto & q^{s_i} \end{array}$$

and

$$d_\Lambda^s(q) = d_\Lambda^{(s;\ A)}(q) = F_s(e^{-\Lambda}\, \mathrm{ch}\, L(\Lambda))$$

is called the specialized character of $L(\Lambda)$ of type s. We set

$$Q(s;\ A) = F_s\Big(\prod_{\alpha \in \Delta_+(A)} (1 - e^{-\alpha})^{\mathrm{mult}\,\alpha} \Big) .$$

Especially in case of $s = 1 = (1, \ldots, 1)$, $d_\Lambda(q) = d_\Lambda^1(q)$ is called the principally specialized character, whose beautiful expression was found by Lepowsky:

Proposition 1.2. Assume that A is symmetrizable and $\Lambda \in P_+$, then

$$d_\Lambda(q) = \frac{Q(s_\Lambda + 1;\ {}^tA)}{Q(1;\ A)}$$

where $s_\Lambda = (\langle \alpha_1^\vee, \Lambda \rangle, \ldots, \langle \alpha_n^\vee, \Lambda \rangle)$.

2. Specialized characters:

The problem we shall concerned with now is to compute specialized characters $d_{\Lambda}^{s}(q)$ for arbitrary specialization s. It seems very difficult to obtain similar general results as in Proposition 1.2 for arbitrary s. However, for some Λ and for some s ($\neq$ 1), we can compute $d_{\Lambda}^{s}(q)$ by using the following two theorems (cf. [22]):

Theorem 2.1. Let $A = (a_{ij})$ be a symmetrizable GCM of degree n, $s = (s_1, \ldots, s_n)$ an n-tuple of positive integers such that $\tilde{A} = (s_i a_{ij} s_j^{-1})$ is a GCM. Then, for every $\Lambda \in P_+$,

$$d_{\Lambda}^{s}(q) = \frac{Q(s_1(\langle \alpha_1^{\vee}, \Lambda \rangle + 1), \ldots, s_n(\langle \alpha_n^{\vee}, \Lambda \rangle + 1);\ {}^t\tilde{A})}{Q(s;\ A)} .$$

Theorem 2.2. Assume that A is a symmetrizable GCM and $\Lambda \in P_+(A)$ and $\Lambda' \in P_+({}^tA)$. Then

$$d_{\Lambda}^{(s_{\Lambda'}+1;\ A)}(q)\ Q(s_{\Lambda'}+1;\ A)$$

$$= d_{\Lambda'}^{(s_{\Lambda}+1;\ {}^tA)}(q)\ Q(s_{\Lambda}+1;\ {}^tA) .$$

Table 1 is a list of specialized characters of affine Lie algebras obtained from the above theorems, where we shall use following notations:

i) $s = (s_0, \ldots, s_{\ell})$ is an (ℓ+1)-tuple of positive integers,

ii) $\Lambda = (n_0, \ldots, n_{\ell})$ is a dominant integral form such that $\langle \alpha_i^\vee, \Lambda \rangle = n_i$ $(i=0,\dots,\ell)$,

iii) n is a positive integer,

iv) $\varphi(q) = \prod_{j=1}^{\infty} (1 - q^j)$.

Table 1

1) $A_1^{(1)}$:

$$d_\Lambda^{(12)}(q) = \frac{Q(n_0+1,\ 2(n_1+1);\ A_2^{(2)})}{\varphi(q)}$$

$$d_\Lambda^{(21)}(q) = \frac{Q(2(n_0+1),\ n_1+1;\ A_2^{(2)})}{\varphi(q)}$$

$$d^s_{(n-1,2n-1)}(q) = \frac{Q(ns_0,\ 2ns_1;\ A_2^{(2)})}{Q(s;\ A_1^{(1)})}$$

$$d^s_{(2n-1,n-1)}(q) = \frac{Q(ns_1,\ 2ns_0;\ A_2^{(2)})}{Q(s;\ A_1^{(1)})}$$

2) $A_2^{(2)}$:

$$d_\Lambda^{(12)}(q) = \frac{Q(n_0+1,\ 2(n_1+1);\ A_1^{(1)})}{\varphi(q)}$$

$$d_\Lambda^{(14)}(q) = \frac{Q(n_0+1,\ 4(n_1+1);\ A_2^{(2)})}{Q(1,4;\ A_2^{(2)})}$$

$$= \frac{\varphi(q^2)\varphi(q^3)\varphi(q^{12})\ Q(n_0+1,\ 4(n_1+1);\ A_2^{(2)})}{\varphi(q)\varphi(q^4)\varphi(q^6)^2}$$

$$d^{s}_{(2n-1,n-1)}(q) = \frac{Q(2ns_0,\ ns_1;\ A_1^{(1)})}{Q(\ s;\ A_2^{(2)})}$$

$$d^{s}_{(4n-1,n-1)}(q) = \frac{Q(ns_1,\ 4ns_0;\ A_2^{(2)})}{Q(s;\ A_2^{(2)})}$$

3) $B_{\ell}^{(1)}$:

$$d_{\Lambda}^{(2\ldots21)}(q) = \frac{Q(2(n_0+1),\ldots,2(n_{\ell-1}+1),n_{\ell}+1;\ B_{\ell}^{(1)})}{Q(2\ldots21;\ B_{\ell}^{(1)})}$$

$$d^{s}_{(n-1,\ldots,n-1,2n-1)}(q) = \frac{Q(ns_0,\ldots,ns_{\ell-1},2ns_{\ell};\ A_{2\ell-1}^{(2)})}{Q(s;\ B_{\ell}^{(1)})}$$

4) $C_{\ell}^{(1)}$:

$$d_{\Lambda}^{(21\ldots1)}(q) = \frac{Q(n_{\ell}+1,\ldots,n_1+1,2(n_0+1);\ A_{2\ell}^{(2)})}{\varphi(q)^{\ell}}$$

$$d_{\Lambda}^{(1\ldots12)}(q) = \frac{Q(n_0+1,\ldots,n_{\ell-1}+1,2(n_{\ell}+1);\ A_{2\ell}^{(2)})}{\varphi(q)^{\ell}}$$

$$d_{\Lambda}^{(21\ldots12)}(q) = \frac{Q(2(n_0+1),n_1+1,\ldots,n_{\ell-1}+1,2(n_{\ell}+1);\ C_{\ell}^{(1)})}{Q(21\ldots12;\ C_{\ell}^{(1)})}$$

$$d^{s}_{(2n-1,\ldots,2n-1,n-1)}(q) = \frac{Q(ns_{\ell},2ns_{\ell-1},\ldots,2ns_0;\ A_{2\ell}^{(2)})}{Q(s;\ C_{\ell}^{(1)})}$$

$$d^{s}_{(n-1,2n-1,\ldots,2n-1)}(q) = \frac{Q(ns_0, 2ns_1, \ldots, 2ns_\ell \;;\; A^{(2)}_{2\ell})}{Q(s;\; C^{(1)}_\ell)}$$

$$d^{s}_{(n-1,2n-1,\ldots,2n-1,n-1)}(q)$$

$$= \frac{Q(ns_0, 2ns_1, \ldots, 2ns_{\ell-1}, ns_\ell \;;\; D^{(2)}_{\ell+1})}{Q(s;\; C^{(1)}_\ell)}$$

5) $F^{(1)}_4$:

$$d^{(22211)}(q) = \frac{Q(2(n_0+1), 2(n_1+1), 2(n_2+1), n_3+1, n_4+1;\; F^{(1)}_4)}{Q(22211;\; F^{(1)}_4)}$$

$$d^{s}_{(n-1,n-1,n-1,2n-1,2n-1)}(q)$$

$$= \frac{Q(ns_0, ns_1, ns_2, 2ns_3, 2ns_4;\; E^{(2)}_6)}{Q(s;\; F^{(1)}_4)}$$

6) $G^{(1)}_2$:

$$d^{(331)}(q) = \frac{Q(3(n_0+1), 3(n_1+1), n_2+1;\; G^{(1)}_2)}{Q(331;\; G^{(1)}_2)}$$

$$= \frac{\varphi(q^2)\,\varphi(q^{12})\; Q(3(n_0+1), 3(n_1+1), n_2+1;\; G^{(1)}_2)}{\varphi(q)\,\varphi(q^3)\,\varphi(q^4)\,\varphi(q^6)}$$

$$d^{s}_{(n-1,n-1,3n-1)}(q) = \frac{Q(ns_0, ns_1, 3ns_2;\; D^{(3)}_4)}{Q(s;\; G^{(1)}_2)}$$

7) $A^{(2)}_{2\ell}$:

$$d^{(12\ldots2)}_{\Lambda}(q) = \frac{Q(n_0+1, 2(n_1+1), \ldots, 2(n_\ell+1);\; D^{(2)}_{\ell+1})}{Q(12\ldots2;\; A^{(2)}_{2\ell})}$$

$$d_{\Lambda}^{(1\ldots12)}(q) = \frac{Q(n_0+1,\ldots,n_{\ell-1}+1,2(n_\ell+1);\ C_\ell^{(1)})}{\varphi(q)^\ell}$$

$$d_{\Lambda}^{(12\ldots24)}(q) = \frac{Q(n_0+1,2(n_1+1),\ldots,2(n_{\ell-1}+1),4(n_\ell+1);\ A_{2\ell}^{(2)})}{Q(12\ldots24;\ A_{2\ell}^{(2)})}$$

$$d^s_{(2n-1,n-1,\ldots,n-1)}(q) = \frac{Q(ns_\ell,\ldots,ns_1,2ns_0;\ C_\ell^{(1)})}{Q(s;\ A_{2\ell}^{(2)})}$$

$$d^s_{(2n-1,\ldots,2n-1,n-1)}(q) = \frac{Q(ns_\ell,2ns_{\ell-1},\ldots,2ns_0;\ D_{\ell+1}^{(2)})}{Q(s;\ A_{2\ell}^{(2)})}$$

$$d^s_{(4n-1,2n-1,\ldots,2n-1,n-1)}(q) = \frac{Q(ns_\ell,2ns_{\ell-1},\ldots,2ns_1,4ns_0;\ A_{2\ell}^{(2)})}{Q(s;\ A_{2\ell}^{(2)})}$$

8) $A_{2\ell-1}^{(2)}$:

$$d_{\Lambda}^{(1\ldots12)}(q) = \frac{Q(n_0+1,\ldots,n_{\ell-1}+1,2(n_\ell+1);\ A_{2\ell-1}^{(2)})}{Q(1\ldots12;\ A_{2\ell-1}^{(2)})}$$

$$d^s_{(2n-1,\ldots,2n-1,n-1)}(q) = \frac{Q(2ns_0,\ldots,2ns_{\ell-1},ns_\ell\ ;\ B_\ell^{(1)})}{Q(s;\ A_{2\ell-1}^{(2)})}$$

9) $D_{\ell+1}^{(2)}$:

$$d_{\Lambda}^{(12\ldots2)}(q) = \frac{Q(n_0+1,2(n_1+1),\ldots,2(n_\ell+1);\ A_{2\ell}^{(2)})}{Q(12\ldots2;\ D_{\ell+1}^{(2)})}$$

$$d_{\Lambda}^{(2\ldots21)}(q) = \frac{Q(n_\ell+1,2(n_{\ell-1}+1),\ldots,2(n_0+1);\ A_{2\ell}^{(2)})}{Q(2\ldots21;\ D_{\ell+1}^{(2)})}$$

$$d_{\Lambda}^{(12\ldots21)}(q) = \frac{Q(n_0+1,2(n_1+1),\ldots,2(n_{\ell-1}+1),n_{\ell}+1;\ D_{\ell+1}^{(2)})}{Q(12\ldots21;\ D_{\ell+1}^{(2)})}$$

$$d^{s}_{(2n-1,n-1,\ldots,n-1)}(q) = \frac{Q(ns_{\ell},\ldots,ns_1,2ns_0;\ A_{2\ell}^{(2)})}{Q(s;\ D_{\ell+1}^{(2)})}$$

$$d^{s}_{(n-1,\ldots,n-1,2n-1)}(q) = \frac{Q(ns_0,\ldots,ns_{\ell-1},2ns_{\ell}\ ;\ A_{2\ell}^{(2)})}{Q(s;\ D_{\ell+1}^{(2)})}$$

$$d^{s}_{(2n-1,n-1,\ldots,n-1,2n-1)}(q)$$
$$= \frac{Q(2ns_0,ns_1,\ldots,ns_{\ell-1},2ns_{\ell}\ ;\ C_{\ell}^{(1)})}{Q(s;\ D_{\ell+1}^{(2)})}$$

10) $E_6^{(2)}$:

$$d_{\Lambda}^{(11122)}(q) = \frac{Q(n_0+1,n_1+1,n_2+1,2(n_3+1),2(n_4+1);\ E_6^{(2)})}{Q(11122;\ E_6^{(2)})}$$

$$d^{s}_{(2n-1,2n-1,2n-1,n-1,n-1)}(q)$$
$$= \frac{Q(2ns_0,2ns_1,2ns_2,ns_3,ns_4;\ F_4^{(1)})}{Q(s;\ E_6^{(2)})}$$

11) $D_4^{(3)}$:

$$d_{\Lambda}^{(113)}(q) = \frac{Q(n_0+1,n_1+1,3(n_2+1);\ D_4^{(3)})}{Q(113;\ D_4^{(3)})}$$
$$= \frac{\varphi(q^2)\,\varphi(q^3)\,\varphi(q^{18})\ Q(n_0+1,n_1+1,3(n_2+1);\ D_4^{(3)})}{\varphi(q)^2\,\varphi(q^6)^2\,\varphi(q^9)}$$

$$d^{s}_{(3n-1,3n-1,n-1)}(q) = \frac{Q(3ns_0,3ns_1,ns_2;\ G_2^{(1)})}{Q(s;\ D_4^{(3)})}$$

We want to note that there is a class of specialized characters which can be computed by making use of suitable inclusions of affine Lie algebras; some of them are shown in Table 2, where we shall use the following notations:

$$\prod_{j}^{(+)}(1-q^{m(j)}) = \prod_{\substack{j\in Z \\ m(j)>0}} (1-q^{m(j)})$$

$$\prod_{j}^{(+)}[(1-q^{m_1(j)})\dots(1-q^{m_r(j)})] = \prod_{i=1}^{r}[\prod_{j}^{(+)}(1-q^{m_i(j)})].$$

<u>Table 2</u>

1) $A_2^{(1)}$:

i) $\Lambda = (n_0-1, n_1-1, n_1-1)$;

$$d_{\Lambda}^{(112)}(q) = \frac{\varphi(q^{|n|})\,\varphi(q^{3|n|})}{\varphi(q)^2} \prod_{j}^{(+)}[(1-q^{j|n|\pm n_0})(1-q^{j|n|\pm n_1}) \times (1-q^{3j|n|\pm 3n_1})]$$

where $|n| = n_0+2n_1$.

ii) $\Lambda = (2n-1, n-1, n-1)$ and $s = (s_0, s_0, s_2)$;

$$d_{\Lambda}^{s}(q) = \frac{\varphi(q^{n|s|})\,\varphi(q^{3n|s|})}{\varphi(q^{|s|})^2}$$

$$\times \prod_{j}^{(+)} \frac{(1-q^{n(j|s|\pm s_0)})(1-q^{n(j|s|\pm s_2)})(1-q^{3n(j|s|\pm s_0)})}{(1-q^{j|s|\pm s_0})^2(1-q^{j|s|\pm s_2})}$$

where $|s| = 2s_0+s_2$.

2) $A_3^{(1)}$:

i) $\Lambda = (2n-1, n-1, n-1, n-1)$;

$$d_{\Lambda}^{(1122)}(q) = \frac{\varphi(q^n)\varphi(q^{2n})\varphi(q^{8n})}{\varphi(q)^2\varphi(q^2)}$$

$$\times \prod_{j}^{(+)} [(1-q^{n(8j\pm3)})(1-q^{n(16j\pm2)})],$$

$$d_{\Lambda}^{(2211)}(q) = \frac{\varphi(q^n)\varphi(q^{2n})\varphi(q^{8n})}{\varphi(q)^2\varphi(q^2)}$$

$$\times \prod_{j}^{(+)} [(1-q^{n(8j\pm1)})(1-q^{n(16j\pm6)})],$$

$$d_{\Lambda}^{s}(q) = \frac{\varphi(q^{n|s|})\varphi(q^{2n|s|})^2}{\varphi(q^{|s|})^3} \prod_{i=1,3} \prod_{j}^{(+)} \frac{(1-q^{n(j|s|\pm s_i)})}{(1-q^{j|s|\pm s_i})}$$

$$\times \prod_{k=1}^{2} \prod_{j}^{(+)} \frac{(1-q^{kn(j|s|\pm s_0)})(1-q^{kn(j|s|\pm(s_0+s_1))})}{(1-q^{j|s|\pm s_0})(1-q^{j|s|\pm(s_0+s_1)})}$$

where $s = (s_0, s_1, s_0, s_3)$ and $|s| = 2s_0+s_1+s_3$.

ii) $\Lambda = (2n-1, 2n-1, n-1, n-1)$;

$$d_{\Lambda}^{(1112)}(q) = \frac{\varphi(q^n)\varphi(q^{2n})\varphi(q^{8n})}{\varphi(q)^3}$$

$$\times \prod_{j}^{(+)} [(1-q^{n(8j\pm3)})(1-q^{n(16j\pm2)})],$$

$$d_{\Lambda}^{(2111)}(q) = \frac{\varphi(q^n)\varphi(q^{2n})\varphi(q^{8n})}{\varphi(q)^3}$$

$$\times \prod_{j}^{(+)} [(1-q^{n(8j\pm1)})(1-q^{n(16j\pm6)})].$$

iii) $\Lambda = (n_0-1, n_0-1, n_2-1, n_2-1)$;

$$d_{\Lambda}^{(1212)}(q) = d_{\Lambda}^{(2121)}(q) = \frac{\varphi(q^6)\varphi(q^{2|n|})^2\varphi(q^{3|n|})}{\varphi(q)^2\varphi(q^3)^2}$$

$$\times \prod_{k=1}^{3} \prod_{j}^{(+)} (1-q^{k(j|n|\pm n_0)})$$

where $|n| = n_0+n_2$.

iv) $\Lambda = (n_0-1, n_1-1, n_0-1, n_3-1)$;

$$d_{\Lambda}^{(2111)}(q) = \frac{\varphi(q^{|n|})\varphi(q^{2|n|})^2}{\varphi(q)^3} \left[\prod_{i=1,3} \prod_{j}^{(+)} (1-q^{j|n|\pm n_i}) \right]$$

$$\times \prod_{k=1}^{2} \prod_{j}^{(+)} [(1-q^{k(j|n|\pm n_0)})(1-q^{k(j|n|\pm(n_0+n_1))})]$$

where $|n| = 2n_0+n_1+n_3$.

v) $\Lambda = (2n-1, n-1, 2n-1, n-1)$ and $s = (s_0, s_0, s_2, s_2)$;

$$d_{\Lambda}^{s}(q) = \frac{\varphi(q^{2n|s|})^2\varphi(q^{3n|s|})}{\varphi(q^{|s|})^2\varphi(q^{2|s|})}$$

$$\times \prod_{k=1}^{3} \prod_{j}^{(+)} (1-q^{kn(j|s|\pm s_0)})$$

$$\times \prod_{j}^{(+)} [(1-q^{j|s|\pm s_0})^2(1-q^{2j|s|\pm 2s_0})]$$

where $|s| = s_0+s_2$.

3) $A_5^{(1)}$:

$$d_{\Lambda_0}^{(111122)}(q) = \frac{\varphi(q^2)\varphi(q^{12})^2}{\varphi(q)\varphi(q^6)\varphi(q^8)}$$

$$d_{\Lambda_0}^{(222211)}(q) = \frac{\varphi(q^6)^2\varphi(q^{12})}{\varphi(q^2)\varphi(q^3)}$$

$$d^{(112112)}_{\Lambda_0}(q) = \frac{\varphi(q^2)\varphi(q^3)\varphi(q^8)\varphi(q^{12})}{\varphi(q)\varphi(q^4)^2\varphi(q^6)}$$

$$d^{(211211)}_{\Lambda_0}(q) = \frac{\varphi(q^6)^5\varphi(q^8)}{\varphi(q^2)\varphi(q^3)^2\varphi(q^4)\varphi(q^{12})}$$

$$d^{(112222)}_{\Lambda_0}(q) = \frac{\varphi(q^{12})}{\varphi(q^2)}\prod_j^{(+)}(1+q^{12j\pm 1})$$

$$d^{(221122)}_{\Lambda_0}(q) = \frac{\varphi(q^{12})}{\varphi(q^2)}\prod_j^{(+)}(1+q^{12j\pm 5})$$

4) $A_7^{(1)}$:

For $s = (s_0,s_1,s_0,s_3,s_0,s_1,s_0,s_3)$,

$$d^{s}_{\Lambda_0}(q) = \frac{\varphi(q^{2|s|})^3}{\varphi(q^{|s|})^3}\prod_j^{(+)}[(1+q^{j|s|\pm s_0})(1+q^{j|s|\pm(s_0+s_1)})],$$

where $|s| = 2s_0+s_1+s_3$.

5) $D_\ell^{(1)}$:

For $s = (s_1,s_1,s_2,s_3,\ldots,s_\ell) \in N^{\ell+1}$,

$$d^{s}_{\Lambda_0}(q) = \frac{\varphi(q^{2|s|})^2}{\varphi(q^{|s|})^2}\prod_{i=1}^{\ell-1}\prod_j^{(+)}(1+q^{j|s|\pm(s_1+\ldots+s_i)})$$

where $|s| = 2(s_1+\ldots+s_{\ell-2})+s_{\ell-1}+s_\ell$.

6) $E_6^{(1)}$:

For $s = \begin{pmatrix} s_0 & s_0 & s_3 & s_0 & s_0 \\ & & s_0 & & \\ & & s_0 & & \end{pmatrix}$,

$$d^{s}_{\Lambda_0}(q) = \frac{\varphi(q^{3|s|})}{\varphi(q^{|s|})}\prod_j^{(+)}\frac{(1-q^{3j|s|\pm 3s_0})}{(1-q^{j|s|\pm s_0})}$$

where $|s| = 3s_0+s_3$.

7) $E_7^{(1)}$:

For $s = \begin{pmatrix} s_0 & s_1 & s_0 & s_3 & s_0 & s_1 & s_0 \\ & & & s_7 & & & \end{pmatrix}$,

$$d^s_{\Lambda_0}(q) = \varphi(q^{2|s|}) \prod_j^{(+)} [(1+q^{j|s|\pm s_0})(1+q^{j|s|\pm(s_0+s_1)}) \times (1+q^{j|s|\pm(2s_0+s_1+s_3)})]$$

where $|s| = 4s_0+2s_1+2s_3+s_7$.

8) $F_4^{(1)}$:

For $\Lambda = (n_0-1,n_1-1,n_0-1,n_3-1,n_4-1)$,

$$d_\Lambda^{(21111)}(q) = \frac{\varphi(q^{|n|})^6}{\varphi(q)^4} [\prod_{i=1}^{4} \prod_j^{(+)} (1-q^{j|n|\pm n_i})]$$

$$\times [\prod_{i=1}^{3} \prod_j^{(+)} (1-q^{j|n|\pm(n_0+n_i)})][\prod_j^{(+)} (1-q^{j|n|\pm 2(n_0+n_1)})]$$

$$\times [\prod_{i=1,3} \prod_j^{(+)} (1-q^{j|n|\pm(2n_0+n_i)})]$$

$$\times [\prod_{k=1}^{3} \prod_j^{(+)} (1-q^{j|n|\pm(kn_0+n_1+n_3)})(1-q^{j|n|\pm((k+1)n_0+2n_1+n_3)})]$$

where $n_2 = n_0$ and $|n| = 4n_0+2n_1+2n_3+n_4$.

9) $G_2^{(1)}$:

i) $s = (s_0,s_1,s_0)$;

$$d^s_{\Lambda_0}(q) = \prod_j^{(+)} \frac{(1+q^{j|s|\pm s_0})}{(1-q^{|s|(5j\pm 2)})} ,$$

$$d^{s}_{\Lambda_2}(q) = \prod_{j}^{(+)} \frac{(1+q^{j|s|\pm s_0})}{(1-q^{|s|(5j\pm 1)})},$$

where $|s| = 2s_0+s_1$.

ii) $\Lambda = (n_0-1,n_0-1,n_2-1)$;

$$d^{(211)}_{\Lambda}(q) = \frac{\varphi(q^{|n|})}{\varphi(q)^2}[\prod_{k=1}^{3}\prod_{j}^{(+)}(1-q^{j|n|\pm kn_0})]$$

$$\times [\prod_{j}^{(+)}(1-q^{3j|n|\pm 3n_0})]$$

where $|n| = 3n_0+n_2$.

10) $A^{(2)}_{2\ell-1}$:

For $s = (s_1,s_1,s_2,s_3,\ldots,s_\ell) \in N^{\ell+1}$,

$$d^{s}_{\Lambda_0}(q) = \prod_{i=0}^{\ell-1}\prod_{j}^{(+)}(1+q^{j|s|\pm(s_1+\ldots+s_i)})$$

where $|s| = 2(s_1+\ldots+s_{\ell-1})+s_\ell$.

11) $D^{(2)}_{3}$:

For $\Lambda = (n_0-1,n_1-1,n_0-1)$,

$$d^{(211)}_{\Lambda}(q) = \frac{\varphi(q^4)\varphi(q^{|n|})\varphi(q^{4|n|})}{\varphi(q)^2\varphi(q^8)}$$

$$\times \prod_{j}^{(+)}[(1-q^{j|n|\pm n_0})(1-q^{4j|n|\pm 2n_0})]$$

where $|n| = n_0+n_1$.

12) $E^{(2)}_{6}$:

For $s = (s_0,s_1,s_0,s_3,s_4)$,

$$d^{s}_{\Lambda_0}(q) = \varphi(q^{|s|})^2 \prod_{j}^{(+)} [(1+q^{j|s|\pm s_0})(1+q^{j|s|\pm(s_0+s_1)}) \times (1+q^{j|s|\pm(2s_0+s_1+s_3)})]$$

where $|s| = 4s_0+2s_1+2s_3+s_4$.

13) $D_4^{(3)}$:

i) $\Lambda = (2n-1,n-1,n-1)$ and $s = (s_0,s_0,s_2)$;

$$d^{s}_{\Lambda}(q) = \frac{\varphi(q^{n|s|})}{\varphi(q^{|s|})} \left[\prod_{j}^{(+)} \frac{(1-q^{3n(j|s|\pm s_0)})}{(1-q^{j|s|\pm s_0})}\right] \times \left[\prod_{k=1}^{3} \prod_{j}^{(+)} \frac{(1-q^{n(j|s|\pm ks_0)})}{(1-q^{j|s|\pm ks_0})}\right]$$

where $|s| = 3s_0+s_2$.

ii) $\Lambda = (n_0-1,n_1-1,n_0-1)$;

$$d^{(211)}_{\Lambda}(q) = \frac{\varphi(q^{|n|})^2}{\varphi(q)^2} \prod_{j}^{(+)} \left[\frac{(1-q^{3(5j\pm 2)})}{(1-q^{|n|(5j\pm 2)})} \times \prod_{i=0,1} (1-q^{j|n|\pm n_i})\right]$$

$$d^{(112)}_{\Lambda}(q) = \frac{\varphi(q^{|n|})^2}{\varphi(q)^2} \prod_{j}^{(+)} \left[\frac{(1-q^{3(5j\pm 1)})}{(1-q^{|n|(5j\pm 1)})} \times \prod_{i=0,1} (1-q^{j|n|\pm n_i})\right]$$

where $|n| = 2n_0+n_1$.

3. Power series identities:

In this section, we are going to compute specialized characters of an integrable highest weight module of an affine Lie algebra $\mathfrak{g}(A)$ in terms of weight system. It may be expected that many things will be comparatively simple if A and Λ satisfy the condition

(*) $\mathrm{Max}_0(\Lambda)$ contains only one element Λ,

where $\mathrm{Max}(\Lambda) = \{$ maximal weights of $L(\Lambda)\}$ and $\mathrm{Max}_0(\Lambda) = P_+ \cap \mathrm{Max}(\Lambda)$.

For such $L(\Lambda)$, the weight system $P(\Lambda)$ is given by

$$P(\Lambda) = \left\{ \lambda \in P;\ \begin{array}{l} |\lambda|^2 \leqq |\Lambda|^2 \\ \langle \lambda, c \rangle = \langle \Lambda, c \rangle \end{array} \right\},$$

where c is the canonical central element and $|\ |$ is the norm defined by the standard bilinear form on $\mathfrak{g}(A)$.

Table 3 is the complete list of $L(\Lambda;\ A)$'s enjoying the property (*) (see also [5]):

<u>Table 3</u>

1) $A_1^{(1)}$: Λ_0, Λ_1, $\Lambda_0+\Lambda_1$.
   $A_\ell^{(1)}$: Λ_i $(i = 0,\ldots,\ell)$.
2) $B_\ell^{(1)}$: Λ_ℓ.
3) $C_2^{(1)}$: Λ_1.
4) $D_\ell^{(1)}$: Λ_i $(i = 0,1,\ell-1,\ell)$.
5) $E_6^{(1)}$: Λ_i $(i = 0,4,6)$.

$E_7^{(1)}$: Λ_0, Λ_6, $\Lambda_0+\Lambda_6$.

$E_8^{(1)}$: Λ_0.

6) $A_{2\ell}^{(2)}(\ell\geqq 1)$: Λ_0.

7) $A_{2\ell-1}^{(2)}$: Λ_0, Λ_1.

8) $D_{\ell+1}^{(2)}$: Λ_0, Λ_ℓ, $\Lambda_0+\Lambda_\ell$.

9) $E_6^{(2)}$: Λ_0.

10) $D_4^{(3)}$: Λ_0.

Now look at the weight system of $L(\Lambda_0;\ A_1^{(1)})$;

$$P(\Lambda_0) = \{ \lambda(m,n) = \Lambda_0+m\alpha_1+n\delta \ ;\ m,n\in Z \text{ and } n\leqq -m^2 \},$$

where $\delta = \alpha_0+\alpha_1$ is the fundamental imaginary root. Since two elements $\lambda(m_1,n_1)$ and $\lambda(m_2,n_2)$ in $P(\Lambda_0)$ are W-conjugate if and only if $n_1+m_1^2 = n_2+m_2^2$, we can put

$$\xi(k) = \text{multiplicity of } \lambda(-m^2-k,m)$$

for $k\in N_0$ and $m\in Z$. Then the character of $L(\Lambda_0;\ A_1^{(1)})$ is written

$$\text{ch } L(\Lambda_0) = e^{\Lambda_0}\sum_{k=0}^{\infty}\sum_{m\in Z}\xi(k)\ e^{m\alpha_1-(m^2+k)\delta}$$

$$= e^{\Lambda_0}[\sum_{k=0}^{\infty}\xi(k)\ e^{-k\delta}][\sum_{m\in Z}e^{-m\alpha_1-m^2\delta}] \quad \ldots (1).$$

First, in order to compute the multiplicity $\xi(k)$, let us take the principal specialization;

$$d_{\Lambda_0}(q) = [\sum_{k=0}^{\infty} \xi(k) q^{2k}][\sum_{m\in Z} q^{2m^2+m}].$$

Noting that $d_{\Lambda_0}(q) = \dfrac{\varphi(q^2)}{\varphi(q)}$

and $\displaystyle\sum_{m\in Z} q^{2m^2+m} = \frac{\varphi(q^2)^2}{\varphi(q)}$ (Gauss 1866),

one obtains

$$\sum_{k=0}^{\infty} \xi(k) q^{2k} = \frac{1}{\varphi(q^2)}$$

i.e.,

$$\sum_{k=0}^{\infty} \xi(k) q^{k} = \frac{1}{\varphi(q)} \qquad \ldots \quad (2).$$

Thus it is proved that the string function of $L(\Lambda_0;\ A_1^{(1)})$ is equal to the generating function of the partition functions (this is a well-known fact in some literatures).

Next, the specialization of (1) of type $s = (s_0, s_1)$ gives

$$d^s_{\Lambda_0}(q) = [\sum_{k=0} \xi(k) q^{|s|k}][\sum_{m\in Z} q^{s_1 m+|s|m^2}]$$

$$= \frac{1}{\varphi(q^{|s|})} \sum_{m\in Z} q^{|s|m^2+s_1 m} \qquad \ldots \quad (3),$$

where $|s| = s_0+s_1$.

Now combining (3) with a formula

$$d^s_{\Lambda_0}(q) = \prod_j^{(+)} [(1+q^{|s|j})(1+q^{2|s|j\pm s_0})]$$

in Table 1, one obtains an identity

$$\sum_{m \in Z} q^{|s|m^2+s_1 m} = \varphi(q^{2|s|}) \prod_j^{(+)} (1+q^{2|s|j \pm s_0}).$$

Note that this formula is equivalent to the Jacobi triple product identity.

Similar arguments are valid for highest weight modules in Table 3, and one can deduce power series identities, some of which are shown in Table 4;

<u>Table 4</u>

1) $L(\Lambda_0;\ A_1^{(1)})$, $L(\Lambda_0;\ A_2^{(2)})$ or $L(\rho\ ;\ A_1^{(1)})$:

For $(s_0, s_1) \in N \times \frac{1}{2}N_0$,

$$\sum_{n \in Z} q^{|s|n^2+s_1 n} = \varphi(q^{2|s|}) \prod_j^{(+)} (1+q^{2j|s| \pm s_0})$$

where $|s| = s_0+s_1$.

2) $L(\Lambda_1;\ C_2^{(1)})$:

For $s = (s_0, s_1, s_2) \in N^3$,

$$\sum_{n \in Z^2} q^{X_s(n)} = \varphi(q^{|s|})^2 \prod_j^{(+)} [(1+q^{j|s| \pm s_1})(1+q^{j|s| \pm (s_0+s_1)})],$$

where $X_s(n) = X_s(n_1, n_2)$

$$= |s|[\frac{n_1{}^2-n_1}{2} + n_2{}^2 - n_1 n_2] + \sum_{i=1}^{2} s_i n_i$$

and $|s| = s_0+2s_1+s_2$.

3) $L(\Lambda_\ell;\ B_\ell^{(1)})$ $(\ell \geqq 3)$:

For $s = (s_0,\dots,s_\ell) \in N^{\ell+1}$,

$$\sum_{n\in Z^\ell} q^{X_s(n)} = \varphi(q^{|s|})^\ell \prod_{i=1}^{\ell}\prod_{j}^{(+)}(1+q^{j|s|\pm(s_i+\dots+s_\ell)}),$$

where $X_s(n) = X_s(n_1,\dots,n_\ell)$

$$= |s|\left[\sum_{i=1}^{\ell-1}(n_i^2-n_in_{i+1}) + \frac{n_\ell^2-n_\ell}{2}\right] + \sum_{i=1}^{\ell} s_in_i$$

and $|s| = s_0+s_1+2(s_2+\dots+s_\ell)$.

4) $L(\Lambda_0;\ A_{2\ell}^{(2)})$ $(\ell \geqq 2)$:

For $s = (s_0,\dots,s_\ell) \in N^{\ell+1}$,

$$\sum_{n\in Z^\ell} q^{X_s(n)} = \frac{1}{\varphi(q^{|s|})^\ell}\prod_{i=0}^{\ell-1}\prod_{j}^{(+)}(1+q^{j|s|\pm(s_0+\dots+s_i)}),$$

where $X_s(n) = X_s(n_0,\dots,n_{\ell-1})$

$$= |s|\left[\sum_{i=1}^{\ell-1}(n_i^2-n_{i-1}n_i) + \frac{n_0^2-n_0}{2}\right] + \sum_{i=0}^{\ell-1} s_in_i$$

and $|s| = 2(s_0+\dots+s_{\ell-1})+s_\ell$.

5) $L(\Lambda_0;\ D_{\ell+1}^{(2)})$ $(\ell \geqq 2)$:

For $s = (s_0,\dots,s_\ell) \in N^{\ell+1}$,

$$\sum_{n\in Z^\ell} q^{X_s(n)} = \varphi(q^{2|s|})^\ell \prod_{i=0}^{\ell-1}\prod_{j}^{(+)}(1+q^{2j|s|\pm(s_0+\dots+s_i)}),$$

where $X_s(n) = X_s(n_1,\dots,n_\ell)$

$$= |s|\left[2\sum_{i=1}^{\ell-1}(n_i^2-n_in_{i+1}) + n_\ell^2\right] + \sum_{i=1}^{\ell} s_in_i$$

and $|s| = s_0+\dots+s_\ell$.

6) $L(\Lambda_0+\Lambda_\ell;\ D_{\ell+1}^{(2)})$ $(\ell \geqq 2)$:

For $s = (s_0,\dots,s_\ell) \in N^{\ell+1}$,

$$\sum_{n\in Z^\ell} q^{X_s(n)} = \varphi(q^{|s|}) \prod_{i=0}^{\ell-1} \prod_{j}^{(+)} (1+q^{j|s|\pm(s_0+\dots+s_i)}),$$

where $X_s(n) = X_s(n_1,\dots,n_\ell)$

$$= |s|\left[\sum_{i=1}^{\ell-1}(n_i^2-n_in_{i+1}) + \frac{n_\ell^2-n_\ell}{2}\right] + \sum_{i=1}^{\ell} s_in_i$$

and $|s| = s_0+\dots+s_\ell$.

7) $L(\Lambda_0;\ A_{2\ell-1}^{(2)})$ $(\ell \geqq 3)$:

For $s = (s_1,\dots,s_\ell) \in N^{\ell}$,

$$\sum_{n\in Z^\ell} q^{X_s(n)} = \varphi(q^{|s|})^{\ell-2}\varphi(q^{2|s|})^2 \prod_{i=1}^{\ell-1} \prod_{j}^{(+)} (1+q^{j|s|\pm(s_1+\dots+s_i)}),$$

where $X_s(n) = X_s(n_1,\dots,n_\ell)$

$$= |s|\left[\sum_{i=1}^{\ell} n_i^2 - \sum_{i=1}^{\ell-2} n_in_{i+1} - n_{\ell-2}n_\ell\right] + \sum_{i=1}^{\ell} s_in_i$$

and $|s| = 2(s_1+\dots+s_{\ell-2})+s_{\ell-1}+s_\ell$.

8) $L(\Lambda_0,\ A_\ell^{(1)})$ $(\ell \geqq 1)$:

For $s = (s_0,\dots,s_\ell) \in N^{\ell+1}$,

$$\sum_{n\in Z^{\ell}} q^{X_s(n)} = \varphi(q^{|s|}) \; d_{\Lambda_0}^{(s;\; A_{\ell}^{(1)})}(q),$$

where $X_s(n) = X_s(n_1,\ldots,n_{\ell})$

$$= |s|\,[\sum_{i=1}^{\ell-1}(n_i^2-n_in_{i+1}) + n_{\ell}^2] + \sum_{i=1}^{\ell} s_in_i$$

and $|s| = s_0+\ldots+s_{\ell}$.

Example 1. $s = 1 = (1,\ldots,1)$;

$$\sum_{n\in Z^{\ell}} q^{X_1(n)} = \frac{\varphi(q^{\ell+1})^{\ell+1}}{\varphi(q)} \quad ,$$

where $X_1(n) = X_1(n_1,\ldots,n_{\ell})$

$$= (\ell+1)[\sum_{i=1}^{\ell-1}(n_i^2-n_in_{i+1}) + n_{\ell}^2] + \sum_{i=1}^{\ell} n_i.$$

Example 2. $\ell = 2$ and $s = (s_1,s_1,s_2)$;

$$\sum_{n\in Z^2} q^{X(n)} = \varphi(q^{|s|})\varphi(q^{3|s|}) \prod_j^{(+)} \frac{(1-q^{3(j|s|\pm s_1)})}{(1-q^{j|s|\pm s_1})} \quad ,$$

where $X(n) = X(n_1,n_2)$

$$= |s|(n_1^2+n_2^2-n_1n_2) + \sum_{i=1}^{2} s_in_i$$

and $|s| = 2s_1+s_2$.

Example 3. $\ell = 3$ and $s = (s_2,s_1,s_2,s_3)$;

$$\sum_{n\in Z^3} q^{X(n)} = \varphi(q^{|s|})\,\varphi(q^{2|s|})^2 \prod_j^{(+)} [(1+q^{j|s|\pm s_2})(1+q^{j|s|\pm(s_1+s_2)})]$$

where $X(n) = X(n_1,n_2,n_3)$

$$= |s|[\sum_{i=1}^{3} n_i^2 - n_2(n_1+n_3)] + \sum_{i=1}^{3} s_i n_i$$

and $|s| = s_1+2s_2+s_3$.

References

[1] A. J. Feingold and J. Lepowsky: The Weyl-Kac character formula and power series identities, Advances in Math., 29(1978), 271-309.

[2] H. Garland and J. Lepowsky: Lie algebra homology and the Macdonald-Kac formulas, Invent. math.,34(1976), 37-76.

[3] 岩堀長慶・横沼健雄： Kac-Moody Lie 環と Macdonald 恒等式, 数学 33 (1981), 193-212

[4] M. Jimbo and T. Miwa: Irreducible decomposition of fundamental modules for $A_{\ell}^{(1)}$ and $C_{\ell}^{(1)}$, and Hecke modular forms, RIMS preprint-434(1983).

[5] V. G. Kac: Infinite-dimensional algebras, Dedekind η-function, classical Möbius function and the very strange formula, Advances in Math., 30(1978), 85-136.

[6] V. G. Kac: Infinite-dimensional Lie algebras (Lectures at M.I.T., 1982 spring).

[7] V. G. Kac and D. H. Peterson: Infinite-dimensional Lie algebras, theta functions and modular forms, to appear in Advances in Math.

[8] B. Kostant: On Macdonald's η-function formula, the

Laplacian and generalized exponents, Advances in Math., 20(1976), 179-212.

[9] J. Lepowsky: Macdonald-type identities, Advances in Math., 27(1978), 230-234.

[10] J. Lepowsky: Lectures on Kac-Moody Lie algebras, Université de Paris VI (spring 1978).

[11] J. Lepowsky: Generalized Verma modules, loop space cohomology and Macdonald-type identities, Ann. Sci. Ecole Norm. Sup., 12(1979), 169-234.

[12] J. Lepowsky: Application of the numerator formula to k-rowed plane partitions, Advances in Math., 35(1980), 179-194.

[13] J. Lepowsky: Affine Lie algebras and combinatorial identities, Lecture Notes in Math., 933(1982), 130-156.

[14] J. Lepowsky and S. Milne: Lie algebras and classical partition identities, Proc. Nat. Acad. Sci. U.S.A., 75(1978), 578-579.

[15] J. Lepowsky and S. Milne: Lie algebra approaches to classical partition identities, Advances in Math., 29(1978), 15-59.

[16] J. Lepowsky and R. L. Wilson: The Rogers-Ramanujan identities: Lie theoretic interpretation and proof, Proc. Nat. Acad. Sci. U.S.A., 78(1981), 699-701.

[17] J. Lepowsky and R. L. Wilson: A new family of algebras underlying the Rogers-Ramanujan identities and generalizations, Proc. Nat. Acad. Sci. U.S.A., 78(1981), 7254-7258.

[18] J. Lepowsky and R. L. Wilson: A Lie theoretic interpretation

and proof of the Rogers-Ramanujan identities, Advances in Math., 45(1982), 21-72.

[19] 小池和彦：Lie環論の最近の話題から，月刊マセマティクス 7(1980), 674-690.

[20] 森田純：Kac-Moody Lie algebras, 数学のあゆみ 23(1981), 5-45.

[21] M. Wakimoto: A note on fundamental representations of an affine Lie algebra $C_2^{(1)}$, preprint.

[22] M. Wakimoto: Two formulae for specialized characters of Kac-Moody Lie algebras, preprint.

[23] M. Wakimoto: Specialized characters of affine Lie algebras and some identities, preprint.

[24] M. Wakimoto: Realization of level-two highest weight modules of an affine Lie algebra $A_1^{(1)}$, preprint.

[25] M. Wakimoto: A note on a realization of a fundamental representation of an affine Lie algebra $B_\ell^{(1)}$, preprint.

[26] M. Wakimoto: Some characters of affine Lie algebras, preprint.

[27] M. Wakimoto: Multi-tuple product identities associated to affine Lie algebras, preprint.