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A method of axiomatizing fragments of intuitionistic theories

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The purpose of this paper is to present a method of axiomatizing fragments of intuitionistic theories. The main theorem is an extension of Motohashi's Axiomatization Theorem which concerns classical theories (N. Motohashi: An Axiomatization Theorem, J. Math. Soc. Japan 34 (1982), PP. 551-560).

Let Q be a set of predicate symbols. A formula is Q-atomic if it is an atomic formula with a predicate symbol in Q .

A sequent is a Q-clause if its sequent formulas are either Q-atomic or Q-free.

A set of Q-clauses is Q-closed if it is closed for substitutions, contractions, interchanges and the following inference rules:

$$\frac{\Gamma \rightarrow A \quad A, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta},$$

where A is Q-atomic.

$$\frac{\Gamma \rightarrow P(\bar{s}) \quad P(\bar{t}), \Gamma \rightarrow \Delta}{\bar{s}=\bar{t}, \Gamma \rightarrow \Delta} ,$$

where $P \in Q$.

$$\frac{A_1(s), \dots, A_m(s) \rightarrow B_1(s), \dots, B_n(s)}{s=t, A_1(t), \dots, A_m(t) \rightarrow B_1(t), \dots, B_n(t)} .$$

Proposition 1. Let C be a finite set of Q -clauses whose sequent formulas are quantifier free. Then there is a Q -closed set \bar{C} of Q -clauses such that $C \subseteq \bar{C}$, the theory $LJ_{=}[\bar{C}]$ is equivalent to the theory $LJ_{=} [C]$, and \bar{C} is primitive recursive.

Let P, N and F be mutually disjoint sets of predicate symbols. A formula is a $(P, N, F)_+$ formula if it is P -positive, N -negative and F -free. A formula is a $(P, N, F)_-$ formula if it is $(N, P, F)_+$.

A (P, N, F) -basic sequent is a sequent of the form

$$A_1 \supset B_1, \dots, A_n \supset B_n, \forall \bar{x} C \rightarrow \exists \bar{y} D,$$

where A_1, \dots, A_n and D are disjunctions of conjunctions of formulas which are either $P \cup F$ -atomic or $P \cup N \cup F$ -free, and B_1, \dots, B_n and C are conjunctions of formulas which are $N \cup F$ -atomic or $P \cup N \cup F$ -free.

A sequence of $(P, N, F)_+$ formulas is a $(P, N, F)_+$ -sequence if it has the form

$$\langle A(\Sigma) \mid \Sigma \text{ is a finite sequence of } P \cup F\text{-atomic formulas} \rangle,$$

where each free variable in $A(\Sigma)$ occurs in Σ for each Σ .

A sequence of $(P,N,F)_-$ formulas is a $(P,N,F)_-$ -sequence if it has the form

$\langle A(\Sigma;R) \mid \Sigma \text{ is a finite sequence of } P \cup F\text{-atomic formulas}$
 $\text{and } R \text{ is a } N \cup F\text{-atomic formula} \rangle,$

where each free variable in $A(\Sigma;R)$ occurs in Σ or R .

Let A_+ and A_- be a $(P,N,F)_+$ -sequence and $(P,N,F)_-$ -sequence, respectively. For each finite sequence Σ of $P \cup F$ -atomic formulas, each disjunction D of conjunctions of formulas which are either $P \cup F$ -atomic or $P \cup N \cup F$ -free, and each conjunction C of formulas which are either $N \cup F$ -atomic or $P \cup N \cup F$ -free, we define formulas $A_+(\Sigma,D)$, $A_-(\Sigma;C)$ and $A_-(\Sigma,D;C)$ by the following way:

If C is $P \cup N \cup F$ -free, then $A_-(\Sigma;C)$ is the formula $A_+(\Sigma) \supset C$.

If C is not $P \cup N \cup F$ -free and has the form $C_1 \wedge C_2$, then $A_-(\Sigma;C)$ is the formula $A_-(\Sigma;C_1) \wedge A_-(\Sigma;C_2)$.

If D is not $P \cup N \cup F$ -free and has the form $D_1 \vee D_2$, then $A_+(\Sigma,D)$ and $A_-(\Sigma,D;C)$ are the formulas $A_+(\Sigma,D_1) \vee A_+(\Sigma,D_2)$ and $A_-(\Sigma,D_1;C) \wedge A_-(\Sigma,D_2;C)$, respectively.

If D has the form $\bigwedge \Sigma_0 \wedge A_1 \wedge \bigwedge \Sigma_1 \wedge \dots \wedge A_n \wedge \bigwedge \Sigma_n$, where $\Sigma_0, \Sigma_1, \dots, \Sigma_n$ are sequences of $P \cup F$ -atomic formulas and A_1, \dots, A_n are $P \cup N \cup F$ -free, then $A_+(\Sigma,D)$ and $A_-(\Sigma,D;C)$ are

$$A_+(\Sigma, \Sigma_0, \Sigma_1, \dots, \Sigma_n) \wedge A_1 \wedge \dots \wedge A_n$$

and

$$A_1 \wedge \dots \wedge A_n \supset A_-(\Sigma, \Sigma_0, \Sigma_1, \dots, \Sigma_n; C),$$

respectively.

Let S be a (P, N, F) -basic sequent of the form

$$A_1(\bar{u}) \supset B_1(\bar{u}), \dots, A_n(\bar{u}) \supset B_n(\bar{u}), \forall \bar{x} C(\bar{x}, \bar{u}) \rightarrow \exists \bar{y} D(\bar{y}, \bar{u}),$$

where \bar{u} is the sequence (lined up by a fixed order) of all the variables occurring free in S . Then we denote $A_+(\Sigma; S)$ and $A_-(\Sigma; R; S)$ the following formulas, respectively:

$$\forall \bar{z} \left(\bigwedge_{1 \leq i \leq n} A_-(\Sigma, A_i(\bar{z}); B_i(\bar{z})) \wedge \forall \bar{x} A_-(\Sigma; C(\bar{x}, \bar{z})) \supset \exists \bar{y} A_+(\Sigma, D(\bar{y}, \bar{z})) \right),$$

where no variable in \bar{z} occurs in Σ or S .

$$\exists \bar{z} \left(\bigwedge_{1 \leq i \leq n} A_-(\Sigma, A_i(\bar{z}); B_i(\bar{z})) \wedge \forall \bar{x} A_-(\Sigma; C(\bar{x}, \bar{z})) \wedge \forall \bar{y} A_-(\Sigma, D(\bar{y}, \bar{z}); R) \right),$$

where no variable in \bar{z} occurs in Σ , R or S .

Let \mathcal{S} be a finite set of (P, N, F) -basic sequents. Then we define the pair $(A_+^{\mathcal{S}, k}, A_-^{\mathcal{S}, k})$ of a $(P, N, F)_+$ -sequence and a $(P, N, F)_-$ -sequence for each natural number k as follows:

$$A_+^{\mathcal{S}, 0}(\Sigma) \text{ is } A_+(\Sigma);$$

$$A_-^{\mathcal{S}, 0}(\Sigma; R) \text{ is } A_-(\Sigma; R).$$

$$A_+^{\mathcal{S}, k+1}(\Sigma) \text{ is the formula}$$

$$\bigwedge \{A_+^{\mathcal{S}, k}(\Sigma; S) \mid S \in \mathcal{S}\} \wedge A_+^{\mathcal{S}, k}(\Sigma);$$

$$A_-^{\mathcal{S}, k+1}(\Sigma; R) \text{ is the formula}$$

$$\bigvee \{A_-^{\mathcal{S}, k}(\Sigma; R; S) \mid S \in \mathcal{S}\} \vee (A_+^{\mathcal{S}, k}(\Sigma) \supset A_-^{\mathcal{S}, k}(\Sigma; R)).$$

Theorem 2. Let \mathcal{C} be a $P \cup N \cup F$ -closed set of $P \cup N \cup F$ -clauses and \mathcal{S} a finite set of (P, N, F) -basic sequents. Let A_+ and A_- be

a $(P, N, F)_+$ -sequence and a $(P, N, F)_-$ -sequence, respectively. Let T be a theory with the following properties:

- (1) For each P-atomic formula A , $\frac{}{T} A_+(A) \rightarrow A$.
- (2) For each N-atomic formula A , $\frac{}{T} A \rightarrow A_-(\varepsilon; A)$, where ε is the empty sequence of formulas.
- (3) For each F-atomic formula A , $\frac{}{T} A_-(A; A)$.
- (4) For each clause $\Sigma, \Pi \rightarrow \Delta$ in C , where Σ is a sequence of $P \cup F$ -atomic formulas and Π is a sequence of N-atomic formulas or $P \cup N \cup F$ -free formulas, if Δ is $P \cup N \cup F$ -free or a P-atomic formula, then $\frac{}{T} A_+^{\mathcal{S}, k}(\Sigma), \Gamma \rightarrow \Delta$ for some number k ; if Δ is a $N \cup F$ -atomic formula, then $\frac{}{T} \Pi \rightarrow A_-^{\mathcal{S}, k}(\Sigma; \Delta)$ for some number k .
- (5) For each pair Σ, Σ' of finite sequences of $P \cup F$ -atomic formulas such that each formula in Σ occurs in Σ' , and for each $N \cup F$ -atomic formula R ,

$$\frac{}{T} A_+(\Sigma') \rightarrow A_+(\Sigma),$$

$$\frac{}{T} A_-(\Sigma; R) \rightarrow A_-(\Sigma'; R),$$

and

$$\frac{}{T} \neg A_+(\Sigma) \rightarrow A_-(\Sigma; R).$$

- (6) For each finite sequence Σ of $P \cup F$ -atomic formulas, each $N \cup F$ -atomic formula R and each substitution θ ,

$$\frac{}{T} A_+(\Sigma)\theta \equiv A_+(\Sigma\theta),$$

and

$$\frac{}{T} A_-(\Sigma; R)\theta \equiv A_-(\Sigma\theta; R\theta).$$

Suppose that the language of T contains all the symbols of $LJ_-[C \cup \mathcal{S}]$ other than symbols in F .

Then, for each finite sequent Σ of $P \cup F$ -atomic formulas

and each $(P, N, F)_+$ sequent $\Gamma \rightarrow \Delta$, $\frac{}{LJ_{=[C \cup S]} \Sigma, \Gamma \rightarrow \Delta}$ implies $\frac{}{T A_+^{S, k}(\Sigma), \Gamma \rightarrow \Delta}$ for some number k .

Colloary 3. Let C, S, A_+, A_- and T be the same as in Theorem 1. Moreover suppose that

(7) for each finite sequence Σ of $P \cup F$ -atomic formulas and each $N \cup F$ -atomic formula R ,

$$\frac{}{LJ_{=[C \cup S]} \Sigma \rightarrow A_+(\Sigma)}$$

and

$$\frac{}{LJ_{=[C \cup S]} \Sigma, A_-(\Sigma; R) \rightarrow R},$$

and

(8) $LJ_{=[C \cup S]}$ is an extension of T .

Then, for each $(P, N, F)_+$ sequent $\Gamma \rightarrow \Delta$,

$$\frac{}{T[\{A_+^{S, k}(\Sigma) \mid k \in \omega\}]} \Gamma \rightarrow \Delta \text{ if and only if } \frac{}{LJ_{=[C \cup S]} \Gamma \rightarrow \Delta}.$$

Let C be a set of $P \cup N \cup F$ -clauses. Let Σ be a finite sequence of $P \cup F$ -atomic formulas, and A a $N \cup F$ -atomic formula.

A formula is a positive (P, N, F) -section of Σ for C if it has the form $\bigwedge \Gamma \supset \bigvee \Delta$, where

(i) Γ is a finite sequence of formulas which are either N -atomic or $P \cup N \cup F$ -free,

(ii) Δ consists of at most one P -atomic or $P \cup N \cup F$ -free formula,

(iii) for some subsequence Σ' of Σ the sequent $\Sigma', \Gamma \rightarrow \Delta$ belongs to C ,

(iv) no formula occurs twice in Γ , and

(v) each variable occurring free in $\Gamma \rightarrow \Delta$ occurs in Σ .

A formula is negative (P,N,F)-section of Σ and R for \mathbb{C} if it has the form $\bigwedge \Gamma$, where

(i) Γ is a finite sequence of formulas which are either N-atomic or P \cup N \cup F-free,

(ii) for some subsequence Σ' of Σ the sequent $\Sigma', \Gamma \rightarrow R$ belongs to \mathbb{C} ,

(iii) no formula occurs twice in Γ ,

(iv) each variable occurring free in Γ occurs in Σ or R.

Let $A_+(\Sigma)$ be the following formula:

$$\bigwedge \{A \mid A \text{ is a positive (P,N,F)-section of } \Sigma \text{ for } \mathbb{C}\} \\ \wedge \bigwedge \{B \mid B \text{ is a P-atomic formula occurring in } \Sigma\}.$$

Let $A_-(\Sigma; R)$ be the following formula:

$$A_+(\Sigma) \supset \bigvee \{A \mid A \text{ is a negative (P,N,F)-section of } \Sigma \text{ and } R \text{ for } \mathbb{C}\} \vee R$$

if R is N-atomic;

$$A_+(\Sigma) \supset \bigvee \{A \mid A \text{ is a negative (P,N,F)-section of } \Sigma \text{ and } R \text{ for } \mathbb{C}\} \\ \vee \bigvee \{\bar{s} = \bar{t} \mid P(\bar{t}) \text{ occurs in } \Sigma\}$$

if R is an F-atomic formula of the form $P(\bar{s})$.

The sequences A_+ and A_- are called the canonical (P,N,F) $_+$ -sequence for \mathbb{C} and the canonical (P,N,F) $_-$ -sequence for \mathbb{C} , respectively.

Proposition 4. Let \mathbb{C} be a P \cup N \cup F-closed set of P \cup N \cup F-clauses. Let A_+ and A_- be the canonical (P,N,F) $_+$ -sequence for \mathbb{C} and (P,N,F) $_-$ -sequence for \mathbb{C} , respectively.

(1) If A is a P-atomic formula, then $\frac{}{LJ} A_+(A) \rightarrow A$.

(2) If A is an N -atomic formula, then $\frac{}{LJ} A \rightarrow A_-(\varepsilon; A)$.

(3) If A is an F -atomic formula, then $\frac{}{LJ} A_-(A; A)$.

(4) If $\Sigma, \Pi \rightarrow \Delta$ is a clause in \mathbb{C} , where Σ is a finite sequence of $P \cup F$ -atomic formulas and $\Pi \rightarrow \Delta$ is a $(P, N, F)_+$ sequent, and if Σ^* is a finite sequence of $P \cup F$ -atomic formulas such that each formula in Σ occurs in Σ^* and each variable free in $\Gamma \rightarrow \Delta$ occurs in Σ^* , then $\frac{}{LJ} A_+(\Sigma^*), \Pi \rightarrow \Delta$.

(5) If $\Sigma, \Pi \rightarrow R$ is a clause in \mathbb{C} , where Σ is a finite sequence of $P \cup F$ -atomic formulas, Π is a sequence of N -atomic or $P \cup N \cup F$ -free formulas and R is a $N \cup F$ -atomic formula, and if Σ^* is a finite sequence of $P \cup F$ -atomic formulas such that each formula in Σ occurs in Σ^* and each variable free in Γ occurs in Σ^* or R , then $\frac{}{LJ} \Pi \rightarrow A_-(\Sigma^*; R)$.

(6) If Σ and Σ' is finite sequences of $P \cup F$ -atomic formulas such that each formula in Σ occurs in Σ' , and if R is a $N \cup F$ -atomic formula, then

$$\frac{}{LJ} A_+(\Sigma') \rightarrow A_+(\Sigma),$$

$$\frac{}{LJ} A_-(\Sigma; R) \rightarrow A_-(\Sigma'; R),$$

and

$$\frac{}{LJ} \neg A_+(\Sigma) \rightarrow A_-(\Sigma; R).$$

(7) If Σ is a finite sequence of $P \cup F$ -atomic formulas, R is a $N \cup F$ -atomic formula and θ is a substitution, then

$$\frac{}{LJ} A_+(\Sigma)\theta \equiv A_+(\Sigma\theta),$$

and

$$\frac{}{LJ} A_-(\Sigma; R)\theta \equiv A_-(\Sigma\theta; R\theta).$$

(8) If Σ is a finite sequence of $P \cup F$ -atomic formulas and

R is a $N \cup F$ -atomic formula, then

$$\frac{}{\text{LJ}_=[\mathbb{C}] \Sigma \rightarrow A_+(\Sigma)},$$

and

$$\frac{}{\text{LJ}_=[\mathbb{C}] \Sigma, A_-(\Sigma; R) \rightarrow R}.$$

Corollary 5. Let \mathbb{C} , A_+ and A_- be the same as in Proposition 4. Let \mathbb{S} be a finite set of (P, N, F) -basic sequents.

Suppose that the sequent $\rightarrow \exists \bar{y} P(\bar{x}, \bar{y})$ belongs to \mathbb{S} , where P is a predicate symbol in $P \cup F$ and \bar{x} is not empty. Then, for each

$(P, N, F)_+$ sequent $\Gamma \rightarrow \Delta$, $\frac{}{\text{LJ}_=[\{A_+^{\mathbb{S}, k}(i) \mid k \in \omega\}] \Gamma \rightarrow \Delta}$ if and only if $\frac{}{\text{LJ}_=[\mathbb{C} \cup \mathbb{S}] \Gamma \rightarrow \Delta}$.

Proposition 6. For each finite set \mathbb{A} of sentences there exist a finite set G of new predicate symbols, a finite set \mathbb{C} of $P \cup N \cup F \cup G$ -clauses whose sequent formulas are quantifier free, and a finite set \mathbb{S} of $(P, N, F \cup G)$ -basic sequents such that $\text{LJ}_=[\mathbb{C} \cup \mathbb{S}]$ is conservative over $\text{LJ}_=[\mathbb{A}]$ and the sequent $\rightarrow \exists \bar{y} P(\bar{x}, \bar{y})$ belongs to \mathbb{S} for some predicate symbol P in $P \cup F$, where \bar{x} is not empty.

Axiomatization Theorem. Let L be a first order language without function symbols. Let \mathbb{A} be a finite set of sentences of L . Let P , N , F and G be mutually disjoint sets of predicate symbols with $P \cup N \cup F \subseteq L$ and $L \cap G = \emptyset$. Let \mathbb{C} and \mathbb{S} be a finite set of $P \cup N \cup F \cup G$ -clauses and a finite set of $(P, N, F \cup G)$ -basic sequents, respectively, such that $\text{LJ}_=[\mathbb{C} \cup \mathbb{S}]$ is conservative over $\text{LJ}_=[\mathbb{A}]$, and for some predicate symbol P in $P \cup F \cup G$ the

sequent $\rightarrow \exists \bar{y} P(\bar{x}, \bar{y})$ belongs to \mathcal{S} , where \bar{x} is not empty. Let A_+ and A_- be the canonical $(P, N, F \cup G)_+$ -sequence for \mathcal{C} and the canonical $(P, N, F \cup G)_-$ -sequence for \mathcal{C} , respectively. Then the $(P, N, F)_+$ part of $LJ_{=} [A]$ is axiomatized by the system of axioms $\{A_+^{S, k}(\varepsilon) \mid k \in \omega\}$.