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京都大学
Powerposets

Takanori Adachi
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Department of
Information Science
Tokyo Institute of Technology
Ookayama, Meguro-ku,
Tokyo 152
JAPAN
C-52 Powerposets
by Takanori Adachi, July 1983.

Abstract. We introduce the notion of powerposets which is a natural generalization of that of powersets with inclusion as their partial ordering. We show that every powerposet is an algebraic semilattice and that every continuous poset can be directed-continuously embeddable into some powerposet. We also discuss the possibility of making powerposets into $\lambda$-models as in the case of Plotkin-Scott's $\mathbb{P}_\omega$ theory.
0. Introduction

The domain $P\omega$ introduced by Dana Scott is a very simple and beautiful structure [9]. It provides a universal circumstance to develop theoretical computer science. Nevertheless, to many of computer scientists, $P\omega$ is too large to handle with in their everydays' work. So we want to select other (possibly partially ordered) set for $\omega$. Powerposets are domains constructed in this way.

In section 1 we introduce the notions of lower ends and upper ends in slightly generalized forms of those usually defined.

Section 2 is devoted to review the fundamental concepts of the theories of continuous lattices and $\lambda$-calculus models. The main results of this note are in section 3, including the theorem which says that every powerposet is an algebraic semilattice. As a corollary of this theorem, we can conclude that $P\omega$ is an algebraic lattice as already mentioned by Scott. We also show that for every continuous poset there is an one-one map from it to some powerposet preserving directed sups.

Finally in section 4 we discuss the possibility of expanding a self-referential powerposet to a $\lambda$-calculus model.

1. Lower Ends and Upper Ends

Let $\pi = (\pi, \preceq)$ and $\pi' = (\pi', \preceq')$ be posets, $a, b, c$ subsets of $\pi$, and $x, y, z$ elements of $\pi$ throughout this note.

Definition 1.1. (i) $a \downarrow x = \{ y \in a \mid y \preceq x \}$.

(ii) $\downarrow x = \pi \downarrow x$. 

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(iii) $a \downarrow b = \bigcup \{ a \downarrow x \mid x \in b \}$.
(iv) $\downarrow a = \pi \downarrow a$.
(v) $a \uparrow x = \{ y \in a \mid x \leq y \}$.
(vi) $\uparrow x = \pi \uparrow x$.
(vii) $a \uparrow b = \bigcup \{ a \uparrow x \mid x \in b \}$.
(viii) $\uparrow a = \pi \uparrow a$.

Proposition 1.2.

$$a \downarrow b \subset \bigcap_{i \in I} \bigcap_{j \in J} (a_i \downarrow b_j).$$

Proposition 1.3. If $\pi$ is discrete (i.e. for every $x$ and $y$ in $\pi$ $x \leq y$ implies $x = y$), $a \downarrow b = a \downarrow b = a \uparrow b$.

Proposition 1.4. (i) $a \downarrow \emptyset = \emptyset = a \uparrow \emptyset$.
(ii) $a \subset b$ implies $a \downarrow b = a = a \uparrow b$.
(iii) $(\bigcup_{i \in I} a_i) \downarrow (\bigcup_{j \in J} b_j) = \bigcup_{i \in I} \bigcup_{j \in J} (a_i \downarrow b_j)$.
(iv) $(\bigcup_{i \in I} a_i) \uparrow (\bigcup_{j \in J} b_j) = \bigcup_{i \in I} \bigcup_{j \in J} (a_i \uparrow b_j)$.
(v) $(\bigcap_{i \in I} a_i) \downarrow b = \bigcap_{i \in I} (a_i \downarrow b)$.
(vi) $(\bigcap_{i \in I} a_i) \uparrow b = \bigcap_{i \in I} (a_i \uparrow b)$.

Corollary 1.5. $a \subset a'$ and $b \subset b'$ imply

(i) $a \downarrow b \subset a' \downarrow b'$,
(ii) $a \uparrow b \subset a' \uparrow b'$.

Proposition 1.6. (i) $a \downarrow b \subset c$ implies $a \downarrow b \subset c \downarrow b$.
(ii) $a \uparrow b \subset c$ implies $a \uparrow b \subset c \uparrow b$.
(iii) $a \downarrow (b \uparrow c) \subset a \downarrow c$.
(iv) $a \uparrow (b \uparrow c) \subset a \uparrow c$.
(v) $a \downarrow (a \downarrow b) = a \downarrow b = (a \downarrow b) \downarrow b$.
(vi) $a \uparrow (a \uparrow b) = a \uparrow b = (a \uparrow b) \uparrow b$. 
The proofs of these propositions are very easy, and so left to readers.

Definition 1.7. (i) a is called a lower end of b (notation: \( a \leq_L b \)) when \( b \uparrow a = a \).

(ii) a is called an upper end of b (notation: \( a \leq_U b \)) when \( b \uparrow a = a \).

Lemma 1.8. (i) a is a lower end of b iff \( b \uparrow a \sqsubseteq a \sqsubseteq b \).

(ii) a is an upper end of b iff \( b \uparrow a \sqsubseteq a \sqsubseteq b \).

Proof. (i) If part: \( a = a \uparrow a \) by 1.4(ii)
\[ \sqsubseteq b \uparrow a \] by 1.5(i).

Only if part: \( a = b \uparrow a \sqsubseteq b \) by 1.2.

(ii) Similar to (i).

Proposition 1.9. If \( \pi \) is discrete, the following three statements are equivalent;

(1) a is a lower end of b.

(2) a is a subset of b.

(3) a is an upper end of b.

Proof. By 1.3 and 1.8.

Proposition 1.10. (i) \( a \sqsubseteq b \leq_L a \).

(ii) a \( \leq_L b \) iff there exists a subset c of \( \pi \) such that a = b \( \sqcup c \).

(iii) a \( \sqcup b \leq_U a \).

(iv) a \( \leq_U b \) iff there exists a subset c of \( \pi \) such that a = b \( \sqcup c \).

Proof. (i) By 1.6(v) \( a \downarrow (a \sqcup b) = a \sqcup b \).
(ii) Only if part: Immediate.

If part: By 1.6(v) $b!a = b!(b!c) = b!c = a$.

(iii) Similar to (i).

(iv) Similar to (ii).

Theorem 1.11. Let $a$, $b$ and $b'$ be subsets of $\pi$ with $b \cup b' = a$ and $b \cap b' = \emptyset$. Then $b \preceq_a b'$ iff $b' \preceq a$.

Proof. Since $b = a!b \subseteq a!b \subseteq a$ and $b' = a\cap b' \subseteq a!b' \subseteq a$ by 1.2, we have $a = b!b' \subseteq a!b \cup a!b' \subseteq a!a = a$. Thus, $a!b \cup a!b' = a$. Moreover for every $x$, $x \in a!b \cap a!b'$ implies the existence of $y \in b$ and $z \in b'$ that satisfy

$$z \in b' \cap a!x \subseteq b' \cap a!y \subseteq b' \cap a!b \quad \text{and} \quad y \in b \cap a!x \subseteq a!z \subseteq b \cap a!b'$$

Thus, $b' \cap a!b = \emptyset$ or $b \cap a!b' = \emptyset$ imply $a!b \cap a!b' = \emptyset$.

Only if part: If $a!b = b$, then $b' \cap a!b = b' \cap b = \emptyset$.

Thus, $a!b \cap a!b' = \emptyset$. Hence $a!b' = a - a!b = a - b = b'$.

If part: If $a!b' = b'$, then $b \cap a!b' = b \cap b' = \emptyset$.

Thus, $a!b \cap a!b' = \emptyset$. Hence $a!b = a - a!b' = a - b' = b$.

Corollary 1.12. For $x \in a \subseteq \pi$,

(i) $x$ is maximal in $a$ iff $\{ x \} \preceq_a a$ iff $a - \{ x \} \preceq_a a$.

(ii) $x$ is minimal in $a$ iff $\{ x \} \preceq_a a$ iff $a - \{ x \} \preceq_a a$.

Proof. Immediate from 1.11.

2. Review

In this section we review the fundamental concepts of the theories of continuous lattices and $\lambda$-calculus models.
Definition 2.1. (i) A subset $d$ of $\pi$ is called directed if every finite subset of $d$ has an upper bound in $d$.

(ii) We say that $x$ is way below $y$ (notation: $x \ll y$), if for every directed subset $d$ of $\pi$ the relation $y \leq \sup d$ always implies the existence of $z$ of $d$ with $x \preceq z$.

(iii) $a\downarrow x = \{ y \in a \mid y \ll x \}$.

(iv) $\downarrow x = \pi \downarrow x$.

(v) $a\downarrow b = \cup\{ a\downarrow x \mid x \in b \}$.

(vi) $\downarrow a = \pi \downarrow a$.

(vii) An element $x \in \pi$ is called compact if $x \ll x$.

(viii) $K(\pi) = \{ x \in \pi \mid x$ is compact $\}$.

Note that every directed set is nonempty.

Proposition 2.2. (i) $x \ll y$ implies $x \preceq y$.

(ii) $w \preceq x \ll y \preceq z$ implies $w \ll z$.

(iii) $x = \sup \{ x_1, \ldots, x_n \}$ and $x_i \ll y$ for all $i = 1, \ldots, n$ imply $x \ll y$.

Definition 2.3. (i) A poset $\pi$ is called up-complete if every directed subset of $\pi$ has a sup in $\pi$.

(ii) [Markowski] An up-complete poset $\pi$ is called continuous if for every $x$ in $\pi$, $\downarrow x$ is directed and $x = \sup \downarrow x$.

(iii) [Hoffman] An up-complete poset $\pi$ is called algebraic if for every $x$ in $\pi$, $K(\pi) \downarrow x$ is directed and $x = \sup (K(\pi) \downarrow x)$.

Iwamura and Markowski's result says that we can replace "directed set" by "nonempty chain" in 2.3(i) [5, 7]. Markowski also suggests the thesis that "continuous posets" are the proper setting for an abstract theory of computation.

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[8].

The following two theorems are due to Markowski.

Theorem 2.4. [Interpolation Theorem] Let $\pi$ be a continuous poset, $x \ll y$ in $\pi$, and $d$ a directed subset of $\pi$ with $y \leq \sup d$. Then there exists $z \in d$ such that $x \ll z$.

Theorem 2.5. Every algebraic poset is continuous.

Definition 2.6. (i) A **semilattice** is a poset in which every nonempty finite subset has an inf.

(ii) A **complete semilattice** is an up-complete poset in which every nonempty subset has an inf.

(iii) An **arithmetic semilattice** is an algebraic semilattice $\pi$ in which $\mathcal{K}(\pi)$ is a semilattice.

Note that every complete semilattice $\pi$ has the least element $\inf \pi$.

Definition 2.7. (i) A **lattice** is a semilattice in which every nonempty finite subset has a sup.

(ii) A lattice is called **complete** if every subset has an inf and a sup.

Theorem 2.8. Every complete semilattice with a greatest element is a complete lattice.

Next we state some concepts of the theory of $\lambda$-calculus models.

Definition 2.9. Let $(X, .)$ be a system with a binary operator $.$ on a set $X$, called an **applicative structure**.
(i) \((X, .)\) is called combinatory complete when there are two elements \(k\) and \(s\) in \(X\) such that \(kxy = x\) and \(sxyz = xz(yz)\) for all \(x, y, z \in X\).

(ii) A function \(f : X \to X\) is called representable if there is an element \(x \in X\) such that for every \(y \in X\) \(f(y) = xy\).

(iii) \([X \to X]\) denotes the set of all representable functions on \(X\).

The notion of \(\lambda\)-models is introduced by Barendregt in order to investigate \(\lambda\)-calculus models formally.

The following theorem is due to Barendregt [2].

Theorem 2.10. Let \((X, .)\) be combinatory complete and define the map \(F : X \to [X \to X]\) by \(F(x)(y) = xy\). Then \((X, .)\) can be expanded to a \(\lambda\)-model iff there exists a \(G : [X \to X] \to X\) such that:

1. \(F \circ G = 1_{[X \to X]}\);
2. \(G \cdot F \in [X \to X]\).

Readers may refer to [4] and [1, 2] for further information on these structures.

3. Powerposets

Theorem 3.1. Two relations \(\preceq_L\) and \(\preceq_U\) are partial order relations on \(\mathcal{P} \mathcal{M}\).

Proof. We only prove for the relation \(\preceq_L\); The other case is analogous.

Reflexivity: a \(\preceq_L\) a by 1.4(ii).

Antisymmetry: a \(\preceq_L\) b and b \(\preceq_L\) a imply a \(\subset\) b and b \(\subset\) a by 1.8(i). Thus, a = b.
Transitivity: Assume that \( a \preceq_L b \preceq_L c \). Then by 1.5(i) \( c \uparrow a \preceq c \uparrow b = b \). Thus, by 1.6(i) \( c \uparrow a \preceq b \uparrow a = a \). On the other hand \( a \preceq c \). Hence by 1.8(i) \( a \preceq_L c \).

According to the above theorem we call these structures \((P\pi, \preceq_L)\) and \((P\pi, \preceq_U)\) powerposets.

Corollary 3.2. Let \( \pi \) be a discrete poset. Then
\[
(P\pi, \preceq_L) = (P\pi, \preceq_U) = (P\pi, \preceq).
\]

Proof. By 1.9.

Proposition 3.3. Let \( \mathcal{F} : \pi \rightarrow \pi' \) be a monotonic function. Then the map \( \mathcal{F}^{-1} : P\pi' \rightarrow P\pi \) is also monotonic with respect to each ordering \( \preceq_L \) and \( \preceq_U \).

Proof. Suppose that \( P\pi \) is partially ordered by \( \preceq_L \). Then it is trivial that \( \mathcal{F}^{-1}(a) \preceq \mathcal{F}^{-1}(b) \) if \( a \preceq_L b \) in \( P\pi' \). So it suffices to show that \( \mathcal{F}^{-1}(b) \uparrow \mathcal{F}^{-1}(a) \preceq \mathcal{F}^{-1}(a) \) by 1.8.

Now let \( x \in \mathcal{F}^{-1}(b) \downarrow \mathcal{F}^{-1}(a) \). Then \( x \in \mathcal{F}^{-1}(b) \) and there is \( y \in \mathcal{F}^{-1}(a) \) with \( x \prec y \). Hence \( \mathcal{F}(x) \in b \), \( \mathcal{F}(y) \in a \) and \( \mathcal{F}(x) \preceq \mathcal{F}(y) \) in \( \pi' \) because \( \mathcal{F} \) is monotonic. Thus, \( \mathcal{F}(x) \in b \uparrow a = a \) since \( a \preceq_L b \).

So \( x \in \mathcal{F}^{-1}(a) \). Therefore \( \mathcal{F}^{-1}(b) \downarrow \mathcal{F}^{-1}(a) \preceq \mathcal{F}^{-1}(a) \).

The proof for the ordering \( \preceq_U \) is similar.

Definition 3.4. (i) \textbf{Poset} denotes the category of all posets with all monotonic functions as arrows.

(ii) The contravariant functor \( P_L : \textbf{Poset} \rightarrow \textbf{Poset} \) is defined by
Theorem 3.5. For every poset $\pi$, $P_{U}(\pi) = P_{L}(\pi^{op})$ where $\pi^{op}$ is an opposite poset, considering $\pi$ as a category.

Proof. Immediate because $a \uparrow_{\pi} b = a \downarrow_{\pi^{op}} b$ for all $a, b \in P_{\pi}$.

By the above theorem we can assume that every powerposet is of the form $P_{L}(\pi) = (P_{\pi}, \prec_{L})$ without loss of generality. So in the rest of this note we concentrate on this form, and write $P_{\pi} = (P_{\pi}, \prec)$ instead of writing $P_{L}(\pi) = (P_{\pi}, \prec_{L})$.

Lemma 3.6. Let $S$ be a subset of $P_{\pi}$ that has an upper bound in $P_{\pi}$. Then $S$ has a sup in $P_{\pi}$ and sup $S = U_{S}$.

Proof. Let $t$ be an upper bound for $S$ in $P_{\pi}$ and $s = U_{S}$. Then for every $a$ in $S$, $s \uparrow a = (U_{S}) \uparrow a = U\{ b \downarrow a \mid b \in S \}$ by 1.4(iii). Now for any $b$ in $S$, since $a, b \leq t$, $b \downarrow a \leq t \downarrow a = a$ by 1.5(i). Hence $s \downarrow a \leq U\{ a \} = a \leq s$. Therefore by 1.8(i) $a \preceq s$, i.e. $s$ is an upper bound for $S$. Next suppose that $u$ is a given upper bound for $S$. Then $u \uparrow S = u \uparrow (U_{S}) = U\{ u \downarrow a \mid a \in S \}$ by 1.4(iii). Here $u \downarrow a = a$ since $a \preceq u$. Thus, $u \uparrow S = U\{ a \mid a \in S \} = s$. Therefore
s ≤ u.

Theorem 3.7. A powerposet $P_\pi$ is a complete semilattice.

Proof. Let $D$ be a directed subset of $P_\pi$, and $d = \cup D$. Then for every $a$ in $D$, $\downarrow a = (\cup D) \downarrow a = \cup \{ b \uparrow a \mid b \in D \}$ by 1.4(iii). Here for any $b$ of $D$, there exists $c$ in $D$ such that $a, b \preceq c$ since $D$ is directed. Then for such $c$, $b \uparrow a \subset c \uparrow a = a$. Thus, $\downarrow a \subset \cup \{ a \} = a \subset d$. Hence by 1.8(i) $a \preceq d$. Therefore by 3.6 $d = \sup D$, i.e. $P_\pi$ is up-complete.

Next let $S$ be a nonempty subset of $P_\pi$, and let $T$ be the set of all lower bounds for $S$. Then since $S$ is nonempty, there is an element $s$ of $S$, and $s$ is an upper bound for $T$. Thus, by 3.6 $T$ has a sup in $P_\pi$. On the other hand, for every $a$ of $S$ since $T \preceq a$, we have $\sup T \preceq a$. Therefore $\sup T = \inf S$.

Corollary 3.8. If $\pi$ is discrete, $P_\pi$ is a complete lattice.

Proof. Since $P_\pi$ has the greatest element $\pi \in P_\pi$, $P_\pi$ is a complete lattice by 3.7 and 2.8.

The converse of this corollary also holds.

Proposition 3.9. If $P_\pi$ is a complete lattice, $\pi$ is discrete.

Proof. By 3.6, $\sup P_\pi = \cup P_\pi = \pi$. Thus, for every $a$ of $P_\pi$, $a \preceq \pi$. Now assume that $x \preceq y$ in $\pi$. Then $x \in \downarrow y = \pi \downarrow \{ y \} = \{ y \}$ since $\{ y \} \preceq \pi$. Hence $x = y$. Therefore $\pi$ is discrete.

Definition 3.10. (i) $B_a = \{ a \uparrow f \mid f$ is a finite subset of $a \}$.
(ii) $B = \cup \{ B_a \mid a \in P_\pi \}$.
Proposition 3.11. (i) $B_a$ is directed.

(ii) $a = \text{sup } B_a$.

Proof. (i) Let $F$ be a finite subset of $B_a$. Then since $F \subseteq B_a \leq a$ by 1.10(i), there exists $\text{sup } F = U_F \in P \cap a$ by 3.6. Now let $F = \{ a f_1, \ldots, a f_n \}$. Then $\text{sup } F = a \downarrow (U \{ f_1, \ldots, f_n \}) \in B_a$ by 1.4(iii). Thus, $B_a$ is directed.

(ii) Since $B_a \leq a$ by 1.10(i),

$$\text{sup } B_a = U_{B_a} = a \downarrow (U \{ f \mid f \text{ is finite subset of } a \})$$

$$= a \downarrow a = a$$ by 1.4(iii) and (ii).

Proposition 3.12. $a \ll b$ iff there exists a finite subset $f$ of $b$ with $a \leq b \downarrow f$.

Proof. If part: Let $D$ be a directed subset of $P \cap a$ with $b \leq \text{sup } D$. Then for every $x \in f$, since $x \in f \subseteq b \subseteq \text{sup } D = U_D$, there is $d_x \in D$ such that $x \in d_x$. Thus, for such $d_x$, $b \downarrow d_x \subseteq (\text{sup } D) \downarrow d_x = d_x$ since $b$, $d_x \leq \text{sup } D$. Therefore $b \downarrow x \subseteq b \downarrow d_x \subseteq d_x$. Moreover $d_x \downarrow (b \downarrow x) \subseteq (\text{sup } D) \downarrow (b \downarrow x) = b \downarrow x$ because $b \downarrow x \leq b \leq \text{sup } D$ by 1.10(i). Hence $b \downarrow x \leq d_x$.

Now, since $D$ is directed and $f$ is finite, $\{ d_x \mid x \in f \}$ has an upper bound $d$ in $D$.

Then $d \downarrow (b \downarrow f) = U \{ d \downarrow (b \downarrow x) \mid x \in f \}$

$$= U \{ b \downarrow x \mid x \in f \} = b \downarrow f$$ since $b \downarrow x \leq d_x \leq d$.

Hence $b \downarrow f \leq d$. Therefore by the assumption $a \leq d$.

Only if part: By 3.11(ii) $b \leq \text{sup } B_b$. Thus, by the assumption and 3.11(i) there is a finite subset $f$ of $b$ such that $a \leq b \downarrow f$.

Proposition 3.13. (i) $B_a = K(P \cap a)$.

(ii) $B = K(P \cap a)$. 

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Proof. (i) For every $a \uparrow f \in B_q$ with a finite subset $f$ of $a$, $a \uparrow f \not\leq a$ by 1.10(i). Moreover $a \uparrow f = (a \uparrow f) \downarrow f$ by 1.6(v). Thus, by 3.12 $a \uparrow f \ll a \uparrow f$, i.e. $a \uparrow f$ is compact. Hence $a \uparrow f \in K(P \pi) \downarrow_{P \pi} a$. Conversely, for every $b \in K(P \pi) \downarrow_{P \pi} a$ $b \ll b \leq a$. Then by 3.12 there is a finite $f \subset b$ with $b \leq b \uparrow f \leq b$. Thus, $b = b \uparrow f$. Now since $b \leq a$, we have $a \uparrow f \subset a \uparrow b = b$. Thus, by 1.6(i) $b \uparrow f \subset a \uparrow f \subset b \uparrow f$. Therefore $b = b \uparrow f = a \uparrow f \in B_q$.

(ii) Immediate from (i).

Theorem 3.14. A powerposet $P \pi$ is an algebraic semilattice.

Proof. By 3.7, 3.11 and 3.13(i).

Proposition 3.15. If $\pi$ is discrete, $P \pi$ is an arithmetic lattice.

Proof. That $P \pi$ is an algebraic lattice is clear from 3.8 and 3.14. So we must show that $K(P \pi)$ is a similattice. But by 3.13(ii) $K(P \pi) = \{ f \mid f \text{ is a finite subset of } \pi \}$. Hence every nonempty finite subset $F \subset K(P \pi)$ has an inf $\cap F$ in $K(P \pi)$.

The following example says that $P \pi$ is not always an arithmetic semilattice.

Example 3.16. Let $\pi = \omega \cup \{ \#, \$\} (\omega = \{ 0, 1, 2, \ldots \}) in which every order relation is of the form $n \leq \#$ or $n \leq \$ for some $n$ of $\omega$. Then by 3.13(ii) $K(P \pi) = \{ a \mid a \text{ is a finite subset of } \omega \}$

$$U \{ a \mid \# \in a \cap \pi \} U \{ a \mid \$ \in a \cap \pi \},$$

and $\downarrow \#$ and $\downarrow \$ are both compact in $P \pi$. But the set of all lower bounds for $\{ \downarrow \#, \downarrow \$ \}$ in $K(P \pi)$ is
\{ a \mid a \text{ is a finite subset of } \omega \},

and clearly this set has no maximum element. Therefore \( K(P_\pi) \) is not a semilattice.

Proposition 3.17. A function \( \varphi : P_\pi \to P_{\pi'} \) is continuous (w.r.t. the Scott topology induced by \( \preceq \)) iff it is monotonic and for every \( a \in P_\pi \) \( \varphi(a) = \bigcup \{ \varphi(e) \mid e \in B_a \} \).

Proof. Only if part: Immediate because
\[
\varphi(a) = \sup \{ \varphi(e) \mid e \in B_a \} = \bigcup \{ \varphi(e) \mid e \in B_a \} \text{ by 3.6.}
\]

If part: For every \( e \in B_a \) we have \( \varphi(e) \preceq \varphi(a) \) since \( \varphi \) is monotonic and \( e \preceq a \). Thus, the set \( \{ \varphi(e) \mid e \in B_a \} \) is upper bounded and its sup is \( \bigcup \{ \varphi(e) \mid e \in B_a \} \) by 3.6. Hence \( \varphi(a) = \sup \{ \varphi(e) \mid e \in B_a \} \).

Corollary 3.18. Let \( \varphi : \pi \to \pi' \) be a monotonic function. Then the map \( P_\varphi : P_{\pi'} \to P_\pi \) is continuous w.r.t. the Scott topology.

Proof. Since \( \varphi^{-1}(\bigcup_i a_i) = \bigcup_i \varphi^{-1}(a_i) \), it is immediate by 3.3 and 3.17.

In the rest of this section we shall show that every continuous poset can be directed-continuously embeddable into its powerposet.

Definition 3.19. For a poset \( \pi \) the function \( \xi_\pi : \pi \to P_\pi \) is defined by \( \xi_\pi(x) = \downarrow x \).

Lemma 3.20. The function \( \xi_\pi \) is monotonic.

Proof. For \( x \) and \( y \) in \( \pi \) with \( x \preceq y \), \( \downarrow x \subseteq \downarrow y \) by 2.2(ii). Moreover for \( z \in (\downarrow y) \downarrow (\downarrow x) \), there is \( t \in \downarrow x \) with \( z \preceq t \). Thus,
z \leq t << x, which implies z \in \downarrow x. Therefore (\downarrow y) \downarrow (\downarrow x) \subset \downarrow x. Hence by 1.8(1) \varepsilon_\pi(x) = \downarrow x \leq \downarrow y = \varepsilon_\pi(y).

Theorem 3.21. For a continuous poset \( \pi \), \( \varepsilon_\pi \) is a one to one function preserving directed sups.

Proof. Assume that \( \varepsilon_\pi(x) = \varepsilon_\pi(y) \) for some \( x, y \in \pi \). Then \( x = \sup \downarrow x = \sup \varepsilon_\pi(x) = \sup \varepsilon_\pi(y) = \sup \downarrow y = y \) since \( \pi \) is continuous. Hence \( \varepsilon_\pi \) is one to one.

Now let \( d \) be a directed subset of \( \pi \) with \( z = \sup d \). Then by 3.20 \( \varepsilon_\pi(d) = \{ \varepsilon_\pi(x) \mid x \in d \} \leq \varepsilon_\pi(z) \). Hence by 3.6 \( \sup \varepsilon_\pi(d) \) exists in \( P_\pi \) and \( \sup \varepsilon_\pi(d) \leq \varepsilon_\pi(z) \). On the other hand, for any \( x \in \varepsilon_\pi(z) = \downarrow z \) \( x \ll z = \sup d \). Thus, by 2.4 there is \( y \in d \) such that \( x \ll y \). Hence \( x \in \downarrow y = \varepsilon_\pi(y) \leq \sup \varepsilon_\pi(d) \). Therefore \( \varepsilon_\pi(z) \subset \sup \varepsilon_\pi(d) \). So we can conclude that \( \sup \varepsilon_\pi(d) = \varepsilon_\pi(\sup d) \).

4. Powerposets as Lambda Calculus Models

In this section our interests is on the posets with coding functions of their compact elements. We will show that such a poset can be made into a \( \lambda \)-model in a natural way iff it is discrete.

Definition 4.1. A poset \( \pi = (\pi, \leq) \) is called self-referential when it is equipped with the two partial functions \( p : \pi \rightarrow K(P\pi) \) and \( q : \pi \rightarrow \pi \) that satisfy:

\[ [SR] \text{ For every } e \in K(P\pi) \text{ and } y \in \pi \text{ there exists } x \in \pi \text{ such that } p(x) = e \text{ and } q(x) = y. \]

All the posets appeared in this section are self-referential. We will write "\( p(x) = e \)" or "\( q(x) \in a \)" instead of
writing "p(x) is defined and p(x) = e" or "q(x) is defined and q(x) ∈ a", and so on.

Definition 4.2. (i) For a, b ∈ Pπ, a·b ∈ Pπ is defined by
\[ a·b = \{ q(x) \mid x ∈ a \text{ and } p(x) ≤ b \}. \]

We write ab and abc for a·b and (a·b)·c, respectively.

(ii) For a ∈ Pπ, a function fun(a) : Pπ → Pπ is defined by
\[ \text{fun}(a)(b) = ab, \text{i.e. fun}(a) \text{ is the function represented by } a. \]

(iii) For a function ψ : Pπ → Pπ, graph(ψ) ∈ Pπ is defined by
\[ \text{graph}(ψ) = \{ x \mid q(x) ∈ ψ(p(x)) \}. \]

Note that the binary operator · on a powerposet defined above is exactly corresponding to that of a Plotkin-Scott-algebra (PSE-algebra, in view of Engeler's approach) [3, 6, 9].

So we have the following theorem:

Theorem 4.3. If π is discrete, (Pπ, ·) can be expanded to a λ-model.

Proof. Since (Pπ, ·) is a PSE-algebra, it is a well-known result.

Proposition 4.4. For a, b ∈ Pπ,

(i) \[ ab = \bigcup \{ ae \mid e ∈ B_b \}. \]

(ii) \[ (\bigcup_{i∈I} a_i)b = \bigcup_{i∈I}(a_i b). \]

Proof. (i) First we show that ae ⊆ ab for all e ∈ B_b. Let y ∈ ae. Then there exists x in a such that p(x) ≤ e and q(x) = y. But since e ≤ b, we have p(x) ≤ b. Hence y ∈ ab.

Conversely, for every y ∈ ab, there exists x ∈ a such that p(x) ≤ b and q(x) = y. Then y ∈ a(p(x)).
Therefore \( ab = \bigcup \{ ae \mid e \in B_b \} \).

(ii) Immediate.

Proposition 4.5. For a function \( \Psi : P\alpha \rightarrow P\alpha \) and \( a \in P\alpha \),
\[
(f_{\mathrm{fun}} \circ \text{graph})(\Psi)(a) = \bigcup \{ \Psi(e) \mid e \in B_a \}.
\]

Proof. \( (f_{\mathrm{fun}} \circ \text{graph})(\Psi)(a) = \text{graph}(\Psi)a \)
\[
= \{ q(x) \mid x \in \text{graph}(\Psi) \text{ and } p(x) \leq a \}
= \{ q(x) \mid q(x) \in \Psi(p(x)) \text{ and } p(x) \leq a \}
= \{ y \mid (\exists e \in B_a) \ y \in \Psi(e) \} \text{ by [SR]}
= \bigcup \{ \Psi(e) \mid e \in B_a \}.
\]

Theorem 4.6. For a function \( \Psi : P\alpha \rightarrow P\alpha \), the following three statements are equivalent:

1. \( \Psi \) is representable.
2. For every \( a \in P\alpha \), \( \Psi(a) = \bigcup \{ \Psi(e) \mid e \in B_a \} \).
3. \( \Psi = (f_{\mathrm{fun}} \circ \text{graph})(\Psi) \).

Proof. (1) \( \Rightarrow \) (2): Let \( \Psi = \text{fun}(b) \).
Then \( \Psi(a) = ba \) and \( \Psi(e) = be \). Thus, (2) holds by 4.4(1).

(2) \( \Rightarrow \) (3): By 4.5, for any \( a \) of \( P\alpha \)
\[
(f_{\mathrm{fun}} \circ \text{graph})(\Psi)(a) = \bigcup \{ \Psi(e) \mid e \in B_a \} = \Psi(a).
\]
Thus, \( (f_{\mathrm{fun}} \circ \text{graph})(\Psi) = \Psi \).

(3) \( \Rightarrow \) (1): Trivial.

Corollary 4.7. Every continuous function from \( P\alpha \) to \( P\alpha \) (w.r.t. the Scott topology induced by \( \varepsilon \)) is representable.


Proposition 4.8. The function \( \text{graph} \circ \text{fun} \) is representable.
Proof. For all \( a \in P_\pi \), \((\text{graph} \circ \text{fun})(a)\)

\[
= \{ x \mid q(x) \in a(p(x)) \} \\
= \{ x \mid q(x) \in \bigcup \{ e(p(x)) \mid e \in B_a \} \} \text{ by 4.4(ii)} \\
= \bigcup \{ \{ x \mid q(x) \in e(p(x)) \} \mid e \in B_a \} \\
= \bigcup \{ (\text{graph} \circ \text{fun})(e) \mid e \in B_a \}.
\]

Therefore by 4.6 \( \text{graph} \circ \text{fun} \) is representable.

Theorem 4.9. A powerposet \((P_\pi, .)\) can be expanded to a \( \lambda \)-model iff it is combinatorial complete.

Proof. Only if part: Trivial.

If part: By 4.6, 4.8 and 2.10.

Proposition 4.10. There exists \( k \in P_\pi \) such that for every \( a, b \in P_\pi \) \( k a b = a \).

Proof. Let \( k = \{ x \mid q(q(x)) \in p(x) \} \). Then

\[
ka = \{ q(x) \mid q(q(x)) \in p(x) \text{ and } p(x) \leq a \} \\
= \{ y \mid (\exists e \in K(P_\pi)) q(y) \in e \text{ and } e \leq a \} \text{ by [SR]} \\
= \{ y \mid q(y) \in a \}.
\]

and \( k a b = \{ q(y) \mid q(y) \in a \text{ and } p(y) \leq b \} = a \) again by [SR].

Although we had the above proposition, there is a self-referential poset whose powerposet is not combinatorial complete. Moreover we can show that the converse of Theorem 4.3 is also valid.

Theorem 4.11. If a powerposet \((P_\pi, .)\) is combinatorial complete, \( \pi \) is discrete.

Proof. By 4.6, for any \( a, b, c_1, c_2 \in P_\pi \) \( c_1 \leq c_2 \) implies \( a(b c_1) \leq a(b c_2) \) since the function \( \lambda c.a(b c) \) is representable.
Now suppose that $\pi$ is not discrete. Then $\pi \in P\pi$ is not a maximum element by 2.8. Hence there exists a compact element $e_1$ such that $e_1 \not\leq \pi$. Let $e_2 = \pi \downarrow e_1$. Then we have
$e_2 \in K(P\pi)$, $e_2 \not\leq \pi$, $e_1 \subset e_2$, $e_1 \not\leq e_2$ and $e_2 \not\leq e_1$.
By [SR] there are $x_1$ and $x_2$ such that $p(x_1) = e_1$, $p(x_2) = e_2$ and $q(x_1) \neq q(x_2)$.
Put $a = \{ x_1, x_2 \}$,
$b = \{ x \mid p(x) = \emptyset \text{ and } q(x) \in e_1 \}$
$\cup \{ x \mid p(x) = e_1 \text{ and } q(x) \in e_2 \}$,
$c_1 = \emptyset \text{ and } c_2 = e_1$.
Then $a(bc_1) = ae_1 = \{ q(x_1) \}$
and $a(bc_2) = a(e_1 \cup e_2) = ae_2 = \{ q(x_2) \}$.
Hence $a(bc_1) \not\leq a(bc_2)$ while $c_1 \not\leq c_2$.
But this is a contradiction. Therefore $\pi$ is discrete.

Corollary 4.12. A powerposet $(P\pi, .)$ can be expanded to a $\lambda$-model iff $\pi$ is discrete.

Proof. By 4.3 and 4.11.

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References