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Countable $J^S_a$-admissible ordinals

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50. Introduction.

In [3], Platek constructs a hierarchy of jumps $J^S_a$ indexed by elements $a$ of a set $0^S$ of ordinal notations. He asserts that a real $X \subseteq \omega$ is recursive in the superjump $S$ if and only if it is recursive in some $J^S_a$. Unfortunately, his assertion is not correct as is shown in [1]. In [1], it also has been shown that an ordinal $>\omega$ is $J^S_a$-admissible if it is $|a|_S$-recursively inaccessible, where $|a|_S$ is the ordinal denoted by $a$.

Let $A$ be an arbitrary set. We say that an ordinal $\alpha$ is $A$-admissible if the structure $<L_\alpha[A], \in, A \cap L_\alpha[A]>$, which we will denote by $L_\alpha[A]$ for simplicity, is admissible, a model of the Kripke-Platek set theory $KP$, where $L_\alpha[A]$ is the sets constructible relative to $A$ in fewer than $\alpha$ steps. We use $\omega^A_1$ or $\omega_1(A)$ to denote the first $A$-admissible ordinal $>\omega$, and use $\omega_1(A_1, \ldots, A_n)$ for $\omega_1(<A_1, \ldots, A_n>)$.

The purpose of this paper is to prove the following theorem.

**Theorem 1.** Suppose $a \in 0^S$ and $\alpha > \omega$ is a countable $|a|_S$-recursively inaccessible ordinal. Then, there exists a real $X \subseteq \omega$ such that $\alpha = \omega_1(J^S_a, X)$.

In the case $|a|_S = 0$, $J^S_a = 2^E$, the Kleene object of type 2, and $\omega_1(2^E, X) = \omega^X_1$ for all reals $X \subseteq \omega$. $\alpha$ is an admissible ordinal if and only if it is $0$-recursively inaccessible. Therefore, Theorem 1 is an extension of the following theorem of Sacks.
Theorem 2. (Sacks [4]). If \( \alpha > \omega \) is a countable admissible ordinal, then there exists a real \( X \) such that \( \alpha = \omega_1^X \).

Sacks also showed that the real \( X \) mentioned in Theorem 2 can be taken to have the minimality property:

\[ \omega_1^Y < \alpha \text{ for every } Y \text{ of lower hyperdegree than } X. \]

Likewise, we can show that for every countable \( |a|_S \)-recursively inaccessible ordinal \( \alpha > \omega \) there is a real \( X \) such that:

\[ \alpha = \omega_1(J_a^S, X); \]

and

\[ \omega_1(J_a^S, Y) < \alpha \text{ for every } Y \text{ of lower } J_a^S \text{-degree than } X. \]

Theorem 1 will be proved by the forcing with \( J_a^S \)-pointed perfect trees. Let \( \alpha > \omega \) be a countable \( |a|_S \)-recursively inaccessible ordinal and \( X \) be a generic real with respect to this forcing relation. Then \( L_\alpha[X] \) is admissible and \( \alpha \leq \omega_1(J_a^S, X) \). To see \( \omega_1(J_a^S, X) < \alpha \), we must show that \( X \) preserves sufficiently many admissible ordinals below \( \alpha \) to make \( \alpha \) to be \( <J_a^S, X>-\)admissible.

\section*{§1. \( |a|_S \)-recursively inaccessible ordinals.}

A normal type 2 object is a total function \( F \) from \( \omega \) to \( \omega \) such that the Kleene object \( \mathcal{Z}_F \) of type 2:

\[ \mathcal{Z}_F = \begin{cases} 0 & \text{if } (\exists n)f(n) = 0, \\ 1 & \text{otherwise}, \end{cases} \]

is recursive in \( F \). The superjump \( S(F) \) of \( F \) is a type 2 object

---

1) For \( J_a^S \)-degrees, the reader may refer to [5].
defined by:

\[ S(F)(<n, f>) = \begin{cases} 
0 & \text{if } \{n\}^F(f) \text{ is defined}, \\
1 & \text{otherwise.} 
\end{cases} \]

Platek [3] defines a hierarchy \( J^S_a \) of type 2 objects along with a set \( O^S \) of ordinal notations, starting from \( 2^\omega \) and iterating the superjump operation transfinitely.

An ordinal \( \alpha \) is \( 0 \)-recursively inaccessible if it is admissible. \( \alpha \) is \( (\sigma + 1) \)-recursively inaccessible if it is \( \sigma \)-recursively inaccessible and a limit of \( \sigma \)-recursively inaccessible ordinals. For limit \( \lambda \), \( \alpha \) is said to be \( \lambda \)-recursively inaccessible if it is \( \sigma \)-recursively inaccessible for all \( \sigma < \lambda \). Let \( X \) be an arbitrary set. \( \sigma \)-recursively-in-\( X \) inaccessible ordinals are defined in the same way starting from \( X \)-admissible ordinals. By \( RI(\sigma, X) \), we denote the class of all \( \sigma \)-recursively-in-\( X \) inaccessible ordinals. In the case \( X = \phi \), \( RI(\sigma, \phi) \) is the class of all \( \sigma \)-recursively inaccessible ordinals.

The following lemma, due to Aczel and Hinman, gives a characterization of \( \omega_1(J^S_a, X) \) for \( X \subseteq \omega \).

**Lemma 3.** (Aczel and Hinman [1]). Suppose \( a \in O^S \) and \( \sigma = |a|_S \), the ordinal denoted by \( a \). Then \( \sigma < \omega_1(J^S_a) \), and for any ordinal \( \alpha > \omega \) and \( X \subseteq \omega \):

\[ \alpha \in RI(\sigma, X) \implies \alpha \text{ is } <J^S_a, X>-\text{admissible.} \]

\( \omega_1(J^S_a, X) \) is the least ordinal in \( RI(\sigma, X) \).

Let \( \lambda_0 \) be the least ordinal \( \lambda \) such that \( \lambda \) is \( \lambda \)-recursively inaccessible. Lemma 3 shows that \( |O^S| = \sup\{|a|_S : a \in O^S\} \leq \lambda_0 \). In [1], it has shown that \( |O^S| = \lambda_0 \).

Let \( \alpha > \omega \) be a countable admissible ordinal. Using the unbounded
Levy forcing over $L_\alpha$, we can add to $L_\alpha$ a generic function $K:(\alpha, \omega) \times \omega \to \alpha$ such that if $\omega \leq \beta < \alpha$ then the function $\lambda n K(\beta, n)$ is a bijection from $\omega$ onto $\beta$. Therefore, in $L_\alpha [K]$ all sets are countable. It has been shown in [4] that $<L_\alpha [K], \in, K>$ is an admissible structure in which $\Sigma_1-DC$ ($\Sigma_1$-Dependent Choice) holds.

Suppose $a \in O^S$. For any $X, Y \subseteq \omega$, $X \leq_{J_a^S} Y$ means $X$ is recursive in $<J_a^S, Y>$, which is equivalent to that $X \in L_\rho [Y]$, where $\rho = \omega_1 (J_a^S, Y)$. $X$ and $Y$ have the same $J_a^S$-degree, $X \equiv_{J_a^S} Y$, if $X \leq_{J_a^S} Y$ and $Y \leq_{J_a^S} X$. $X <_{J_a^S} Y$ if $X \leq_{J_a^S} Y$ but $X \neq_{J_a^S} Y$.

**Lemma 4.** Suppose $\alpha > \omega$ is a countable $|a|_S$-recursively inaccessible ordinal and $K$ is a generic function with respect to the unbounded Levy forcing over $L_\alpha$. Then for any $X, Y \subseteq \omega$:

$$X \leq_{J_a^S} Y \land Y \in L_\alpha [K] \longrightarrow X \in L_\alpha [K].$$

**Proof.** The unbounded Levy forcing preserves admissible ordinals. That is, if $\beta < \alpha$ is an admissible ordinal then $\beta$ is $K$-admissible. This is because for admissible $\beta$, $K \upharpoonright (\beta, \omega) \times \omega$ is generic with respect to the unbounded Levy forcing over $L_\beta$. Therefore, if $Y \in L_\alpha [K]$ then $\alpha$ is $|a|_S$-recursively-in-$Y$ inaccessible, so $L_\rho [Y] \subseteq L_\alpha [K]$, where $\rho = \omega_1 (J_a^S, Y)$. Thus we have the lemma. \[ \Box \]

§2. $J_a^S$-pointed perfect trees.

Let $a$ be an element of $O^S$ such that $|a|_S > 0$. We put $J = J_a^S$ for simplicity.
A perfect tree is a set \( P \) of finite sequences of 0's and 1's such that:

(1) \( p \in P \) \& \( q \subseteq p \rightarrow q \in P \);

and

(2) \( (\forall p \in P)(\exists q, r \in P) (q \text{ and } r \text{ are incompatible extensions of } p) \),

where \( q \subseteq p \) denotes that \( p \) is an extension of \( q \). For a perfect tree \( P \), \([P]\) denotes the set of all infinite paths through \( P \):

\[
[P] = \{ f \in 2^\omega : (\forall n) \bar{f}(n) \in P \}.
\]

We say that \( P \) is \( J \)-pointed if:

(3) \( (\forall f \in [P])(\omega_1(J, P) \leq \omega_1(J, f) \& P \in L_{\omega_1(J, P)}[f]) \).

Note that if \( P \) is \( J \)-pointed then it is \( \leq_J \)-pointed in the sense of Sacks [4: 2.1], but not vice versa.

Lemma 5. Suppose \( P \) is \( J \)-pointed. If \( X \subseteq \omega \) and \( P \leq_J X \), then there exists a \( J \)-pointed \( Q \subseteq P \) such that \( Q =_J X \).

Proof. In [4: 2.3], Sacks constructed a perfect subtree \( Q \) of \( P \) such that:

(4) \( Q \) is recursive in \( P \) and \( f \) for every \( f \in [Q] \);

and

(5) \( Q =_J X \).

To see \( Q \) is \( J \)-pointed in our sense, fix \( f \in [Q] \). Since \( P \) is \( J \)-pointed and \( f \in [P] \), by (3), we have:

(6) \( P \in L_{\omega_1(J, P)}[f] \).

Clearly:

(7) \( f \in L_{\omega_1(J, P)}[f] \).
From (4) (6) and (7), we obtain:

(8) \( Q \in L_{\omega_1}(J,P)[f] \).

From (5) and the assumption \( P \leq J^X \), we see:

(9) \( \omega_1(J,P) \leq \omega_1(J,Q) \).

From (8) and (9), we obtain \( Q \in L_{\omega_1(J,Q)}[f] \).

For any ordinal \( \delta \), \( \{\delta\}_f^P \) denotes the \( \delta \)-th element of \( L[f] \) in the canonical wellordering on \( L[f] \). A perfect tree \( P \) is said to be uniformly \( J \)-pointed if there exists an ordinal \( \delta \) such that:

(10) \( \forall f \in [P] \, (P = \{\delta\}_f^P \& \delta < \omega_1(J,f)) \).

Obviously, uniformly \( J \)-pointed perfect trees are \( J \)-pointed. Let \( \alpha > \omega \) be a countable \( |a|_S \)-recursively inaccessible ordinal and \( K \) a generic function over \( L_\alpha \) in the sense of the unbounded Levy forcing. Observe that if \( P \) is uniformly \( J \)-pointed and \( P \in L_\alpha[K] \) then there exists a \( \delta < \alpha \) which satisfies (10) since the leftmost path \( f_P \) through \( P \) is recursive in \( P \) and so \( \omega_1(J,f_P) \leq \omega_1(J,P) < \alpha \).

Let \( M \) be a countable admissible set and \( P \) be a perfect tree in \( M \). Then \( P \) becomes a partially ordered set as usual. The forcing with \( P \) as the set of conditions is called the local Cohen forcing over \( M \) and denoted by \( \|P\|^M_M \), or simply by \( \|P\|^M \). If \( f \in [P] \) is generic with respect to \( \|P\|^M \), then \( M[f] \) is an admissible set, and so is \( L_{\mu}[f] \), where \( \mu = M \cap \text{On} \).

Lemma 6. For any \( \xi < \alpha \) and any \( J \)-pointed perfect tree \( P \) in \( L_\alpha[K] \), there exists a uniformly \( J \)-pointed perfect tree \( Q \subseteq P \) such that \( \xi < \omega_1(J,Q) \) and \( Q \in L_\alpha[K] \).
Proof. Since $\xi$ is countable in $L_\alpha[K]$, there is a real $X \in L_\alpha[K]$ such that $\xi$ is recursive in $X$. By Lemma 5, there is a $J$-pointed perfect subtree $P_1$ of $P$ such that $P_1 \equiv J X$. Then we see $\xi \lt \omega_1(J, P_1)$, and $P_1 \in L_\alpha[K]$ by Lemma 4. Thus, we may assume $\xi \lt \omega_1(J, P)$ from the beginning. Put $M = L_{\omega_1(J, P)}[P]$. Consider the local Cohen forcing relation $\Vdash_{M}^{P} \varphi$ over $M$. Since $P$ is $J$-pointed, we have:

(11) $(\forall f \in [P]) \omega_1(J, P) \leq \omega_1(J, f)$;

and

(12) $(\forall f \in [P]) (\exists \gamma \lt \omega_1(J, P)) \{ \gamma \}^P = P$.

By (12), there exists a $p_0 \in P$ and $\gamma \lt \omega_1(J, P)$ such that:

(13) $p_0 \Vdash_{M}^{P} \{ \gamma \}^P = P$,

where $\mathcal{J}$ is the canonical name which denotes the generic reals. As in [4:2.10], we can construct a perfect tree $Q \subseteq P$ such that:

(14) $Q \in L_{\omega_1(J, P)}[P]$;

and

(15) $(\forall f \in [Q]) \{ \gamma \}^P = P$.

From (14), we can find a $\delta \lt \omega_1(J, P)$ such that $\{ \delta \}^P = Q$. So, by (15), there is an $\varepsilon \lt \omega_1(J, P)$ such that:

(16) $(\forall f \in [Q]) \{ \varepsilon \}^P = Q$.

Let $f_Q$ be the leftmost branch of $Q$. Then, by (11):

(17) $\omega_1(J, P) \leq \omega_1(J, f_Q) \leq \omega_1(J, Q)$.

Hence, from (16), we see that $Q$ is uniformly $J$-pointed. By (17),
we also see \( \xi < \omega_1(J, Q) \). Since \( P \in L_\alpha[K] \), we have \( \omega_1(J, P) \leq \alpha \), and so \( Q \in L_{\omega_1(J, P)}[P] \leq L_\alpha[K] \).

Let \( \mathcal{L} \) be a first-order language. A \( \Pi^1_1 \) formula in \( \mathcal{L} \) is a second-order formula of the form:

\[
(\forall S_1) \cdots (\forall S_m) \psi,
\]

where \( S_1, \ldots, S_m \) are predicate variables and \( \psi \) is first-order formula in the expanded language \( \mathcal{L} \cup \{S_1, \ldots, S_m\} \).

**Lemma 7.** Suppose \( A \) is a countable admissible set such that \( \omega \in A \) and \( \mathcal{L} \in A \) is a first-order language. Let \( \theta(x_1, \ldots, x_n) \) be a \( \Pi^1_1 \) formula in \( \mathcal{L} \). Then there exists a \( \Sigma_1 \) formula \( \Phi(x_1, \ldots, x_n, y) \) such that for any structure \( \mathcal{M} = \langle M, \ldots \rangle \in A \) for \( \mathcal{L} \) and any \( a_1, \ldots, a_n \in M \):

\[
A \models \Phi(a_1, \ldots, a_n, \mathcal{M}) \iff \mathcal{M} \models \theta(a_1, \ldots, a_n).
\]

**Proof.** This is well-known. See, e.g., Barwise [2: IV.3.1].

Using this lemma, we obtain the following lemma.

**Lemma 8.** The set of all uniformly \( J \)-pointed perfect trees in \( L_\alpha[K] \) is \( \Sigma^1_1 \) over \( L_\alpha[K] \).

**Proof.** Put \( \sigma = |a|_S \), (recall that \( J = \mathcal{J}_a \)). Let \( P \) be a perfect tree in \( L_\alpha[K] \) and \( \delta < \alpha \). Let \( \beta(P, \delta, \sigma) \) denote the least admissible ordinal \( \beta < \alpha \) such that \( \max(\delta, \sigma, \omega) < \beta \) and \( P \in L_\beta[K] \). The function \( \beta \) is \( \Sigma^1_1 \) over \( L_\alpha[K] \). We can easily find a \( \Pi^1_1 \) formula \( \theta \) in the language of set theory such that for any perfect tree \( P \in L_\alpha[K] \):

\[
P \text{ is uniformly } J \text{-pointed } \iff (\exists \delta < \alpha) L_\beta(P, \delta, \sigma)[K] \models \theta(P, \delta, \sigma).
\]

Thus the lemma follows from Lemma 7. \( \square \)
§3. Forcing with uniform $J^{S}_a$-pointed perfect trees.

Suppose $|a|_S > 0$ and put $J = J^{S}_a$. Let $\alpha > \omega$ be a countable $|a|_S$-recursively inaccessible ordinal and $K$ a generic function with respect to the unbounded Levy forcing over $L_\alpha$, which we fix throughout this section.

Let $L(\alpha, \mathcal{T})$ be a ramified language containing names for all members of $L_\alpha[f]$. $L(\alpha, \mathcal{T})$ includes: a numeral $\bar{n}$ for each $n \in \omega$, unranked variables $x, y, z, \cdots$; ranked variables $x^\beta, y^\beta, z^\beta, \cdots$ for each $\beta < \alpha$; and abstraction operator $\hat{\cdot}$. It is intended that $\mathcal{T}$ denotes $\{n \in \omega : f(n) = 1\}$, that $x$ ranges over $L_\alpha[f]$, that $x^\beta$ ranges over $L_\beta[f]$, and that $\hat{x^\beta} \phi(x^\beta)$ denotes the set:

$$\{x \in L_\beta[f] : L_\beta[f] \models \phi(x)\}.$$  

$C(\beta)$ is the set of names for elements of $L_\beta[f]$ and $C = \bigcup_{\beta < \alpha} C(\beta)$.

Let $\mathcal{P}$ denote the set of all uniformly $J$-pointed perfect trees in $L_\alpha[K]$. $P, Q, R, \cdots$ denote the members of $\mathcal{P}$. For a ranked sentence $\phi$ of $L(\alpha, \mathcal{T})$ and $P \in \mathcal{P}$, let $\rho(P, \phi)$ be the least admissible ordinal $\rho < \alpha$ such that $P \in L_\rho[K]$ and rank($\phi$) $< \rho$. The function $\rho$ is $\Sigma_1$ over $L_\alpha[K]$. The forcing relation $P \models \phi$, where $\phi$ is a sentence of $L(\alpha, \mathcal{T})$, is defined inductively:

1. $\phi$ is ranked. $P \models \phi \iff (\forall f \in [P]) L_{\rho(P, \phi)}[f] \models \phi$;

2. $\phi \lor \psi$ is not ranked. $P \models \phi \lor \psi \iff P \models \phi$ or $P \models \psi$;

3. $(\exists x^\beta) \phi(x^\beta)$ is not ranked. $P \models (\exists x^\beta) \phi(x^\beta)$ if $P \models \phi(c)$ for some $c \in C(\beta)$;

4. $P \models (\exists x) \phi(x)$ iff $P \models \phi(c)$ for some $c \in C$;

5. $\phi$ is not ranked. $P \models \neg \phi \iff (\forall Q \subseteq P) \neg (Q \models \phi)$.  

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Using Lemma 7 and 8, it is easy to see that the forcing relation $P \models \phi$, restricted $\Sigma_1$ sentences $\phi$, is $\Sigma_1$ over $L_\alpha[K]$.

**Lemma 9.** For each $P$ and $\phi$, there exists a $Q \subseteq P$ such that $Q \models \phi$ or $Q \models \neg \phi$.

**Proof.** In view of (5), we may assume that $\phi$ is ranked. By Lemma 6, we may also assume that $\phi \in L^\delta[P]$ for some $P$-admissible $\delta$ such that $\delta < \omega_1(J, P)$. Then, in $L^\delta[P]$, all sets are countable. Thus, in $L^\delta[P]$, we can enumerate all ranked sentences of rank $\leq \text{rank}(\phi)$:

$$\phi = \phi_0, \phi_1, \ldots, \phi_n, \ldots \quad (n \in \omega).$$

Let $\|P\|$ be the local Cohen forcing relation over $L^\delta[P]$. In $L^\delta[P]$, we can construct a family $\langle q_s : s \in \text{Seq}(2) \rangle$ of elements of $P$ such that:

(6) $q_s \models^P \phi_n$ or $q_s \models^P \neg \phi_n$, where $n = \text{lh}(s)$;

and

(7) $q_s^{<0}$ and $q_s^{<1}$ are incompatible extensions of $q_s$,

where $\text{Seq}(2)$ is the set of all finite sequences of 0's and 1's.

Let $Q = \{ q \in P : (\exists s) q \subseteq q_s \}$. Then by (7) $Q$ is a perfect subtree of $P$. By (6), it is easy to see that $Q \models \phi$ or $Q \models \neg \phi$. Since $Q \in L^\delta[P]$, $Q = \{ \gamma \}$ for some $\gamma < \delta$. Therefore $Q$ is uniformly $J$-pointed because $P$ is.

A real $f \in 2^\omega$ is said to be generic if for every dense subset $\mathcal{D}$ of $P$ which is definable over $L_\alpha[K]$ there is a $P \in \mathcal{D}$ such that $f \in [P]$. For every $P \in \mathcal{D}$, there is a generic $f$ such that $f \in [P]$. From Lemma 9, it follows that for every generic $f$ and sentence $\phi$:

$$L_\alpha[f] \models \phi \iff (\exists P)(f \in [P] \land P \models \phi).$$
Lemma 10. If \( f \) is generic, then \( L_\alpha[f] \) is admissible.

Proof. We need to show that \( L_\alpha[f] \) satisfies the \( \Delta_0 \) Collection. Let \( \phi(x, y) \) be a formula of \( \mathcal{L}(\alpha, \mathcal{J}) \) with no unranked quantifiers. We claim that if \( P \models (\forall n)(\exists y)\phi(n, y) \) then there exists a \( Q \subseteq P \) and a \( \beta < \alpha \) such that \( Q \models (\forall n)(\exists y^\beta)\phi(n, y^\beta) \). The proof of this claim is almost the same as that of [4: 3.12] with some notational changes. So, we omit the proof here. From the claim, it follows that \( L_\alpha[f] \) satisfies the \( \Delta_0 \) Collection. \qed

Proof of Theorem 1. Let \( \alpha > \omega \) be a countable \( |a|_S \)-recursively inaccessible ordinal and \( K \) be as before. Put \( \sigma = |a|_S \) and \( J = J^S_a \).

In the case \( \sigma = 0 \), Theorem 1 is exactly Theorem 2, which has already been established by Sacks [4]. So we may assume \( \sigma > 0 \). Let \( f_0 \in 2^\omega \) be a generic real over \( L_\alpha[K] \) with respect to the forcing with uniform J-pointed perfect trees. By Lemma 6, for each \( \xi < \alpha \), the set \( \{ P \in \mathcal{P} : \xi < \omega_1(J, P) \} \) is dense in \( \mathcal{P} \). It is obviously definable over \( L_\alpha[K] \). Therefore there is a \( P \in \mathcal{P} \) such that \( f_0 \in [P] \) and \( \xi < \omega_1(J, P) \).

Since \( P \) is J-pointed, we have:

\[ \xi < \omega_1(J, P) \leq \omega_1(J, f_0). \]

Thus, we have \( \alpha \leq \omega_1(J, f_0) \). To see \( \alpha = \omega_1(J, f_0) \), we must show that \( \alpha \in RI(\sigma, f_0) \). At first we consider the case where \( \sigma = \tau + 1 \) for some \( \tau \).

It is sufficient to prove that \( \alpha \) is a limit of ordinals in \( RI(\tau, f_0) \), since then by induction on \( \tau \) we can show that \( \alpha \in RI(\tau, f_0) \), (note that \( \alpha \in RI(0, f_0) \) by Lemma 10). Suppose \( \xi < \alpha \). We shall show that the following set \( \mathcal{D}_\xi \) is dense in \( \mathcal{P} \):

\[ \mathcal{D}_\xi = \{ P \in \mathcal{P} : (\exists \delta < \alpha)(\xi < \delta \quad \& \quad (\forall f \in [P])\delta \in RI(\tau, f)) \}. \]
Assume this can be done. Using Lemma 7, it is easy to see that \( \xi \) is \( \xi_1 \) over \( \lambda_1^\alpha[K] \). Therefore, for every \( \xi < \alpha \), there exists a \( \delta < \alpha \) such that \( \xi < \delta \) and \( \delta \in \text{RI}(\tau, f_0) \).

To show that \( \mathcal{D}_\xi \) is dense in \( \mathcal{G} \), take an arbitrary \( \mathcal{P} \in \mathcal{G} \).

By Lemma 6, we may assume \( \xi < \omega_1(J, \mathcal{P}) \). Take a \( \delta \in \text{RI}(\tau, \mathcal{P}) \) so that \( \xi < \delta < \omega_1(J, \mathcal{P}) \). Such a \( \delta \) exists because \( \omega_1(J, \mathcal{P}) \) is a limit of ordinals in \( \text{RI}(\tau, \mathcal{P}) \). Consider the local Cohen forcing relation \( \Vdash^{\mathcal{P}} \) over \( \lambda_\delta^\alpha[\mathcal{P}] \). Let \( \delta^+ \) be the next \( \mathcal{P} \)-admissible ordinal of \( \delta \). Then, \( \lambda_\delta^\alpha[\mathcal{P}] \) is countable in \( \lambda_{\delta^+}[\mathcal{P}] \). So we can enumerate inside \( \lambda_{\delta^+}[\mathcal{P}] \) all sentences of the appropriate forcing language:

\[
\phi_0, \phi_1, \ldots, \phi_n, \ldots \quad (n \in \omega).
\]

As in the proof of Lemma 9, we can construct a perfect subtree \( \mathcal{Q} \in \lambda_{\delta^+}[\mathcal{P}] \) of \( \mathcal{P} \) such that:

\[(\forall f \in \mathcal{Q}) f \text{ is generic with respect to } \Vdash^{\mathcal{P}}.\]

\( \mathcal{Q} \) is uniformly \( J \)-pointed since \( \mathcal{Q} \in \lambda_{\delta^+}[\mathcal{P}], \delta^+ < \omega_1(J, \mathcal{P}) \) and \( \mathcal{P} \) is uniformly \( J \)-pointed. To show that \( \delta \in \text{RI}(\tau, f) \) for all \( f \in \mathcal{Q} \), take \( f \in \mathcal{Q} \). Let \( \beta < \delta \) be an arbitrary \( \mathcal{P} \)-admissible ordinal \( > \omega \), and \( \Vdash^{\mathcal{P}}_\beta \) be the local Cohen forcing relation over \( \lambda_\beta^\alpha[\mathcal{P}] \). It is easy to see that \( f \) is generic with respect to \( \Vdash^{\mathcal{P}}_\beta \), and so \( \beta \) is \( f \)-admissible.

From this, by induction on \( \tau \), we see that \( \delta \in \text{RI}(\tau, f) \).

Now we consider the case where \( \sigma \) is a limit ordinal. The proof is carried out in the same way. For any \( \xi < \alpha \) and any \( \tau < \sigma \), let \( \mathcal{D}_{\xi^T} \) be the set:

\[
\{ \mathcal{P} \in \mathcal{G} : (\exists \delta < \alpha)(\xi < \delta \quad \& \quad (\forall f \in [\mathcal{P}]) \delta \in \text{RI}(\tau, f) \} \).
\]

Then \( \mathcal{D}_{\xi^T} \) is dense in \( \mathcal{G} \) and definable over \( \lambda_\alpha[K] \). Therefore, we have that \( \alpha = \omega_1(J, f_0) \) for any generic \( f_0 \) with respect to \( \Vdash^\tau \). □
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