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Powerposets

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C-52 Powerposets
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Abstract. We introduce the notion of powerposets which is a natural generalization of that of powersets with inclusion as their partial ordering. We show that every powerposet is an algebraic semilattice and that every continuous poset can be directed-continuously embeddable into some powerposet.
We also discuss the possibility of making powerposets into λ-models as in the case of Plotkin-Scott's $P\omega$ theory.

0. Introduction

The domain $P\omega$ introduced by Dana Scott is a very simple and beautiful structure [9]. It provides a universal circumstance to develop theoretical computer science. Nevertheless, to many of computer scientists, $P\omega$ is too large to handle with in their everydays' work. So we want to select other (possibly partially ordered) set for ω. Powerposets are domains constructed in this way.

In section 1 we introduce the notions of lower ends and upper ends in slightly generalized forms of those usually defined.

Section 2 is devoted to review the fundamental concepts of the theories of continuous lattices and λ-calculus models.

The main results of this note are in section 3, including the theorem which says that every powerposet is an algebraic semilattice. As a corollary of this theorem, we can conclude that $P\omega$ is an algebraic lattice as already mentioned by Scott. We also show that for every continuous poset there is an one-one map from it to some powerposet preserving directed sups.

Finally in section 4 we discuss the possibility of expanding a self-referential powerposet to a λ-calculus model.

1. Lower Ends and Upper Ends

Let $\pi = (\pi, \leq)$ and $\pi' = (\pi', \leq')$ be posets, a, b, c subsets of π, and x, y, z elements of π throughout this note.

Definition 1.1. (i) $a{\downarrow}x = \{ y \in a \mid y \leq x \}$.

(ii) ${\downarrow}x = \pi{\downarrow}x$.

(iii) $a{\downarrow}b = \bigcup\{\, a{\downarrow}x \mid x \in b \,\}$.

(iv) ${\downarrow}a = \pi{\downarrow}a$.

(v) $a{\uparrow}x = \{\, y \in a \mid x \leq y \,\}$.

(vi) ${\uparrow}x = \pi{\uparrow}x$.

(vii) $a{\uparrow}b = \bigcup\{\, a{\uparrow}x \mid x \in b \,\}$.

(viii) ${\uparrow}a = \pi{\uparrow}a$.

Proposition 1.2.

$$\begin{tikzcd}[column sep=small, row sep=small]
 & a{\downarrow}b \arrow[dr, phantom, sloped, "\subset"] & \\
a\cap b \arrow[ur, phantom, sloped, "\subset"] \arrow[dr, phantom, sloped, "\supset"] & & a\,. \\
 & a{\uparrow}b \arrow[ur, phantom, sloped, "\subset"] &
\end{tikzcd}$$

Proposition 1.3. If π is discrete (i.e. for every x and y in π $x \leq y$ implies $x = y$), $a{\downarrow}b = a\cap b = a{\uparrow}b$.

Proposition 1.4. (i) $a{\downarrow}\emptyset = \emptyset = a{\uparrow}\emptyset$.

(ii) $a \subset b$ implies $a{\downarrow}b = a = a{\uparrow}b$.

(iii) $(\bigcup_{i\in I} a_i){\downarrow}(\bigcup_{j\in J} b_j) = \bigcup_{i\in I}\bigcup_{j\in J}(a_i{\downarrow}b_j)$.

(iv) $(\bigcup_{i\in I} a_i){\uparrow}(\bigcup_{j\in J} b_j) = \bigcup_{i\in I}\bigcup_{j\in J}(a_i{\uparrow}b_j)$.

(v) $(\bigcap_{i\in I} a_i){\downarrow}b = \bigcap_{i\in I}(a_i{\downarrow}b)$.

(vi) $(\bigcap_{i\in I} a_i){\uparrow}b = \bigcap_{i\in I}(a_i{\uparrow}b)$.

Corollary 1.5. $a \subset a'$ and $b \subset b'$ imply

(i) $a{\downarrow}b \subset a'{\downarrow}b'$,

(ii) $a{\uparrow}b \subset a'{\uparrow}b'$.

Proposition 1.6. (i) $a{\downarrow}b \subset c$ implies $a{\downarrow}b \subset c{\downarrow}b$.

(ii) $a{\uparrow}b \subset c$ implies $a{\uparrow}b \subset c{\uparrow}b$.

(iii) $a{\downarrow}(b{\downarrow}c) \subset a{\downarrow}c$.

(iv) $a{\uparrow}(b{\uparrow}c) \subset a{\uparrow}c$.

(v) $a{\downarrow}(a{\downarrow}b) = a{\downarrow}b = (a{\downarrow}b){\downarrow}b$.

(vi) $a{\uparrow}(a{\uparrow}b) = a{\uparrow}b = (a{\uparrow}b){\uparrow}b$.

The proofs of these propositions are very easy, and so left to readers.

Definition 1.7. (i) a is called a <u>lower end</u> of b (notation: $a \lesssim_L b$) when $b{\downarrow}a = a$.

(ii) a is called an <u>upper end</u> of b (notaion: $a \lesssim_U b$) when $b{\uparrow}a = a$.

Lemma 1.8. (i) a is a lower end of b iff $b{\downarrow}a \subset a \subset b$.

(ii) a is an upper end of b iff $b{\uparrow}a \subset a \subset b$.

Proof. (i) If part: $a = a{\downarrow}a$ by 1.4(ii)

$\subset b{\downarrow}a$ by 1.5(i).

Only if part: $a = b{\downarrow}a \subset b$ by 1.2.

(ii) Similar to (i).

Proposition 1.9. If π is discrete, the following three statements are equivalent;

(1) a is a lower end of b.

(2) a is a subset of b.

(3) a is an upper end of b.

Proof. By 1.3 and 1.8.

Proposition 1.10. (i) $a{\downarrow}b \lesssim_L a$.

(ii) $a \lesssim_L b$ iff there exists a subset c of π such that $a = b{\downarrow}c$.

(iii) $a{\uparrow}b \lesssim_U a$.

(iv) $a \lesssim_U b$ iff there exists a subset c of π such that $a = b{\uparrow}c$.

Proof. (i) By 1.6(v) $a{\downarrow}(a{\downarrow}b) = a{\downarrow}b$.

(ii) Only if part: Immediate.

If part: By 1.6(v) $b{\downarrow}a = b{\downarrow}(b{\downarrow}c) = b{\downarrow}c = a$.

(iii) Similar to (i).

(iv) Similar to (ii).

Theorem 1.11. Let a, b and b' be subsets of π with $b \cup b' = a$ and $b \cap b' = \emptyset$. Then $b \lesssim_L a$ iff $b' \lesssim_U a$.

Proof. Since $b = a \cap b \subset a{\downarrow}b \subset a$ and $b' = a \cap b' \subset a{\uparrow}b' \subset a$ by 1.2, we have $a = b \cup b' \subset a{\downarrow}b \cup a{\uparrow}b' \subset a \cup a = a$. Thus, $a{\downarrow}b \cup a{\uparrow}b' = a$. Moreover for every x, $x \in a{\downarrow}b \cap a{\uparrow}b'$ implies the existence of $y \in b$ and $z \in b'$ that satisfy

$$z \in b' \cap a{\downarrow}x \subset b' \cap a{\downarrow}y \subset b' \cap a{\downarrow}b$$

and $$y \in b \cap a{\uparrow}x \subset b \cap a{\uparrow}z \subset b \cap a{\uparrow}b'.$$

Thus, $b' \cap a{\downarrow}b = \emptyset$ or $b \cap a{\uparrow}b' = \emptyset$ imply $a{\downarrow}b \cap a{\uparrow}b' = \emptyset$.

Only if part: If $a{\downarrow}b = b$, then $b' \cap a{\downarrow}b = b' \cap b = \emptyset$. Thus, $a{\downarrow}b \cap a{\uparrow}b' = \emptyset$. Hence $a{\uparrow}b' = a - a{\downarrow}b = a - b = b'$.

If part: If $a{\uparrow}b' = b'$, then $b \cap a{\uparrow}b' = b \cap b' = \emptyset$. Thus, $a{\downarrow}b \cap a{\uparrow}b' = \emptyset$. Hence $a{\downarrow}b = a - a{\uparrow}b' = a - b' = b$.

Corollary 1.12. For $x \in a \subset \pi$,

(i) x is maximal in a iff $\{x\} \lesssim_U a$ iff $a - \{x\} \lesssim_L a$.

(ii) x is minimal in a iff $\{x\} \lesssim_L a$ iff $a - \{x\} \lesssim_U a$.

Proof. Immediate from 1.11.

2. Review

In this section we review the fundamental concepts of the theories of continuous lattices and λ-calculus models.

Definition 2.1. (i) A subset d of π is called <u>directed</u> if every finite subset of d has an upper bound in d.

(ii) We say that x is <u>way below</u> y (notation: $x \ll y$), if for every directed subset d of π the relation $y \leq \sup d$ always implies the existence of z of d with $x \leq z$.

(iii) $a{\downarrow}x = \{ y \in a \mid y \ll x \}$.

(iv) ${\downarrow}x = \pi{\downarrow}x$.

(v) $a{\downarrow}b = \cup\{ a{\downarrow}x \mid x \in b \}$.

(vi) ${\downarrow}a = \pi{\downarrow}a$.

(vii) An element $x \in \pi$ is called <u>compact</u> if $x \ll x$.

(viii) $K(\pi) = \{ x \in \pi \mid x \text{ is compact} \}$.

Note that every directed set is nonempty.

Proposition 2.2. (i) $x \ll y$ implies $x \leq y$.

(ii) $w \leq x \ll y \leq z$ implies $w \ll z$.

(iii) $x = \sup \{ x_1, \ldots, x_n \}$ and $x_i \ll y$ for all $i = 1,\ldots,n$ imply $x \ll y$.

Definition 2.3. (i) A poset π is called <u>up-complete</u> if every directed subset of π has a sup in π.

(ii) [Markowski] An up-complete poset π is called <u>continuous</u> if for every x in π, ${\downarrow}x$ is directed and $x = \sup {\downarrow}x$.

(iii) [Hoffman] An up-complete poset π is called <u>algebraic</u> if for every x in π, $K(\pi){\downarrow}x$ is directed and $x = \sup (K(\pi){\downarrow}x)$.

Iwamura and Markowski's result says that we can replace "directed set" by "nonempty chain" in 2.3(i) [5, 7].

Markowski also suggests the thesis that "continuous posets" are the proper setting for an abstract theory of computation

[8].

The following two theorems are due to Markowski.

Theorem 2.4. [Interpolation Theorem] Let π be a continuous poset, $x \ll y$ in π, and d a directed subset of π with $y \leq \sup d$. Then there exists $z \in d$ such that $x \ll z$.

Theorem 2.5. Every algebraic poset is continuous.

Definition 2.6. (i) A <u>semilattice</u> is a poset in which every nonempty finite subset has an inf.

(ii) A <u>complete semilattice</u> is an up-complete poset in which every nonempty subset has an inf.

(iii) An <u>arithmetic semilattice</u> is an algebraic semilattice π in which $K(\pi)$ is a semilattice.

Note that every complete semilattice π has the least element $\inf \pi$.

Definition 2.7. (i) A <u>lattice</u> is a semilattice in which every nonempty finite subset has a sup.

(ii) A lattice is called <u>complete</u> if every subset has an inf and a sup.

Theorem 2.8. Every complete semilattice with a greatest element is a complete lattice.

Next we state some concepts of the theory of λ-calculus models.

Definition 2.9. Let $(X, .)$ be a system with a binary operator . on a set X, called an <u>applicative structure</u>.

(i) (X, .) is called <u>combinatory complete</u> when there are two elements k and s in X such that $kxy = x$ and $sxyz = xz(yz)$ for all $x, y, z \in X$.

(ii) A function $f : X \to X$ is called <u>representable</u> if there is an element $x \in X$ such that for every $y \in X$ $f(y) = xy$.

(iii) $[X \to X]$ denotes the set of all representable functions on X.

The notion of <u>λ-models</u> is introduced by Barendregt in order to investigate λ-calculus models formally.
The following theorem is due to Barendregt [2].

Theorem 2.10. Let (X, .) be combinatory complete and define the map $F : X \to [X \to X]$ by $F(x)(y) = xy$. Then (X, .) can be expanded to a λ-model iff there exists a $G : [X \to X] \to X$ such that:

(1) $F \circ G = 1_{[X \to X]}$;

(2) $G \circ F \in [X \to X]$.

Readers may refer to [4] and [1, 2] for further information on these structures.

3. Powerposets

Theorem 3.1. Two relations $\leq_L$ and $\leq_U$ are partial order relations on Pπ.

Proof. We only prove for the relation $\leq_L$; The other case is analogous.

Reflexivity: $a \leq_L a$ by 1.4(ii).

Antisymmetricity: $a \leq_L b$ and $b \leq_L a$ imply $a \subset b$ and $b \subset a$ by 1.8(i). Thus, $a = b$.

Transitivity: Assume that $a \leq_L b \leq_L c$. Then by 1.5(i) $c{\downarrow}a \subset c{\downarrow}b = b$. Thus, by 1.6(i) $c{\downarrow}a \subset b{\downarrow}a = a$. On the other hand $a \subset c$. Hence by 1.8(i) $a \leq_L c$.

According to the above theorem we call these structures $(P\pi, \leq_L)$ and $(P\pi, \leq_U)$ <u>powerposets</u>.

Corollary 3.2. Let π be a discrete poset. Then $(P\pi, \leq_L) = (P\pi, \leq_U) = (P\pi, \subset)$.

Proof. By 1.9.

Proposition 3.3. Let $\varphi : \pi \to \pi'$ be a monotonic function. Then the map $\varphi^{-1} : P\pi' \to P\pi$ is also monotonic with respect to each ordering $\leq_L$ and $\leq_U$.

Proof. Suppose that $P\pi$ is partially ordered by $\leq_L$. Then it is trivial that $\varphi^{-1}(a) \subset \varphi^{-1}(b)$ if $a \leq_L b$ in $P\pi'$. So it suffices to show that $\varphi^{-1}(b){\downarrow}\varphi^{-1}(a) \subset \varphi^{-1}(a)$ by 1.8.
Now let $x \in \varphi^{-1}(b){\downarrow}\varphi^{-1}(a)$. Then $x \in \varphi^{-1}(b)$ and there is $y \in \varphi^{-1}(a)$ with $x \leq y$. Hence $\varphi(x) \in b$, $\varphi(y) \in a$ and $\varphi(x) \leq \varphi(y)$ in π' because φ is monotonic. Thus, $\varphi(x) \in b{\downarrow}a = a$ since $a \leq_L b$. So $x \in \varphi^{-1}(a)$. Therefore $\varphi^{-1}(b){\downarrow}\varphi^{-1}(a) \subset \varphi^{-1}(a)$.

The proof for the ordering $\leq_U$ is similar.

Definition 3.4. (i) <u>Poset</u> denotes the category of all posets with all monotonic functions as arrows.

(ii) The contravariant functor P_L : <u>Poset</u> $\to$ <u>Poset</u> is defined by

$$\begin{array}{rcccl} & \pi & \longmapsto & (P\pi, \leq_L) & \\ P_L : \varphi & \downarrow & & \uparrow & \varphi^{-1} \\ & \pi' & \longmapsto & (P\pi', \leq'_L). & \end{array}$$

(iii) The contravariant functor P_U : Poset → Poset is defined by

$$\begin{array}{rcccl} & \pi & \longmapsto & (P\pi, \leq_U) & \\ P_U : \varphi & \downarrow & & \uparrow & \varphi^{-1} \\ & \pi' & \longmapsto & (P\pi', \leq'_U). & \end{array}$$

Note that the above functors are well-defined by 3.3.

Theorem 3.5. For every poset π $P_U(\pi) = P_L(\pi^{op})$ where π^{op} is an opposite poset, considering π as a category.

Proof. Immediate because $a{\uparrow_\pi}b = a{\downarrow_{\pi^{op}}}b$ for all a, b $\in P\pi$.

By the above theorem we can assume that every powerposet is of the form $P_L(\pi) = (P\pi, \leq_L)$ without loss of generality. So in the rest of this note we concentrate on this form, and write $P\pi = (P\pi, \leq)$ instead of writing $P_L(\pi) = (P\pi, \leq_L)$.

Lemma 3.6. Let S be a subset of $P\pi$ that has an upper bound in $P\pi$. Then S has a sup in $P\pi$ and sup S = $\cup S$.

Proof. Let t be an upper bound for S in $P\pi$ and $s = \cup S$. Then for every a in S, $s{\downarrow}a = (\cup S){\downarrow}a = \cup\{ b{\downarrow}a \mid b \in S \}$ by 1.4(iii). Now for any b in S, since a, b $\leq$ t, $b{\downarrow}a \subset t{\downarrow}a = a$ by 1.5(i). Hence $s{\downarrow}a \subset \cup\{ a \} = a \subset s$. Therefore by 1.8(i) $a \leq s$, i.e. s is an upper bound for S. Next suppose that u is a given upper bound for S. Then $u{\downarrow}s = u{\downarrow}(\cup S) = \cup\{ u{\downarrow}a \mid a \in S \}$ by 1.4(iii). Here $u{\downarrow}a = a$ since $a \leq u$. Thus, $u{\downarrow}s = \cup\{ a \mid a \in S \} = s$. Therefore

$s \leq u$.

Theorem 3.7. A powerposet $P\pi$ is a complete semilattice.

Proof. Let D be a directed subset of $P\pi$, and $d = \cup D$. Then for every a in D, $d{\downarrow}a = (\cup D){\downarrow}a = \cup\{\, b{\downarrow}a \mid b \in D \,\}$ by 1.4(iii). Here for any b of D, there exists c in D such that $a, b \leq c$ since D is directed. Then for such c, $b{\downarrow}a \subset c{\downarrow}a = a$. Thus, $d{\downarrow}a \subset \cup\{\, a \,\} = a \subset d$. Hence by 1.8(i) $a \leq d$. Therefore by 3.6 $d = \sup D$, i.e. $P\pi$ is up-complete.

Next let S be a nonempty subset of $P\pi$, and let T be the set of all lower bounds for S. Then since S is nonempty, there is an element s of S, and s is an upper bound for T. Thus, by 3.6 T has a sup in $P\pi$. On the other hand, for every a of S since $T \leq a$, we have $\sup T \leq a$. Therefore $\sup T \in T$. Hence $\sup T = \inf S$.

Corollary 3.8. If π is discrete, $P\pi$ is a complete lattice.

Proof. Since $P\pi$ has the greatest element $\pi \in P\pi$, $P\pi$ is a complete lattice by 3.7 and 2.8.

The converse of this corollary also holds.

Proposition 3.9. If $P\pi$ is a complete lattice, π is discrete.

Proof. By 3.6, $\sup P\pi = \cup P\pi = \pi$. Thus, for every a of $P\pi$, $a \leq \pi$. Now assume that $x \leq y$ in π. Then $x \in {\downarrow}y = \pi{\downarrow}\{\, y \,\} = \{\, y \,\}$ since $\{\, y \,\} \leq \pi$. Hence $x = y$. Therefore π is discrete.

Definition 3.10. (i) $B_a = \{\, a{\downarrow}f \mid f \text{ is a finite subset of } a \,\}$.
(ii) $B = \cup\{\, B_a \mid a \in P\pi \,\}$.

Proposition 3.11. (i) B_a is directed.

(ii) $a = \sup B_a$.

Proof. (i) Let F be a finite subset of B_a. Then since $F \subset B_a \leq a$ by 1.10(i), there exists $\sup F = \cup F \in P\pi$ by 3.6. Now let $F = \{ a{\downarrow}f_1, \dots, a{\downarrow}f_n \}$. Then $\sup F = a{\downarrow}(\cup\{ f_1, \dots, f_n \}) \in B_a$ by 1.4(iii). Thus, B_a is directed.

(ii) Since $B_a \leq a$ by 1.10(i),

$\sup B_a = \cup B_a = a{\downarrow}(\cup\{ f \mid f \text{ is finite subset of } a \})$

$= a{\downarrow}a = a$ by 1.4(iii) and (ii).

Proposition 3.12. $a \ll b$ iff there exists a finite subset f of b with $a \leq b{\downarrow}f$.

Proof. If part: Let D be a directed subset of $P\pi$ with $b \leq \sup D$. Then for every $x \in f$, since $x \in f \subset b \subset \sup D = \cup D$, there is $d_x \in D$ such that $x \in d_x$. Thus, for such d_x $b{\downarrow}d_x \subset (\sup D){\downarrow}d_x = d_x$ since $b, d_x \leq \sup D$. Therefore $b{\downarrow}x \subset b{\downarrow}d_x \subset d_x$. Moreover $d_x{\downarrow}(b{\downarrow}x) \subset (\sup D){\downarrow}(b{\downarrow}x) = b{\downarrow}x$ because $b{\downarrow}x \leq b \leq \sup D$ by 1.10(i). Hence $b{\downarrow}x \leq d_x$.

Now, since D is directed and f is finite, $\{ d_x \mid x \in f \}$ has an upper bound d in D.

Then $d{\downarrow}(b{\downarrow}f) = \cup\{ d{\downarrow}(b{\downarrow}x) \mid x \in f \}$

$= \cup\{ b{\downarrow}x \mid x \in f \} = b{\downarrow}f$ since $b{\downarrow}x \leq d_x \leq d$.

Hence $b{\downarrow}f \leq d$. Therefore by the assumption $a \leq d$.

Only if part: By 3.11(ii) $b \leq \sup B_b$. Thus, by the assumption and 3.11(i) there is a finite subset f of b such that $a \leq b{\downarrow}f$.

Proposition 3.13. (i) $B_a = K(P\pi){\downarrow}_{P\pi}a$.

(ii) $B = K(P\pi)$.

Proof. (i) For every $a{\downarrow}f \in B_a$ with a finite subset f of a, $a{\downarrow}f \leq a$ by 1.10(i). Moreover $a{\downarrow}f = (a{\downarrow}f){\downarrow}f$ by 1.6(v). Thus, by 3.12 $a{\downarrow}f \ll a{\downarrow}f$, i.e. $a{\downarrow}f$ is compact. Hence $a{\downarrow}f \in K(P\pi){\downarrow}_{P\pi}a$. Conversely, for every $b \in K(P\pi){\downarrow}_{P\pi}a$ $b \ll b \leq a$. Then by 3.12 there is a finite $f \subset b$ with $b \leq b{\downarrow}f \leq b$. Thus, $b = b{\downarrow}f$. Now since $b \leq a$, we have $a{\downarrow}f \subset a{\downarrow}b = b$. Thus, by 1.6(i) $b{\downarrow}f \subset a{\downarrow}f \subset b{\downarrow}f$. Therefore $b = b{\downarrow}f = a{\downarrow}f \in B_a$.

(ii) Immediate from (i).

Theorem 3.14. A powerposet $P\pi$ is an algebraic semilattice.

Proof. By 3.7, 3.11 and 3.13(i).

Proposition 3.15. If π is discrete, $P\pi$ is an arithmetic lattice.

Proof. That $P\pi$ is an algebraic lattice is clear from 3.8 and 3.14. So we must show that $K(P\pi)$ is a similattice. But by 3.13(ii) $K(P\pi) = \{ f \mid f \text{ is a finite subset of } \pi \}$. Hence every nonempty finite subset $F \subset K(P\pi)$ has an inf $\cap F$ in $K(P\pi)$.

The following example says that $P\pi$ is not always an arithmetic semilattice.

Example 3.16. Let $\pi = \omega \cup \{ \#, \$ \}$ $(\omega = \{ 0, 1, 2, \ldots \})$ in which every order relation is of the form $n \leq \#$ or $n \leq \$$ for some n of ω. Then by 3.13(ii)

$$K(P\pi) = \{ a \mid a \text{ is a finite subset of } \omega \} \cup \{ a \mid \# \in a \subset \pi \} \cup \{ a \mid \$ \in a \subset \pi \},$$

and ${\downarrow}\#$ and ${\downarrow}\$$ are both compact in $P\pi$. But the set of all lower bounds for $\{ {\downarrow}\#, {\downarrow}\$ \}$ in $K(P\pi)$ is

$\{ a \mid a \text{ is a finite subset of } \omega \}$,

and clearly this set has no maximum element. Therefore $K(P\pi)$ is not a semilattice.

Proposition 3.17. A function $\varphi : P\pi \to P\pi'$ is continuous (w.r.t. the Scott topology induced by $\leq$) iff it is monotonic and for every $a \in P\pi$ $\varphi(a) = \cup\{ \varphi(e) \mid e \in B_a \}$.

Proof. Only if part: Immediate because
$\varphi(a) = \sup \{ \varphi(e) \mid e \in B_a \} = \cup\{ \varphi(e) \mid e \in B_a \}$ by 3.6.

If part: For every $e \in B_a$ we have $\varphi(e) \leq \varphi(a)$ since φ is monotonic and $e \leq a$. Thus, the set $\{ \varphi(e) \mid e \in B_a \}$ is upper bounded and its sup is $\cup\{ \varphi(e) \mid e \in B_a \}$ by 3.6. Hence $\varphi(a) = \sup \{ \varphi(e) \mid e \in B_a \}$.

Corollary 3.18. Let $\varphi : \pi \to \pi'$ be a monotonic function. Then the map $P\varphi : P\pi' \to P\pi$ is continuous w.r.t. the Scott topology.

Proof. Since $\varphi^{-1}(\bigcup_i a_i) = \bigcup_i \varphi^{-1}(a_i)$, it is immediate by 3.3 and 3.17.

In the rest of this section we shall show that every continuous poset can be directed-continuously embeddable into its powerposet.

Definition 3.19. For a poset π the function $\varepsilon_\pi : \pi \to P\pi$ is defined by $\varepsilon_\pi(x) = \downarrow x$.

Lemma 3.20. The function ε_π is monotonic.

Proof. For x and y in π with $x \leq y$, $\downarrow x \subset \downarrow y$ by 2.2(ii). Moreover for $z \in (\downarrow y) \downarrow (\downarrow x)$, there is $t \in \downarrow x$ with $z \leq t$. Thus,

$z \leq t \ll x$, which implies $z \in \Downarrow x$. Therefore $(\Downarrow y)\downarrow(\Downarrow x) \subset \Downarrow x$. Hence by 1.8(i) $\varepsilon_\pi(x) = \Downarrow x \leq \Downarrow y = \varepsilon_\pi(y)$.

Theorem 3.21. For a continuous poset π, ε_π is a one to one function preserving directed sups.

Proof. Assume that $\varepsilon_\pi(x) = \varepsilon_\pi(y)$ for some $x, y \in \pi$. Then
$x = \sup \Downarrow x = \sup \varepsilon_\pi(x) = \sup \varepsilon_\pi(y) = \sup \Downarrow y = y$
since π is continuous. Hence ε_π is one to one.

Now let d be a directed subset of π with $z = \sup d$. Then by 3.20 $\varepsilon_\pi(d) = \{ \varepsilon_\pi(x) \mid x \in d \} \leq \varepsilon_\pi(z)$. Hence by 3.6 $\sup \varepsilon_\pi(d)$ exists in $P\pi$ and $\sup \varepsilon_\pi(d) \leq \varepsilon_\pi(z)$. On the other hand, for any $x \in \varepsilon_\pi(z) = \Downarrow z$ $x \ll z = \sup d$. Thus, by 2.4 there is $y \in d$ such that $x \ll y$. Hence $x \in \Downarrow y = \varepsilon_\pi(y) \leq \sup \varepsilon_\pi(d)$. Therefore $\varepsilon_\pi(z) \subset \sup \varepsilon_\pi(d)$. So we can conclude that $\sup \varepsilon_\pi(d) = \varepsilon_\pi(\sup d)$.

4. Powerposets as Lambda Calculus Models

In this section our interests is on the posets with coding functions of their compact elements. We will show that such a poset can be made into a λ-model in a natural way iff it is discrete.

Definition 4.1. A poset $\pi = (\pi, \leq)$ is called <u>self-referential</u> when it is equipped with the two partial functions $p : \pi \to K(P\pi)$ and $q : \pi \to \pi$ that satisfy:

[SR] For every $e \in K(P\pi)$ and $y \in \pi$ there exists $x \in \pi$ such that $p(x) = e$ and $q(x) = y$.

All the posets appeared in this section are self-referential. We will write "$p(x) = e$" or "$q(x) \in a$" instead of writing "p(x) is defined and p(x) = e" or "q(x) is defined and q(x) $\in$ a", and so on.

Definition 4.2. (i) For a, b $\in P\pi$, a.b $\in P\pi$ is defined by

$$a.b = \{ q(x) \mid x \in a \text{ and } p(x) \leq b \}.$$

We write ab and abc for a.b and (a.b).c, respectively.

(ii) For a $\in P\pi$, a function fun(a) : $P\pi \to P\pi$ is defined by fun(a)(b) = ab, i.e. fun(a) is the function represented by a.

(iii) For a function $\varphi : P\pi \to P\pi$, graph(φ) $\in P\pi$ is defined by graph(φ) = $\{ x \mid q(x) \in \varphi(p(x)) \}$.

Note that the binary operator . on a powerposet difined above is exactly correspoding to that of a Plotkin-Scott-algebra (PSE-algebra, in view of Engeler's approach) [3, 6, 9].

So we have the following theorem:

Theorem 4.3. If π is discrete, ($P\pi$, .) can be expanded to a λ-model.

Proof. Since ($P\pi$, .) is a PSE-algebra, it is a well-known result.

Proposition 4.4. For a, b $\in P\pi$,

(i) $ab = \bigcup\{ ae \mid e \in B_b \}$.

(ii) $(\bigcup_{i \in I} a_i)b = \bigcup_{i \in I}(a_i b)$.

Proof. (i) First we show that ae $\subset$ ab for all e $\in B_b$. Let y $\in$ ae. Then there exists x in a such that p(x) $\leq$ e and q(x) = y. But since e $\leq$ b, we have p(x) $\leq$ b. Hence y $\in$ ab.

Conversely for every y $\in$ ab, there exists x $\in$ a such that p(x) $\leq$ b and q(x) = y. Then y $\in$ a(p(x)).

Therefore $ab = \bigcup\{ ae \mid e \in B_b \}$.

(ii) Immediate.

Proposition 4.5. For a function $\varphi : P\pi \to P\pi$ and $a \in P\pi$,

$$(\text{fun} \circ \text{graph})(\varphi)(a) = \bigcup\{ \varphi(e) \mid e \in B_a \}.$$

Proof. $(\text{fun} \circ \text{graph})(\varphi)(a) = \text{graph}(\varphi)a$

$= \{ q(x) \mid x \in \text{graph}(\varphi) \text{ and } p(x) \leq a \}$

$= \{ q(x) \mid q(x) \in \varphi(p(x)) \text{ and } p(x) \leq a \}$

$= \{ y \mid (\exists e \in B_a)\ y \in \varphi(e) \}$ by [SR]

$= \bigcup\{ \varphi(e) \mid e \in B_a \}$.

Theorem 4.6. For a function $\varphi : P\pi \to P\pi$, the following three statements are equivalent:

(1) φ is representable.

(2) For every $a \in P\pi$, $\varphi(a) = \bigcup\{ \varphi(e) \mid e \in B_a \}$.

(3) $\varphi = (\text{fun} \circ \text{graph})(\varphi)$.

Proof. (1) => (2): Let $\varphi = \text{fun}(b)$.

Then $\varphi(a) = ba$ and $\varphi(e) = be$. Thus, (2) holds by 4.4(i).

(2) => (3): By 4.5, for any a of $P\pi$

$$(\text{fun} \circ \text{graph})(\varphi)(a) = \bigcup\{ \varphi(e) \mid e \in B_a \} = \varphi(a).$$

Thus, $(\text{fun} \circ \text{graph})(\varphi) = \varphi$.

(3) => (1): Trivial.

Corollary 4.7. Every continous function from $P\pi$ to $P\pi$ (w.r.t. the Scott topology induced by $\leq$) is representable.

Proof. By 3.17 and 4.6.

Proposition 4.8. The function graph $\circ$ fun is representable.

Proof. For all $a \in P\pi$, $(\mathrm{graph} \circ \mathrm{fun})(a)$

$$= \{ x \mid q(x) \in a(p(x)) \}$$
$$= \{ x \mid q(x) \in \cup\{ e(p(x)) \mid e \in B_a \} \} \text{ by 4.4(ii)}$$
$$= \cup\{ \{ x \mid q(x) \in e(p(x)) \} \mid e \in B_a \}$$
$$= \cup \{ (\mathrm{graph} \circ \mathrm{fun})(e) \mid e \in B_a \}.$$

Therefore by 4.6 graph $\circ$ fun is representable.

Theorem 4.9. A powerposet $(P\pi, .)$ can be expanded to a λ-model iff it is combinatory complete.

Proof. Only if part: Trivial.

If part: By 4.6, 4.8 and 2.10.

Proposition 4.10. There exists $k \in P\pi$ such that for every $a, b \in P\pi$ $kab = a$.

Proof. Let $k = \{ x \mid q(q(x)) \in p(x) \}$. Then

$$ka = \{ q(x) \mid q(q(x)) \in p(x) \text{ and } p(x) \leq a \}$$
$$= \{ y \mid (\exists e \in K(P\pi))\ q(y) \in e \text{ and } e \leq a \} \text{ by [SR]}$$
$$= \{ y \mid q(y) \in a \}.$$

and $kab = \{ q(y) \mid q(y) \in a \text{ and } p(y) \leq b \} = a$ again by [SR].

Although we had the above proposition, there is a self-referential poset whose powerposet is not combinatory complete. Moreover we can show that the converse of Theorem 4.3 is also valid.

Theorem 4.11. If a powerposet $(P\pi, .)$ is combinatory complete, π is discrete.

Proof. By 4.6, for any $a, b, c_1, c_2 \in P\pi$ $c_1 \leq c_2$ implies $a(bc_1) \subset a(bc_2)$ since the function $\lambda c.a(bc)$ is representable.

Now suppose that π is not discrete. Then $\pi \in P\pi$ is not a maximum element by 2.8. Hence there exists a compact element e_1 such that $e_1 \nleq \pi$. Let $e_2 = \pi{\downarrow}e_1$. Then we have

$$e_2 \in K(P\pi),\ e_2 \leq \pi,\ e_1 \subset e_2,\ e_1 \nleq e_2 \text{ and } e_2 \nleq e_1.$$

By [SR] there are x_1 and x_2 such that

$$p(x_1) = e_1,\ p(x_2) = e_2 \text{ and } q(x_1) \neq q(x_2).$$

Put $a = \{ x_1, x_2 \}$,

$$b = \{ x \mid p(x) = \emptyset \text{ and } q(x) \in e_1 \} \cup \{ x \mid p(x) = e_1 \text{ and } q(x) \in e_2 \},$$

$c_1 = \emptyset$ and $c_2 = e_1$.

Then $a(bc_1) = ae_1 = \{ q(x_1) \}$

and $a(bc_2) = a(e_1 \cup e_2) = ae_2 = \{ q(x_2) \}$.

Hence $a(bc_1) \not\subset a(bc_2)$ while $c_1 \leq c_2$.

But this is a contradiction. Therefore π is discrete.

Corollary 4.12. A powerposet $(P\pi, .)$ can be expanded to a λ-model iff π is discrete.

Proof. By 4.3 and 4.11.

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References

[1] H. Barendregt, The Lambda Calculus: Its Syntax and Semantics, (North-Holland, Amsterdam, 1981).