Title	A normal form theorem for first order formulas and its application to Gaifman's splitting theorem(LOGIC AND THE FOUNDATIONS OF MATHEMATICS)
Author(s)	MOTOHASHI, Nobuyoshi
Citation	数理解析研究所講究録 (1984), 516: 40-52
Issue Date	1984-03
URL	http://hdl.handle.net/2433/98389
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
516巻 1984年 40-52

A normal form theorem for first order formulas and its application to Gaifman's splitting theorem

By

Nobuyoshi MOTOHASHI
(本 橋 信 義)

Let L be a first order predicate calculus with equality which has a fixed binary predicate symbol $<$. In this paper, we shall deal with quantifiers $\overset{\bullet}{\forall} x$, $\forall x \leqq y$, $\exists x \leqq y$ defined by ; $\overset{\bullet}{\forall} x A(x)$ is $\forall y \exists x (y \leqq x \wedge A(x))$, $\forall x \leqq y A(x)$ is $\forall x (x \leqq y \supset A(x))$, and $\exists x \leqq y A(x)$ is $\exists x (x \leqq y \wedge A(x))$. The expressions $\bar{x}$, $\bar{y}$, ... will be used to denote sequences of those symbols, e.g. $\bar{x}$ is $\langle x_1, x_2, \ldots, x_n \rangle$ and $\bar{y}$ is $\langle y_1, y_2, \ldots, y_m \rangle$. Also, $\exists \bar{x}$, $\forall \bar{x} \leqq \bar{y}$, ... will be used to denote $\exists x_1 \exists x_2 \ldots . \exists x_n$, $\forall x_1 \leqq y_1 \forall x_2 \leqq y_2 \ldots \forall x_n \leqq y_n$, ..., respectively. Let X be a set of formulas in L such that; X contains every atomic formulas and is closed under substitutions of free variables and applications of propositional connectives $\neg$(not), $\wedge$ (and), $\vee$ (or). Then, $\Sigma(X)$ is the set of formulas of the form $\exists \bar{x} B(\bar{x})$, where $B \in X$, and $\Phi(X)$ is the set of formulas of the form;

$$\overset{\bullet}{\forall} x_1 \exists y_1 \ldots\ldots \overset{\bullet}{\forall} x_n \exists y_n \exists \bar{z} \forall \bar{u} \leqq \bar{x} \exists \bar{v} \leqq \bar{z} B(\bar{u}, \bar{y}, \bar{v}), \text{ where } B(\bar{x}, \bar{y}, \bar{z}) \in X .$$

Since X is closed under $\wedge$, $\vee$, two sets $\Sigma(X)$ and $\Phi(X)$ are closed under $\wedge$, $\vee$ in the following sense: for any formulas A and B in $\Sigma(X)$ [$\Phi(X)$], there are formulas in $\Sigma(X)$ [$\Phi(X)$] which are obtained from $A \wedge B$ and $A \vee B$ by prefixing some quantifiers in them in the usual manner.

Let $W(x,y,z)$ be a formula in $\Sigma(X)$ which has no free variables except x,y,z. Then, the theory T_W in L consists of the following sentences;

Tr : $\forall x \forall y \leq x \forall z \leq y (z \leq x)$,

Ex(W) : $\forall x \forall y \exists z W(x,y,z)$,

Un(W) : $\forall x \forall y \forall z \forall w (W(x,y,z) \wedge W(x,y,w) . \supset z=w)$,

Bn(W) : $\forall x \forall y \forall z (W(x,y,z) \supset z \leq x)$,

and

Col(W) : $\forall \bar{w} [\forall u \leq x \exists v A(u,v,\bar{w}) \supset \exists y \forall u \leq x \exists v (W(y,u,v) \wedge A(u,v,\bar{w}))]$,

where $A(x,v,\bar{w})$ is a formula in L.

Since $Col(W)$ is a schema, T_W is an infinite set of sentences. A mapping f from a set of formulas in L (domain of f) on to a set of formulas in L (range of f) is called a formula-mapping if $f(A)$ and A have the same set of free variables for each formula A in the domain of f.

In this paper, we shall give a concrete method to construct a formula-mapping f_W whose domain is the set of formulas in L and whose range is a subset of $\Phi(X)$, and prove the following fact.

THEOREM A. For any formula A in L, the formula $A \supset f_W(A)$ is provable from T_W in L, and the formula $f_W(A) \supset A$ is provable in L, i.e. $T_W \vdash_L A \supset f_W(A)$ and $\vdash_L f_W(A) \supset A$.

This theorem shows that any formula A in L is equivalent to $f_W(A)$ in $\Phi(X)$ with respect to the theory T_W, and furthermore the implication from $f_W(A)$ to A is provable logically. This is a normal form theorem for first order formulas, of a new type.

In §1 below, we shall show some applications of Theorem A above and one of which, Corollary E below, is a generalization of Gaifman's splitting theorem, Corollary G below, in Gaifman [1].

The construction of f_W requires two auxiliary formula-mapping h and g_W, where h is a formula-mapping whose domain is the set of formulas (denoted by $\Pi_2(X)$) of the form $\forall\bar{x}\exists\bar{y}B$, $B \in X$, and whose range is a subset of $\Phi(X)$, and g_W is a formula-mapping whose domain is the set of formulas in L and whose range is a subset of $\Phi(X)$. Moreover, we can prove;

LEMMA 1. For any formula A in $\Pi_2(X)$, the formula $A \supset h(A)$ is provable from T_W in L and the formula $h(A) \supset A$ is provable in L, i.e. $T_W \vdash_L A \supset h(A)$ and $\vdash_L h(A) \supset A$.

LEMMA 2. For any formula A in L, the formula $A \supset g_W(A)$ is provable from T_W in L and the formula $g_W(A) \supset A$ is provable from $Tr, Ex(W), Un(W)$ in L, i.e. $T_W \vdash_L A \supset g_W(A)$ and $Tr, Ex(W), Un(W) \vdash_L g_W(A) \supset A$.

Although Tr, $Ex(W)$, $Un(W)$ are not formulas in $\Pi_2(X)$, there are formulas Tr°, $Ex(W)^\circ$, $Un(W)^\circ$ in $\Pi_2(X)$ which are obtained from Tr, $Ex(W)$, $Un(W)$ by prefixing some quantifiers in them, respectively. Therefore, they are equivalent each other. Let $f_W(A)$ be one of formulas in $\Phi(X)$ which are obtained from $g_W(A) \wedge h(Tr^\circ) \wedge h(Ex(W)^\circ) \wedge h(Un(W)^\circ)$ by prefixing some quantifiers in it. Then, Theorem A clearly holds from Lemma 1 and

and Lemma 2. So, in order to prove Theorem A above, it is sufficient to construct h and g_W and prove Lemma 1 and Lemma 2 , which will be done in §2 below.

1

§1. Some applications.

In this section, we shall show some applications of Theorem A to normal form theorems and splitting theorems. From Theorem A, we have the following fact, immediately.

COROLLARY B. It T is a theory in L such that $T_W \subseteq T$ for some formula W in $\Sigma(X)$, then for any formula A in L, there is a formula B in $\Phi(X)$ such that $T \vdash_L A \supset B$ and $\vdash_L B \supset A$.

If T = PA, the first order axioms of Peano Arithmetic (cf. p.68-69 in Takeuti [4]), $L = L_{PA}$, the logic for PA, and X = the set of open formulas in L, then the assumptions of Corollary B hold by Matijasevic's theorem ([2]). Therefore, we have;

COROLLARY C. For any formula A in L_{PA}, there is a formula B of the form $\forall^* x_1 \exists y_1 \ldots . \forall^* x_n \exists y_n \exists \bar{z} \forall \bar{u} \leq \bar{x} \exists \bar{v} \leq \bar{z} C(\bar{u},\bar{y},\bar{z})$, where $C(\bar{x},\bar{y},\bar{z})$ is an open formula, such that PA $\vdash_L A \supset B$ and $\vdash_L B \supset A$.

A weak form of Corollary C is proved in Motohashi [3] (Theorem F in [3]). Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are two L-structures such that $\mathfrak{B}$ is an extension of $\mathfrak{A}$. Then, $\mathfrak{B}$ is an outer extension of $\mathfrak{A}$ (denoted by $\mathfrak{A} \subseteq_o \mathfrak{B}$) if $\mathfrak{B} \models a \leq b$ and $b \in |\mathfrak{A}|$ imply $a \in |\mathfrak{A}|$, for any elements a, b in $|\mathfrak{B}|$. $\mathfrak{B}$ is a cofinal extension of $\mathfrak{A}$ (denoted by $\mathfrak{A} \subseteq_c \mathfrak{B}$) if $\mathfrak{B}$ is a model of the sentence Tr and for any b in $|\mathfrak{B}|$, there is an element a in $|\mathfrak{A}|$ such that $\mathfrak{B} \models b \leq a$. Let S be a set of formulas in L . Then, $\mathfrak{B}$ is an S-extension of $\mathfrak{A}$ (denoted by $\mathfrak{A} \subseteq_S \mathfrak{B}$) if $\mathfrak{A} \models A[\bar{a}]$ implies $\mathfrak{B} \models A[\bar{a}]$, for any formula $A(\bar{x})$ in S and any sequence $\bar{a}$ of elements in $|\mathfrak{A}|$. Let Δ_0 be the set of bounded formulas in L (cf. p.133 in [1]) and $\Phi_0(X)$ be the set of formulas of the form; $\forall \bar{u} \leq \bar{x} \exists \bar{v} \leq \bar{y} B(\bar{x},\bar{y},\bar{u},\bar{v})$, where $B \in X$.

From these definitions, the following facts follow immediately.

LEMMA 3. (i) If $X \subseteq \Delta_0$, then $\Phi_0(X) \subseteq \Delta_0$.

(ii) If $\mathfrak{A} \subseteq_e \mathfrak{B}$, then $\mathfrak{A} \subseteq_{\Delta_0} \mathfrak{B}$.

(iii) If $\mathfrak{A} \subseteq_{\Phi_0(X)} \mathfrak{B}$ and $\mathfrak{A} \subseteq_c \mathfrak{B}$, then $\mathfrak{A} \subseteq_{\Phi(X)} \mathfrak{B}$.

From Theorem A and (iii) of Lemma 3, we have:

COROLLARY D. Suppose that $\mathfrak{A}$ is a model of T_W and $\mathfrak{B}$ is a $\Phi_0(X)$-extension of $\mathfrak{A}$. If $\mathfrak{B}$ is a cofinal extension of $\mathfrak{A}$, then $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$.

(Proof). Assume that $\mathfrak{A} \models T_W$, $\mathfrak{A} \subseteq_{\Phi_0(X)} \mathfrak{B}$ and $\mathfrak{A} \subseteq_c \mathfrak{B}$. Let $A(\bar{x})$ be an arbitrary formula in L and $\bar{a}$ a sequence of elements in $|\mathfrak{A}|$ such that $\mathfrak{A} \models A[\bar{a}]$. Let B be the formula $f_W(A)$. Since $T_W \vdash_L A \supset B$, we have that $\mathfrak{A} \models B[\bar{a}]$. By (iii) of Lemma 3 , $\mathfrak{B}$ is a $\Phi(X)$-extension of $\mathfrak{A}$. Hence, $\mathfrak{B} \models B[\bar{a}]$, because $B \in \Phi(X)$. Since $\vdash_L B \supset A$, we have that $\mathfrak{B} \models A[\bar{a}]$. This means that $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$. (q.e.d)

From Corollary D and (i), (ii) of Lemma 3, we have:

THEOREM E. Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are models of T_W, and X is a subset of Δ_0. If $\mathfrak{B}$ is a $\Phi_0(X)$-extension of $\mathfrak{A}$, then there is an elementary extension $\mathfrak{C}$ of $\mathfrak{A}$ such that $\mathfrak{A} \subseteq_e \mathfrak{C} \subseteq_e \mathfrak{B}$.

(Proof). Assume that $\mathfrak{A} \models T_W$, $\mathfrak{B} \models T_W$, $X \subseteq \Delta_0$, and $\mathfrak{A} \subseteq_{\Phi_0(X)} \mathfrak{B}$. Let C be the set $\{b \in |\mathfrak{B}|;\ \mathfrak{B} \models b \leqq a$ for some a in $|\mathfrak{A}|\}$. Then, clearly $|\mathfrak{A}| \subseteq C$. Suppose that f is an n-ary function symbol in L and $b_1, b_2, \ldots, b_n$ are elements in C. Let $a_1, a_2, \ldots, a_n$ be elements in $|\mathfrak{A}|$ such that $\mathfrak{B} \models b_1 \leqq a_1$, $\mathfrak{B} \models b_2 \leqq a_2, \ldots,$ $\mathfrak{B} \models b_n \leqq a_n$. Since $\mathfrak{A} \models T_W$ and $\mathfrak{A} \models \forall x_1 \leqq a_1 \ldots \forall x_n \leqq a_n \exists y (f(x_1, \ldots, x_n) = y)$, there is an element a in $|\mathfrak{A}|$ such that $\mathfrak{A} \models \forall x_1 \leqq a_1 \ldots \forall x_n \leqq a_n \exists y \leqq a (f(x_1, \ldots, x_n) = y)$. Since the formula $\forall x_1 \leqq u_1 \ldots \forall x_n \leqq u_n \exists y \leqq v (f(x_1, \ldots, x_n) = y)$ belongs to $\Phi_0(X)$, $\mathfrak{B} \models \forall x_1 \leqq a_1 \ldots \forall x_n \leqq a_n \exists y \leqq a (f(x_1, \ldots, x_n) = y)$. Hence, we have that $\mathfrak{B} \models \exists y \leqq a (f(b_1, \ldots, b_n) = y)$, because $\mathfrak{B} \models b_1 \leqq a_1, \ldots, \mathfrak{B} \models b_n \leqq a_n$. This means that $\mathfrak{B} \models f(b_1, \ldots, b_n) \leqq a$. Therefore, the value of the interpretation of the function symbol f in $\mathfrak{B}$ at $b_1, b_2, \ldots, b_n$ belongs to the set C. So, the set C is closed under functions which are interpretations of function symbols of L in $\mathfrak{B}$. Therefore, there is a substructure $\mathfrak{C}$ of $\mathfrak{B}$ whose universe is C. By the definition of $\mathfrak{C}$, $\mathfrak{A} \subseteq_e \mathfrak{C} \subseteq_e \mathfrak{B}$. By (ii) of Lemma 3, $\mathfrak{C} \subseteq_{\Delta_0} \mathfrak{B}$. On the other hand, $\Phi_0(X) \subseteq \Delta_0$ by $X \subseteq \Delta_0$ and (i) of Lemma 3. Hence, we have that $\mathfrak{A} \subseteq_{\Phi_0(X)} \mathfrak{C}$. Therefore, we conclude that $\mathfrak{C}$ is an elementary extension of $\mathfrak{A}$ by Corollary D. (q.e.d.)

From Theorem E, we have:

COROLLARY F. Suppose that X is a subset of Δ_0 and T is a theory in L such that $T_W \subseteq T$ for some W in $\Sigma(X)$. If $\mathfrak{A}$ and $\mathfrak{B}$ are models of T such that $\mathfrak{B}$ is an $\Phi_0(X)$-extension of $\mathfrak{A}$, then there is an elementary extension $\mathfrak{C}$ of $\mathfrak{A}$ such that $\mathfrak{A} \subseteq_c \mathfrak{C} \subseteq_0 \mathfrak{B}$.

Let T = PA and X = the set of open formulas. Then, we have the following theorem from Corollary F and Matijasevic's theorem.

COROLLARY G (Gaifman's Splitting Theorem). If $\mathfrak{A}$ and $\mathfrak{B}$ are models of PA such that $\mathfrak{B}$ is an extension of $\mathfrak{A}$, then there is an elementary extension $\mathfrak{C}$ of $\mathfrak{A}$ such that $\mathfrak{A} \subseteq_c \mathfrak{C} \subseteq_0 \mathfrak{B}$.

§2. Some proofs. In this section, we shall construct two formula-mappings h and g_W , and prove Lemma 1 and Lemma 2 in the introduction of this paper.

LEMMA 4. $\vdash_L \forall x A(x) \equiv \forall^* x \forall u \leqq x A(u)$.

Lemma 4 is an obvious consequence of the definition of $\forall^*$.

LEMMA5. (i) $\vdash_L \exists \bar{z} \forall \bar{u} \leqq \bar{x} \exists \bar{v} \leqq \bar{z} A(\bar{u},\bar{v}) \supset \forall \bar{u} \leqq \bar{x} \exists \bar{z} A(\bar{u},\bar{z})$.

(ii) $\mathrm{Tr}, \mathrm{Bn}(W), \mathrm{Col}(W) \vdash_L \forall \bar{u} \leqq \bar{x} \exists \bar{z} A(\bar{u},\bar{z}) \supset \exists \bar{z} \forall \bar{u} \leqq \bar{x} \exists \bar{v} \leqq \bar{z} A(\bar{u},\bar{v})$.

(Proof). Since (i) is obvious, we prove (ii) only. For the sake of simplicity, we assume that the lenghts of $\bar{x}$ and $\bar{z}$ are the same number 2 . Let B , C, D , E be the following formulas;

B : $\forall u_1 \leqq x_1 \forall u_2 \leqq x_2 \exists z_1 \exists z_2 A(u_1,u_2,z_1,z_2)$,

C : $\forall u_1 \leqq x_1 \exists z_1 \exists z_2 \forall u_2 \leqq x_2 \exists v_1 \exists v_2 (W(z_1,u_2,v_1) \wedge W(z_2,u_2,v_2) \wedge A(u_1,u_2,v_1,v_2))$,

D : $\exists z_1 \exists z_2 \forall u_1 \leqq x_1 \exists w_1 \exists w_2 \forall u_2 \leqq x_2 \exists v_1 \exists v_2 (W(z_1,u_1,w_1) \wedge W(z_2,u_1,w_2) \wedge W(w_1,u_2,v_1) \wedge W(w_2,u_2,v_2) \wedge A(u_1,u_2,v_1,v_2))$,

E : $\exists z_1 \exists z_2 \forall u_1 \leqq x_1 \forall u_2 \leqq x_2 \exists v_1 \leqq z_1 \exists v_2 \leqq z_2 A(u_1.u_2,v_1,v_2)$.

It is sufficient to prove $\mathrm{Tr}, \mathrm{Bn}(W), \mathrm{Col}(W) \vdash_L B \supset E$.

But, this is obvious because $\mathrm{Col}(W) \vdash_L B \supset C$, $\mathrm{Col}(W) \vdash_L C \supset D$, and $\mathrm{Tr}, \mathrm{Bn}(W) \vdash_L D \supset E$. (q.e.d.)

LEMMA 6. Suppose that A is a formula;

$$\forall u \leq x\ \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \forall \bar{u} \leq \bar{v} C(\bar{u},u,x_1,\ldots,x_n,y_1,\ldots,y_n)$$

and B is a formula;

$$\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \forall u \leq x \forall \bar{u} \leq \bar{v} \exists z_1 \ldots \exists z_n (W(y_1,u,z_1) \wedge W(y_2,u,z_2) \wedge \ldots \wedge W(y_n,u,z_n) \wedge C(\bar{u},u,x_1,\ldots,x_n,z_1,\ldots,z_n)).$$

Then, $\mathrm{Col}(W) \vdash_L A \supset B$ and $\mathrm{Ex}(W),\mathrm{Un}(W) \vdash_L B \supset A$.

(Proof). For each $i=1,2,\ldots,n+1$, let A_i be the formula;

$$\forall x_1 \exists y_1 \ldots \forall x_{i-1} \exists y_{i-1} \forall u \leq x \forall x_i \exists y_i \ldots \forall x_n \exists y_n \forall \bar{u} \leq \bar{v} \exists z_1 \ldots \exists z_{i-1} (W(y_1,u,z_1) \wedge \ldots \wedge W(y_{i-1},u,z_{i-1}) \wedge C(\bar{u},u,x_1,\ldots,x_n,z_1,\ldots,z_{i-1},y_i,\ldots,y_n)).$$

Then, A_1 is A, A_{n+1} is B , $\mathrm{Col}(W) \vdash_L A_i \supset A_{i+1}$, and $\mathrm{Ex}(W),\mathrm{Un}(W) \vdash_L A_{i+1} \supset A_i$, for each $i=1,2,\ldots,n$. Therefore, we have that $\mathrm{Col}(W) \vdash_L A \supset B$ and $\mathrm{Ex}(W),\mathrm{Un}(W) \vdash_L B \supset A$. (q.e.d.)

Now, we define the formula-mapping h and prove Lemma 1. For each formula A of the form $\forall x_1 \ldots \forall x_n \exists \bar{z} B(x_1,\ldots,x_n,\bar{z})$, $B \in X$, let $h(A)$ be the formula;

$$\forall^* x_1 \exists y_1 \ldots \forall^* x_n \exists y_n \exists \bar{z} \forall \bar{u} \leq \bar{x} \exists \bar{v} \leq \bar{z} B(\bar{u},\bar{v}).$$

By Lemma 4, A is equivalent to the formula $\forall^* \bar{x} \forall \bar{u} \leq \bar{x} \exists \bar{z} B(\bar{u},\bar{z})$. On the other hand, $\mathrm{Tr},\mathrm{Bn}(W),\mathrm{Col}(W) \vdash_L \forall^* \bar{x} \forall \bar{u} \leq \bar{x} \exists \bar{z} B(\bar{u},\bar{z}) \supset h(A)$, and $\vdash_L h(A) \supset \forall^* \bar{x} \forall \bar{u} \leq \bar{x} \exists \bar{z} B(\bar{u},\bar{z})$ by Lemma 5. Therefore, we have that $T_W \vdash_L A \supset h(A)$ and $\vdash_L h(A) \supset A$. This completes our proof of Lemma 1.

In order to construct g_W and prove Lemma 2 , we require some preliminaries.

A quasi $\Phi(X)$-formula A of degree k (k=0,1,2,...) is a formula of the form;

$$\forall^* x_1 \exists y_1 \ldots \forall^* x_n \exists y_n \forall x_{n+1} \exists y_{n+1} \ldots \forall x_{n+k} \exists y_{n+k} \forall u_1 \leq x_1 \ldots \forall u_n \leq x_n$$
$$B(u_1,\ldots,u_n,x_{n+1},\ldots,x_{n+k},y_1,\ldots,y_{n+k}),$$

where $B(x_1,\ldots,x_{n+k},y_1,\ldots,y_{n+k}) \in \Sigma(X)$.

If A is a quasi $\Phi(X)$-formula of degree k of the above form and k is a positive integer, let $j_W(A)$ be the formula;

$$\forall^* x_1 \exists y_1 \ldots \forall^* x_n \exists y_n \forall^* x_{n+1} \exists y_{n+1} \forall x_{n+2} \exists y_{n+2} \ldots \forall x_{n+k} \exists y_{n+k}$$
$$\forall u_1 \leq x_1 \ldots \forall u_n \leq x_n \forall u_{n+1} \leq x_{n+1} \exists z_{n+1} \ldots \exists z_{n+k} ($$
$$W(y_{n+1},u_{n+1},z_{n+1}) \wedge \cdots\cdots \wedge W(y_{n+k},u_{n+1},z_{n+k}) \wedge$$
$$B(u_1,\ldots,u_n,u_{n+1},x_{n+2},\ldots,x_{n+k},y_1,\ldots,y_n,z_{n+1},\ldots,z_{n+k})).$$

Then, $j_W(A)$ is a quasi $\Phi(X)$-formula of degree k-1 because $\Sigma(\bar{X})$ is closed under $\wedge$ (cf. introduction of this paper).

If A is a quasi $\Phi(X)$-formula of degree 0 , then A has the form;

$$\forall^* x_1 \exists y_1 \ldots \forall^* x_n \exists y_n \forall u_1 \leq x_1 \ldots \forall u_n \leq x_n \exists \bar{z} C(\bar{u},\bar{y},\bar{z}) \ , \ C(\bar{x},\bar{y},\bar{z}) \in X.$$

Let $j_W(A)$ be the formula;

$$\forall^* x_1 \exists y_1 \ldots \forall^* x_n \exists y_n \exists \bar{z} \forall \bar{u} \leq \bar{x} \exists \bar{v} \leq \bar{z} C(\bar{u},\bar{y},\bar{v}) \ .$$

Then, $j_W(A)$ is a formula in $\Phi(X)$.

From Lemma 4, Lemma 5 ,and Lemma 6, we can obtain the following lemma.

LEMMA 7. $T_W \vdash_L A \supset j_W(A)$ and $Tr, Ex(W), Un(W) \vdash_L j_W(A) \supset A$.

Now, we can define g_W by the following; For each formula A in L, let $A°$ be one of formulas which are equivalent to A and have the form $\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n B(\bar{x},\bar{y})$, where B is an open formula. Then, $A°$ is a quasi-$\Phi(X)$-formula of degree n. Let $g_W(A)$ be the formula $j_W^{n+1}(A°)$, where $j_W^0(A°)$ is $A°$ and $j_W^{i+1}(A°)$ is $j_W(j_W^i(A°))$ for each $i=0,1,\ldots$. Then, clearly g_W is a formula-mapping whose domain is the set of formulas in L and whose range is a sub-set of $\Phi(X)$. Moreover, Lemma 2 holds by Lemma 7. This completes our proofs of Lemma 1 and Lemma 2.

5.

References

[1] H.Gaifman, A note on models and submodels of Arithmetic, Conference in Mathematical Logic London'70, Springer Lecture Note No.255, 128-144.

[2] Ju.V.Matijasevic, Diophantine representation of enumerable predicates, Mathematics of the USSR-Izvestija, vol.5(1971), 1-28.

[3] N.Motohashi, Preservation Theorem and Relativization Theorem for Cofinal Extensions, to appear.

[4] G.Takeuti, Proof Theory, North-Holland, Amsterdam, 1975.