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<thead>
<tr>
<th>Title</th>
<th>Countable $J^S_a$-admissible ordinals (LOGIC AND THE FOUNDATIONS OF MATHEMATICS)</th>
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</thead>
<tbody>
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Countable $J_S^a$-admissible ordinals

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§0. Introduction.

In [3], Platek constructs a hierarchy of jumps $J_S^a$ indexed by elements $a$ of a set $0^S$ of ordinal notations. He asserts that a real $X \subseteq \omega$ is recursive in the superjump $S$ if and only if it is recursive in some $J_S^a$. Unfortunately, his assertion is not correct as is shown in [1]. In [1], it also has been shown that an ordinal $\alpha>\omega$ is $J_S^a$-admissible if it is $|a|_S$-recursively inaccessible, where $|a|_S$ is the ordinal denoted by $a$.

Let $A$ be an arbitrary set. We say that an ordinal $\alpha$ is $A$-admissible if the structure $\langle \mathbb{L}_\alpha[A], \in, A \cap \mathbb{L}_\alpha[A] \rangle$, which we will denote by $\mathbb{L}_\alpha[A]$ for simplicity, is admissible, a model of the Kripke-Platek set theory $KP$, where $\mathbb{L}_\alpha[A]$ is the sets constructible relative to $A$ in fewer than $\alpha$ steps. We use $\omega^A_1$ or $\omega_1(A)$ to denote the first $A$-admissible ordinal $\alpha>\omega$, and use $\omega_1(A_1, \ldots, A_n)$ for $\omega_1(\langle A_1, \ldots, A_n \rangle)$.

The purpose of this paper is to prove the following theorem.

Theorem 1. Suppose $a \in 0^S$ and $\alpha>\omega$ is a countable $|a|_S$-recursively inaccessible ordinal. Then, there exists a real $X \subseteq \omega$ such that $\alpha=\omega_1(J_S^a, X)$.

In the case $|a|_S = 0$, $J_S^a = \mathcal{2}E$, the Kleene object of type 2, and $\omega_1(\mathcal{2}E, X) = \omega_1^X$ for all reals $X \subseteq \omega$. $\alpha$ is an admissible ordinal if and only if it is 0-recursively inaccessible. Therefore, Theorem 1 is an extension of the following theorem of Sacks.
Theorem 2. (Sacks [4]). If $\alpha > \omega$ is a countable admissible ordinal, then there exists a real $X$ such that $\alpha = \omega_1^X$.

Sacks also showed that the real $X$ mentioned in Theorem 2 can be taken to have the minimality property:

$$\omega_1^Y < \alpha \text{ for every } Y \text{ of lower hyperdegree than } X.$$ 

Likewise, we can show that for every countable $|a|_S$-recursively inaccessible ordinal $\alpha > \omega$ there is a real $X$ such that:

$$\alpha = \omega_1(J^S_a, X);$$ 

and

$$\omega_1(J^S_a, Y) < \alpha \text{ for every } Y \text{ of lower } J^S_a \text{-degree than } X.\)  \)

Theorem 1 will be proved by the forcing with $J^S_a$-pointed perfect trees. Let $\alpha > \omega$ be a countable $|a|_S$-recursively inaccessible ordinal and $X$ be a generic real with respect to this forcing relation. Then $L_\alpha[X]$ is admissible and $\alpha \leq \omega_1(J^S_a, X)$. To see $\omega_1(J^S_a, X) < \alpha$, we must show that $X$ preserves sufficiently many admissible ordinals below $\alpha$ to make $\alpha$ to be $<J^S_a, X>$-admissible.

§1. $|a|_S$-recursively inaccessible ordinals.

A normal type 2 object is a total function $F$ from $\omega$ to $\omega$ such that the Kleene object $2_E$ of type 2:

$$2_E(f) = \begin{cases} 
0 & \text{if } (\exists n)f(n) = 0, \\
1 & \text{otherwise}, 
\end{cases}$$

is recursive in $F$. The superjump $S(F)$ of $F$ is a type 2 object.

1) For $J^S_a$-degrees, the reader may refer to [5].
defined by:

\[ S(\mathcal{F})(\langle n, f \rangle) = \begin{cases} 0 & \text{if } \{n\}^\mathcal{F}(f) \text{ is defined,} \\ 1 & \text{otherwise.} \end{cases} \]

Platek [3] defines a hierarchy \( J^S_a \) of type 2 objects along with a set \( O^S \) of ordinal notations, starting from \( 2^\mathcal{E} \) and iterating the superjump operation transfinitely.

An ordinal \( \alpha \) is 0-recursively inaccessible if it is admissible. \( \alpha \) is \((\sigma+1)\)-recursively inaccessible if it is \( \sigma \)-recursively inaccessible and a limit of \( \sigma \)-recursively inaccessible ordinals. For limit \( \lambda \), \( \alpha \) is said to be \( \lambda \)-recursively inaccessible if it is \( \sigma \)-recursively inaccessible for all \( \sigma < \lambda \). Let \( X \) be an arbitrary set. \( \sigma \)-recursively-in-\( X \) inaccessible ordinals are defined in the same way starting from \( X \)-admissible ordinals. By \( RI(\sigma, X) \), we denote the class of all \( \sigma \)-recursively-in-\( X \) inaccessible ordinals. In the case \( X = \emptyset \), \( RI(\sigma, \emptyset) \) is the class of all \( \sigma \)-recursively inaccessible ordinals.

The following lemma, due to Aczel and Hinman, gives a characterization of \( \omega_1(J^S_a, X) \) for \( X \subseteq \omega \).

**Lemma 3.** (Aczel and Hinman [1]). Suppose \( a \in O^S \) and \( \sigma = |a|_S \), the ordinal denoted by \( a \). Then \( \sigma < \omega_1(J^S_a) \), and for any ordinal \( \alpha > \omega \) and \( X \subseteq \omega \):

\[ \alpha \in RI(\sigma, X) \implies \alpha \text{ is } \langle J^S_a, X \rangle \text{-admissible.} \]

\( \omega_1(J^S_a, X) \) is the least ordinal in \( RI(\sigma, X) \).

Let \( \lambda_0 \) be the least ordinal \( \lambda \) such that \( \lambda \) is \( \lambda \)-recursively inaccessible. Lemma 3 shows that \(|O^S| = \sup\{|a|_S : a \in O^S\} \leq \lambda_0\). In [1], it has shown that \(|O^S| = \lambda_0\).

Let \( \alpha > \omega \) be a countable admissible ordinal. Using the unbounded
Levy forcing over $L_\alpha$, we can add to $L_\alpha$ a generic function $K: (\alpha - \omega) \times \omega \rightarrow \alpha$ such that if $\omega \leq \beta < \alpha$ then the function $\lambda n K(\beta, n)$ is a bijection from $\omega$ onto $\beta$. Therefore, in $L_\alpha[K]$ all sets are countable. It has been shown in [4] that $<L_\alpha[K], \in, K>$ is an admissible structure in which $\Sigma_1$-DC ($\Sigma_1$-Dependent Choice) holds.

Suppose $a \in O^S$. For any $X, Y \subseteq \omega$, $X \leq_{J_a} Y$ means $X$ is recursive in $<J_a^S, Y>$, which is equivalent to that $X \in L_\rho[Y]$, where $\rho = \omega_1(J_a^S, Y)$.

$X$ and $Y$ have the same $J_a^S$-degree, $X \equiv_{J_a} Y$, if $X \leq_{J_a} Y$ and $Y \leq_{J_a} X$.

$X <_{J_a} Y$ if $X \leq_{J_a} Y$ but $X \neq_{J_a} Y$.

Lemma 4. Suppose $\alpha > \omega$ is a countable $|a|^S_\alpha$-recursively inaccessible ordinal and $K$ is a generic function with respect to the unbounded Levy forcing over $L_\alpha$. Then for any $X, Y \subseteq \omega$:

$$X \leq_{J_a} Y \quad \& \quad Y \in L_\alpha[K] \quad \rightarrow \quad X \in L_\alpha[K].$$

Proof. The unbounded Levy forcing preserves admissible ordinals. That is, if $\beta < \alpha$ is an admissible ordinal then $\beta$ is $K$-admissible.

This is because for admissible $\beta$, $K \upharpoonright (\beta - \omega) \times \omega$ is generic with respect to the unbounded Levy forcing over $L_\beta$. Therefore, if $Y \in L_\alpha[K]$ then $\alpha$ is $|a|^S_\alpha$-recursively-in-$Y$ inaccessible, so $L_\rho[Y] \subseteq L_\alpha[K]$, where $\rho = \omega_1(J_a^S, Y)$. Thus we have the lemma.

§2. $J_a^S$-pointed perfect trees.

Let $a$ be an element of $O^S$ such that $|a|^S > 0$. We put $J = J_a^S$ for simplicity.
A perfect tree is a set $P$ of finite sequences of $0$'s and $1$'s such that:

1. $p \in P$ and $q \subseteq p \rightarrow q \in P$;

and

2. $(\forall p \in P)(\exists q, r \in P)$ (q and r are incompatible extensions of p),

where $q \subseteq p$ denotes that p is an extension of q. For a perfect tree $P$, $[P]$ denotes the set of all infinite paths through $P$:

$$[P] = \{ f \in 2^\omega : (\forall n)\bar{f}(n) \in P \}.$$ 

We say that $P$ is J-pointed if:

3. $(\forall f \in [P])(\omega_1(J, P) \leq \omega_1(J, f) \& P \in L_{\omega_1(J, P)}[f])$.

Note that if $P$ is J-pointed then it is $\leq_J$-pointed in the sense of Sacks [4:2.1], but not vice versa.

**Lemma 5.** Suppose $P$ is J-pointed. If $X \subseteq \omega$ and $P \leq_J X$, then there exists a J-pointed $Q \subseteq P$ such that $Q =_J X$.

**Proof.** In [4:2.3], Sacks constructed a perfect subtree $Q$ of $P$ such that:

4. $Q$ is recursive in $P$ and $f$ for every $f \in [Q]$;

and

5. $Q =_J X$.

To see $Q$ is J-pointed in our sense, fix $f \in [Q]$. Since $P$ is J-pointed and $f \in [P]$, by (3), we have:

6. $P \in L_{\omega_1(J, P)}[f]$.

Clearly:

7. $f \in L_{\omega_1(J, P)}[f]$. 

- 5 -
From (4), (6) and (7), we obtain:

\[(8) \quad Q \in L_{\omega_1(J,P)}[f].\]

From (5) and the assumption \( P \leq J X\), we see:

\[(9) \quad \omega_1(J, P) \leq \omega_1(J, Q).\]

From (8) and (9), we obtain \( Q \in L_{\omega_1(J,Q)}[f]. \)

For any ordinal \( \delta, \{\delta\}^f \) denotes the \( \delta \)-th element of \( L[f] \) in the canonical wellordering on \( L[f] \). A perfect tree \( P \) is said to be uniformly \( J \)-pointed if there exists an ordinal \( \delta \) such that:

\[(10) \quad (\forall f \in \{P\})(P = \{\delta\}^f \& \, \delta < \omega_1(J, f)).\]

Obviously, uniformly \( J \)-pointed perfect trees are \( J \)-pointed. Let \( \alpha > \omega \) be a countable \( |a|_S \)-recursively inaccessible ordinal and \( K \) a generic function over \( L_\alpha \) in the sense of the unbounded Levy forcing. Observe that if \( P \) is uniformly \( J \)-pointed and \( P \in L_\alpha[K] \) then there exists a \( \delta < \alpha \) which satisfies (10) since the leftmost path \( f_P \) through \( P \) is recursive in \( P \) and so \( \omega_1(J, f_P) \leq \omega_1(J, P) < \alpha \).

Let \( M \) be a countable admissible set and \( P \) be a perfect tree in \( M \). Then \( P \) becomes a partially ordered set as usual. The forcing with \( P \) as the set of conditions is called the local Cohen forcing over \( M \) and denoted by \( \mathbb{P}_M \), or simply by \( \mathbb{P} \). If \( f \in \{P\} \) is generic with respect to \( \mathbb{P} \), then \( M[f] \) is an admissible set, and so is \( L_\mu[f] \), where \( \mu = M \cap \text{On} \).

\textbf{Lemma 6.} For any \( \xi < \alpha \) and any \( J \)-pointed perfect tree \( P \) in \( L_\alpha[K] \), there exists a uniformly \( J \)-pointed perfect tree \( Q \leq P \) such that \( \xi < \omega_1(J, Q) \) and \( Q \in L_\alpha[K] \).
Proof. Since $\xi$ is countable in $L_\alpha(K)$, there is a real $X \in L_\alpha(K)$ such that $\xi$ is recursive in $X$. By Lemma 5, there is a $J$-pointed perfect subtree $P_1$ of $P$ such that $P_1 \equiv J X$. Then we see $\xi < \omega_1(J, P_1)$, and $P_1 \in L_\alpha[K]$ by Lemma 4. Thus, we may assume $\xi < \omega_1(J, P)$ from the beginning. Put $M = L_{\omega_1(J, P)}[P]$. Consider the local Cohen forcing relation $\Vdash_M^{P}$ over $M$. Since $P$ is $J$-pointed, we have:

(11) $(\forall f \in [P]) \omega_1(J, P) \leq_1 \omega_1(J, f)$;

and

(12) $(\forall f \in [P]) (\exists \gamma < \omega_1(J, P)) \{\gamma\}^P = P$.

By (12), there exists a $p_0 \in P$ and $\gamma < \omega_1(J, P)$ such that:

(13) $p_0 \Vdash_M^{P} \{\gamma\}^P = P$,

where $\mathcal{J}$ is the canonical name which denotes the generic reals. As in [4:2.10], we can construct a perfect tree $Q \subseteq P$ such that:

(14) $Q \in L_{\omega_1(J, P)}[P]$;

and

(15) $(\forall f \in [Q]) \{f\}^P = P$.

From (14), we can find a $\delta < \omega_1(J, P)$ such that $\{\delta\}^P = Q$. So, by (15), there is an $\varepsilon < \omega_1(J, P)$ such that:

(16) $(\forall f \in [Q]) \{\varepsilon\}^P = Q$.

Let $f_Q$ be the leftmost branch of $Q$. Then, by (11):

(17) $\omega_1(J, P) \leq \omega_1(J, f_Q) \leq \omega_1(J, Q)$.

Hence, from (16), we see that $Q$ is uniformly $J$-pointed. By (17),
we also see $\xi < \omega_1(J, Q)$. Since $P \in L_\alpha[K]$, we have $\omega_1(J, P) \leq \alpha$, and so $Q \in L_{\omega_1(J, P)}[P] \subseteq L_\alpha[K]$. Let $\mathcal{L}$ be a first-order language. A $\Pi^1_1$ formula in $\mathcal{L}$ is a second-order formula of the form:

$$(\forall S_1) \cdots (\forall S_m) \psi,$$

where $S_1, \ldots, S_m$ are predicate variables and $\psi$ is first-order formula in the expanded language $\mathcal{L} \cup \{S_1, \ldots, S_m\}$.

**Lemma 7.** Suppose $A$ is a countable admissible set such that $\omega \in A$ and $\mathcal{L} \in A$ is a first-order language. Let $\theta(x_1, \ldots, x_n)$ be a $\Pi^1_1$ formula in $\mathcal{L}$. Then there exists a $\Sigma^1_1$ formula $\phi(x_1, \ldots, x_n, y)$ such that for any structure $\mathcal{M} = \langle M, \ldots \rangle \in A$ for $\mathcal{L}$ and any $a_1, \ldots, a_n \in M$:

$$A \vdash \phi(a_1, \ldots, a_n, \mathcal{M}) \iff \mathcal{M} \models \theta(a_1, \ldots, a_n).$$

**Proof.** This is well-known. See, e.g., Barwise [2: IV. 3.1].

Using this lemma, we obtain the following lemma.

**Lemma 8.** The set of all uniformly $J$-pointed perfect trees in $L_\alpha[K]$ is $\Sigma^1_1$ over $L_\alpha[K]$.

**Proof.** Put $\sigma = \vert a \vert_S$, (recall that $J = J^S_\alpha$). Let $P$ be a perfect tree in $L_\alpha[K]$ and $\delta < \alpha$. Let $\beta(P, \delta, \sigma)$ denote the least admissible ordinal $\beta < \alpha$ such that $\max(\delta, \sigma, \omega) < \beta$ and $P \in L_\beta[K]$. The function $\beta$ is $\Sigma^1_1$ over $L_\alpha[K]$. We can easily find a $\Pi^1_1$ formula $\theta$ in the language of set theory such that for any perfect tree $P \in L_\alpha[K]$:

$$P \text{ is uniformly } J \text{-pointed } \iff (\exists \delta < \alpha) L_\beta(P, \delta, \sigma)[K] \models \theta(P, \delta, \sigma).$$

Thus the lemma follows from Lemma 7.
§3. Forcing with uniform $J^S_a$-pointed perfect trees.

Suppose $|a|^S > 0$ and put $J = J^S_a$. Let $\alpha > \omega$ be a countable $|a|^S$-recursively inaccessible ordinal and $K$ a generic function with respect to the unbounded Levy forcing over $L_\alpha$, which we fix throughout this section.

Let $\mathcal{L}(\alpha, \mathcal{T})$ be a ramified language containing names for all members of $L_\alpha[f]$. $\mathcal{L}(\alpha, \mathcal{T})$ includes: a numeral $n$ for each $n \in \omega$, unranked variables $x, y, z, \ldots$; ranked variables $x^\beta, y^\beta, z^\beta, \ldots$ for each $\beta < \alpha$; and abstraction operator $^\phi$. It is intended that $\mathcal{T}$ denotes $\{n \in \omega: f(n) = 1\}$, that $x$ ranges over $L_\alpha[f]$, that $x^\beta$ ranges over $L_\beta[f]$, and that $^\phi(x^\beta)$ denotes the set:

$$\{x \in L_\beta[f] : L_\beta[f] \models \phi(x)\}.$$

$C(\beta)$ is the set of names for elements of $L_\beta[f]$ and $C = \bigcup_{\beta<\alpha} C(\beta)$.

Let $\mathcal{F}$ denote the set of all uniformly $J$-pointed perfect trees in $L_\alpha[K]$. $P, Q, R, \ldots$ denote the members of $\mathcal{F}$. For a ranked sentence $\phi$ of $\mathcal{L}(\alpha, \mathcal{T})$ and $P \in \mathcal{F}$, let $\rho(P, \phi)$ be the least admissible ordinal $\rho < \alpha$ such that $P \in L_\rho[K]$ and rank($\phi$) $\leq \rho$. The function $\rho$ is $\Sigma_1$ over $L_\alpha[K]$. The forcing relation $P \models \phi$, where $\phi$ is a sentence of $\mathcal{L}(\alpha, \mathcal{T})$, is defined inductively:

1. $\phi$ is ranked. $P \models \phi$ iff $(\forall \phi \in [P]) L_{\rho(P, \phi)}[f] \models \phi$;

2. $\phi \lor \psi$ is not ranked. $P \models \phi \lor \psi$ iff $P \models \phi$ or $P \models \psi$;

3. $(\exists x^\beta)\phi(x^\beta)$ is not ranked. $P \models (\exists x^\beta)\phi(x^\beta)$ if $P \models \phi(c)$ for some $c \in C(\beta)$;

4. $P \models (\exists x)\phi(x)$ iff $P \models \phi(c)$ for some $c \in C$;

5. $\phi$ is not ranked. $P \models \neg \phi$ iff $(\forall \psi \subseteq P) \neg (Q \models \phi)$.
Using Lemma 7 and 8, it is easy to see that the forcing relation $P \models \phi$, restricted $\Sigma_1$ sentences $\phi$, is $\Sigma_1$ over $L_\alpha[K]$.

**Lemma 9.** For each $P$ and $\phi$, there exists a $Q \subseteq P$ such that $Q \models \phi$ or $Q \models \neg \phi$.

**Proof.** In view of (5), we may assume that $\phi$ is ranked. By Lemma 6, we may also assume that $\phi \in L_\delta[P]$ for some $P$-admissible $\delta$ such that $\delta < \omega_1(J, P)$. Then, in $L_\delta[P]$, all sets are countable. Thus, in $L_\delta[P]$, we can enumerate all ranked sentences of rank $\leq \text{rank}(\phi)$:

$$\phi = \phi_0, \phi_1, \ldots, \phi_n, \ldots \ (n \in \omega).$$

Let $\models_P^P$ be the local Cohen forcing relation over $L_\delta[P]$. In $L_\delta[P]$, we can construct a family $\langle q_s : s \in \text{Seq}(2) \rangle$ of elements of $P$ such that:

(6) $q_s \models_P^P \phi_n$ or $q_s \models_P \neg \phi_n$, where $n = \text{lh}(s)$; and

(7) $q_s^{<0}$ and $q_s^{<1}$ are incompatible extensions of $q_s$,

where $\text{Seq}(2)$ is the set of all finite sequences of 0's and 1's.

Let $Q = \{ q \in P : (\exists s) q \subseteq q_s \}$. Then by (7) $Q$ is a perfect subtree of $P$. By (6), it is easy to see that $Q \models \phi$ or $Q \models \neg \phi$. Since $Q \in L_\delta[P]$, $Q = \{ \gamma \}^P$ for some $\gamma < \delta$. Therefore $Q$ is uniformly $J$-pointed because $P$ is.

A real $f \in \omega$ is said to be generic if for every dense subset $\mathcal{D}$ of $\mathcal{P}$ which is definable over $L_\alpha[K]$ there is a $P \in \mathcal{D}$ such that $f \in [P]$. For every $P \in \mathcal{P}$, there is a generic $f$ such that $f \in [P]$. From Lemma 9, it follows that for every generic $f$ and sentence $\phi$:

$$L_\alpha[f] \models \phi \text{ iff } (\exists P)(f \in [P] \& P \models \phi).$$
Lemma 10. If \( f \) is generic, then \( L_\alpha[f] \) is admissible.

Proof. We need to show that \( L_\alpha[f] \) satisfies the \( \Delta_0 \) Collection. Let \( \phi(x, y) \) be a formula of \( L(\alpha, \mathcal{J}) \) with no unranked quantifiers. We claim that if \( P \models (\forall n)(\exists y)\phi(n, y) \) then there exists a \( Q \subseteq P \) and a \( \beta < \alpha \) such that \( Q \models (\forall n)(\exists y^\beta)\phi(n, y^\beta) \). The proof of this claim is almost the same as that of [4: 3.12] with some notational changes. So, we omit the proof here. From the claim, it follows that \( L_\alpha[f] \) satisfies the \( \Delta_0 \) Collection. \( \square \)

Proof of Theorem 1. Let \( \alpha > \omega \) be a countable \( |a|_S \)-recursively inaccessible ordinal and \( K \) be as before. Put \( \sigma = |a|_S \) and \( J = J^S_a \).

In the case \( \sigma = 0 \), Theorem 1 is exactly Theorem 2, which has already been established by Sacks [4]. So we may assume \( \sigma > 0 \). Let \( f_0 \in 2^\omega \) be a generic real over \( L_\alpha[K] \) with respect to the forcing with uniform J-pointed perfect trees. By Lemma 6, for each \( \xi < \alpha \), the set \( \{P \in \mathcal{P} : \xi < \omega_1(J, P) \} \) is dense in \( \mathcal{P} \). It is obviously definable over \( L_\alpha[K] \). Therefore there is a \( P \in \mathcal{P} \) such that \( f_0 \in [P] \) and \( \xi < \omega_1(J, P) \).

Let \( P \) be J-pointed, we have:

\[ \xi < \omega_1(J, P) \leq \omega_1(J, f_0). \]

Thus, we have \( \alpha \leq \omega_1(J, f_0) \). To see \( \alpha = \omega_1(J, f_0) \), we must show that \( \alpha \in \text{RI}(\sigma, f_0) \). At first we consider the case where \( \sigma = \tau + 1 \) for some \( \tau \).

It is sufficient to prove that \( \alpha \) is a limit of ordinals in \( \text{RI}(\tau, f_0) \), since then by induction on \( \tau \) we can show that \( \alpha \in \text{RI}(\tau, f_0) \), (note that \( \alpha \in \text{RI}(0, f_0) \) by Lemma 10). Suppose \( \xi < \alpha \). We shall show that the following set \( D_\xi \) is dense in \( \mathcal{P} \):

\[ D_\xi = \{ P \in \mathcal{P} : (\exists \delta < \alpha)(\xi < \delta \ \& \ (\forall f \in [P])\delta \in \text{RI}(\tau, f) \} \].

- 11 -
Assume this can be done. Using Lemma 7, it is easy to see that $\mathcal{D}_\xi$ is $\Sigma_1$ over $L_\alpha[K]$. Therefore, for every $\xi < \alpha$, there exists a $\delta < \alpha$ such that $\xi < \delta$ and $\delta \in \text{RI}(\tau, f_0)$.

To show that $\mathcal{D}_\xi$ is dense in $\mathcal{P}$, take an arbitrary $P \in \mathcal{P}$. By Lemma 6, we may assume $\xi < \omega_1(J, P)$. Take a $\delta \in \text{RI}(\tau, P)$ so that $\xi < \delta < \omega_1(J, P)$. Such a $\delta$ exists because $\omega_1(J, P)$ is a limit of ordinals in $\text{RI}(\tau, P)$. Consider the local Cohen forcing relation $|\cdot|^P$ over $L_\delta[P]$. Let $\delta^+$ be the next $P$-admissible ordinal of $\delta$. Then, $L_\delta[P]$ is countable in $L_{\delta^+}[P]$. So we can enumerate inside $L_{\delta^+}[P]$ all sentences of the appropriate forcing language:

$$\phi_0, \phi_1, \ldots, \phi_n, \ldots \quad (n \in \omega).$$

As in the proof of Lemma 9, we can construct a perfect subtree $Q \in L_{\delta^+}[P]$ of $P$ such that:

$$(\forall f \in [Q]) f \text{ is generic with respect to } |\cdot|^P.$$  
Q is uniformly $J$-pointed since $Q \in L_{\delta^+}[P]$, $\delta^+ < \omega_1(J, P)$ and $P$ is uniformly $J$-pointed. To show that $\delta \in \text{RI}(\tau, f)$ for all $f \in [Q]$, take $f \in [Q]$. Let $\beta < \delta$ be an arbitrary $P$-admissible ordinal $> \omega$, and $|\cdot|^P_{\beta}$ be the local Cohen forcing relation over $L_\beta[P]$. It is easy to see that $f$ is generic with respect to $|\cdot|^P_{\beta}$, and so $\beta$ is $f$-admissible.

From this, by induction on $\tau$, we see that $\delta \in \text{RI}(\tau, f)$.

Now we consider the case where $\sigma$ is a limit ordinal. The proof is carried out in the same way. For any $\xi < \alpha$ and any $\tau < \sigma$, let $\mathcal{D}_{\xi \tau}$ be the set:

$$\{P \in \mathcal{P} : (\exists \delta < \alpha)(\xi < \delta \& (\forall f \in [P]) \delta \in \text{RI}(\tau, f)).\}$$

Then $\mathcal{D}_{\xi \tau}$ is dense in $\mathcal{P}$ and definable over $L_\alpha[K]$. Therefore, we have that $\alpha = \omega_1(J, f_0)$ for any generic $f_0$ with respect to $|\cdot|$. $\square$
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