A normal form theorem for first order formulas and its application to Gaifman's splitting theorem

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Let \( L \) be a first order predicate calculus with equality which has a fixed binary predicate symbol \(<\). In this paper, we shall deal with quantifiers \( \forall x, \forall x \leq y, \exists x \leq y \) defined by; \( \forall x A(x) \) is \( \forall y \exists x (y \leq x \land A(x)) \), \( \forall x \leq y A(x) \) is \( \forall x (x \leq y \lor A(x)) \), and \( \exists x \leq y A(x) \) is \( \exists x (x \leq y \land A(x)) \). The expressions \( \bar{x}, \bar{y}, \ldots \) will be used to denote sequences of those symbols, e.g. \( \bar{x} \) is \( \langle x_1, x_2, \ldots, x_n \rangle \) and \( \bar{y} \) is \( \langle y_1, y_2, \ldots, y_m \rangle \). Also, \( \exists \bar{x}, \forall \bar{x} \leq \bar{y}, \ldots \) will be used to denote \( \exists x_1 \exists x_2 \ldots \exists x_n, \forall x_1 \leq y_1 \forall x_2 \leq y_2 \ldots \forall x_n \leq y_n \ldots \ldots \), respectively.

Let \( X \) be a set of formulas in \( L \) such that; \( X \) contains every atomic formulas and is closed under substitutions of free variables and applications of propositional connectives \( \neg \) (not), \( \land \) (and), \( \lor \) (or).

Then, \( \Sigma(X) \) is the set of formulas of the form \( \exists \bar{x} B(\bar{x}) \), where \( B \in X \), and \( \Xi(X) \) is the set of formulas of the form;

\[
\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \exists z \forall \bar{u} \exists \bar{x} \exists \bar{v} \leq \bar{z} B(\bar{u}, \bar{x}, \bar{v}), \quad \text{where } B(\bar{x}, \bar{y}, \bar{z}) \in X.
\]

Since \( X \) is closed under \( \land, \lor \), two sets \( \Sigma(X) \) and \( \Xi(X) \) are closed under \( \land, \lor \) in the following sense: for any formulas \( A \) and \( B \) in \( \Sigma(X) \) [ \( \Xi(X) \) ], there are formulas in \( \Sigma(X) \) [ \( \Xi(X) \) ] which are obtained from \( A \land B \) and \( A \lor B \) by prefixing some quantifiers in them in the usual manner.
Let $W(x, y, z)$ be a formula in $\Sigma(X)$ which has no free variables except $x, y, z$. Then, the theory $T_W$ in $L$ consists of the following sentences:

**$T_r$**: $\forall x \forall y \leq x \forall z \leq y (z \leq x)$,

**$Ex(W)$**: $\forall x \forall y \exists z W(x, y, z)$,

**$Un(W)$**: $\forall x \forall y \forall z \forall w (W(x, y, z) \land W(x, y, w) \land z \neq w)$,

**$Bn(W)$**: $\forall x \forall y \forall z (W(x, y, z) \lor z \leq x)$,

and

**$Col(W)$**: $\forall \bar{w} [ \forall u \leq x \exists v A(u, v, \bar{w}) \lor \exists y \forall u \leq x \exists v (W(y, u, v) \land A(u, v, \bar{w})) ]$,

where $A(x, v, \bar{w})$ is a formula in $L$.

Since $Col(W)$ is a schema, $T_W$ is an infinite set of sentences.

A mapping $f$ from a set of formulas in $L$ (domain of $f$) on to a set of formulas in $L$ (range of $f$) is called a formula-mapping if $f(A)$ and $A$ have the same set of free variables for each formula $A$ in the domain of $f$.

In this paper, we shall give a concrete method to construct a formula-mapping $f_W$, whose domain is the set of formulas in $L$ and whose range is a subset of $\Sigma(X)$, and prove the following fact.

**THEOREM A.** For any formula $A$ in $L$, the formula $A \supseteq f_W(A)$ is provable from $T_W$ in $L$, and the formula $f_W(A) \supseteq A$ is provable in $L$, i.e. $T_W, A \vdash_L f_W(A)$ and $f_W(A) \vdash_L A$.

This theorem shows that any formula $A$ in $L$ is equivalent to $f_W(A)$ in $\Sigma(X)$ with respect to the theory $T_W$, and furthermore the implication from $f_W(A)$ to $A$ is provable logically. This is a normal
form theorem for first order formulas, of a new type.

In §1 below, we shall show some applications of Theorem A above and one of which, Corollary E below, is a generalization of Gaifman's splitting theorem, Corollary G below, in Gaifman [1].

The construction of $f_w$ requires two auxiliary formula-mapping $h$ and $g_w$, where $h$ is a formula-mapping whose domain is the set of formulas (denoted by $\Pi_2(X)$) of the form $\forall \bar{x} \exists \bar{y} B$, $B \in X$, and whose range is a subset of $\Phi(X)$, and $g_w$ is a formula-mapping whose domain is the set of formulas in $L$ and whose range is a subset of $\Phi(X)$. Moreover, we can prove:

**Lemma 1.** For any formula $A$ in $\Pi_2(X)$, the formula $A \supset h(A)$ is provable from $T_w$ in $L$ and the formula $h(A) \supset A$ is provable in $L$, i.e. $T_w \vdash_L A \supset h(A)$ and $\vdash_L h(A) \supset A$.

**Lemma 2.** For any formula $A$ in $L$, the formula $A \supset g_w(A)$ is provable from $T_w$ in $L$ and the formula $g_w(A) \supset A$ is provable from $T_r, Ex(W), Un(W)$ in $L$, i.e. $T_w \vdash_L A \supset g_w(A)$ and $T_r, Ex(W), Un(W)$ $\vdash_L g_w(A) \supset A$.

Although $T_r, Ex(W), Un(W)$ are not formulas in $\Pi_2(X)$, there are formulas $T_r^o, Ex(W)^o, Un(W)^o$ in $\Pi_2(X)$ which are obtained from $T_r, Ex(W), Un(W)$ by prefixing some quantifiers in them, respectively. Therefore, they are equivalent each other. Let $f_w(A)$ be one of formulas in $\Phi(X)$ which are obtained from $g_w(A) \land h(T_r^o) \land h(Ex(W)^o) \land h(Un(W)^o)$ by prefixing some quantifiers in it. Then, Theorem A clearly holds from Lemma 1 and
and Lemma 2. So, in order to prove Theorem A above, it is sufficient to construct $h$ and $g_w$ and prove Lemma 1 and Lemma 2, which will be done in §2 below.
§1. Some applications.

In this section, we shall show some applications of Theorem A to normal form theorems and splitting theorems. From Theorem A, we have the following fact immediately.

COROLLARY B. It T is a theory in L such that $T_W \subseteq T$ for some formula W in $\Sigma(X)$, then for any formula A in L, there is a formula B in $\Xi(X)$ such that $T \models_L A \supseteq B$ and $T \models_L B \supseteq A$.

If $T = PA$, the first order axioms of Peano Arithmetic (cf. p.68-69 in Takeuti [4]), $L = L_{PA}$, the logic for PA, and $X$ = the set of open formulas in L, then the assumptions of Corollary B hold by Matijasevic's theorem ([2]). Therefore, we have:

COROLLARY C. For any formula $\forall \mathbf{A}$ in $L_{PA}$, there is a formula $B$ of the form $\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \exists z \forall u \exists v \exists w \exists c(u, y, z)$, where $C(x, y, z)$ is an open formula, such that $PA \models_L A \supseteq B$ and $L \models L B \supseteq A$.

A weak form of Corollary C is proved in Motohashi [3] (Theorem F in [3]). Suppose that $\mathfrak{a}$ and $\mathfrak{b}$ are two L-structures such that $\mathfrak{b}$ is an extension of $\mathfrak{a}$. Then, $\mathfrak{b}$ is an outer extension of $\mathfrak{a}$ (denoted by $\mathfrak{a} \subseteq^o \mathfrak{b}$) if $\mathfrak{b} \models a \leq b$ and $b \in |\mathfrak{a}|$ imply $a \in |\mathfrak{a}|$, for any elements $a, b$ in $|\mathfrak{b}|$. $\mathfrak{b}$ is a cofinal extension of $\mathfrak{a}$ (denoted by $\mathfrak{a} \subseteq^c \mathfrak{b}$) if $\mathfrak{b}$ is a model of the sentence $Tr$ and for any $b$ in $|\mathfrak{b}|$, there is an element $a$ in $|\mathfrak{a}|$ such that $\mathfrak{b} \models b \leq a$. Let $S$ be a set
of formulas in $L$. Then, $\mathcal{A}$ is an S-extension of $\mathcal{A}$ (denoted by $\mathcal{A} \subseteq^S \mathcal{A}$) if $\mathcal{A} \models A[\bar{a}]$ implies $\mathcal{L} \models A[\bar{a}]$, for any formula $A(\bar{x})$ in $S$ and any sequence $\bar{a}$ of elements in $|\mathcal{A}|$. Let $\Delta_0$ be the set of bounded formulas in $L$ (cf. p.133 in [1]) and $F_0(X)$ be the set of formulas of the form: $\forall \bar{u} \leq X \exists \bar{v} \leq yB(\bar{x}, \bar{y}, \bar{u}, \bar{v})$, where $B \in X$.

From these definitions, the following facts follow immediately.

**Lemma 3.**

(i) If $X \subseteq \Delta_0$, then $F_0(X) \subseteq \Delta_0$.

(ii) If $\mathcal{A} \subseteq \mathcal{A}$, then $\mathcal{A} \subseteq \mathcal{A}$.

(iii) If $\mathcal{A} \subseteq F_0(X)$ and $\mathcal{A} \subseteq \mathcal{A}$, then $\mathcal{A} \subseteq F(X)$.

From Theorem A and (iii) of Lemma 3, we have:

**Corollary D.** Suppose that $\mathcal{A}$ is a model of $T_W$ and $\mathcal{A}$ is a $F_0(X)$-extension of $\mathcal{A}$. If $\mathcal{A}$ is a cofinal extension of $\mathcal{A}$, then $\mathcal{A}$ is an elementary extension of $\mathcal{A}$.

*(Proof)*. Assume that $\mathcal{A} \models T_W$, $\mathcal{A} \subseteq F_0(X)$ and $\mathcal{A} \subseteq \mathcal{A}$. Let $A(\bar{x})$ be an arbitrary formula in $L$ and $\bar{a}$ a sequence of elements in $|\mathcal{A}|$ such that $\mathcal{A} \models A[\bar{a}]$. Let $B(\bar{x})$ be the formula $f_W(A)$. Since $T_W \models A \supseteq B$, we have that $\mathcal{A} \models B[\bar{a}]$. By (iii) of Lemma 3, $\mathcal{A}$ is a $F_0(X)$-extension of $\mathcal{A}$. Hence, $\mathcal{A} \models B[\bar{a}]$, because $B \in F(X)$.

Since $L \models B \supseteq A$, we have that $\mathcal{A} \models A[\bar{a}]$. This means that $\mathcal{A}$ is an elementary extension of $\mathcal{A}$.

(q.e.d.)
Theorem E. Suppose that $\mathfrak{A}$ and $\mathfrak{U}$ are models of $T_W$, and $X$ is a subset of $\Delta_0$. If $\mathfrak{A}$ is a $\mathbb{P}_0(X)$-extension of $\mathfrak{U}$, then there is an elementary extension $\mathfrak{C}$ of $\mathfrak{A}$ such that $\mathfrak{A} \subseteq \mathfrak{C} \subseteq \mathfrak{U}$.

(Proof). Assume that $\mathfrak{A} \models T_W$, $\mathfrak{A} \models X \subseteq \Delta_0$, and $\mathfrak{A} \subseteq \mathbb{P}_0(X) \mathfrak{C}$.

Let $C$ be the set $\{b \in |\mathfrak{A}| : \mathfrak{A} \models b \leq a \text{ for some } a \text{ in } |\mathfrak{A}|\}$. Then, clearly $|\mathfrak{A}| \subseteq C$. Suppose that $f$ is an $n$-ary function symbol in $L$ and $b_1, b_2, \ldots, b_n$ are elements in $C$. Let $a_1, a_2, \ldots, a_n$ be elements in $|\mathfrak{A}|$ such that $\mathfrak{A} \models b_1 \leq a_1$, $\mathfrak{A} \models b_2 \leq a_2$, $\ldots$, $\mathfrak{A} \models b_n \leq a_n$. Since $\mathfrak{A} \models T_W$ and $\mathfrak{A} \models \forall x_1 \leq a_1 \ldots \forall x_n \leq a_n \exists y \in \mathcal{F}(\mathfrak{A}) \mathfrak{C} \models y \leq a(f(x_1, \ldots, x_n)) = y$), there is an element $a$ in $|\mathfrak{A}|$ such that $\mathfrak{A} \models \forall x_1 \leq a_1 \ldots \forall x_n \leq a_n \exists y \leq a(f(x_1, \ldots, x_n)) = y$. Since the formula $\forall x_1 \leq u_1 \ldots \forall x_n \leq u_n \exists y \leq v(f(x_1, \ldots, x_n)) = y$ belongs to $\mathbb{P}_0(X)$, $\mathfrak{A} \models \forall x_1 \leq a_1 \ldots \forall x_n \leq a_n \exists y \leq a(f(x_1, \ldots, x_n)) = y$.

Hence, we have that $\mathfrak{A} \models \exists y \leq a(f(b_1, \ldots, b_n) = y)$, because $\mathfrak{A} \models b_1 \leq a_1$, $\ldots$, $\mathfrak{A} \models b_n \leq a_n$. This means that $\mathfrak{A} \models f(b_1, \ldots, b_n) \leq a$. Therefore, the value of the interpretation of the function symbol $f$ in $\mathfrak{A}$ at $b_1, b_2, \ldots, b_n$ belongs to the set $C$. So, the set $C$ is closed under functions which are interpretations of function symbols of $L$ in $\mathfrak{A}$.

Therefore, there is a substructure $\mathfrak{C}$ of $\mathfrak{U}$ whose universe is $C$. By the definition of $\mathfrak{C}$, $\mathfrak{A} \subseteq \mathfrak{C} \subseteq \mathfrak{U}$. By (ii) of Lemma 3, $\mathfrak{A} \subseteq \mathfrak{A} \mathfrak{C} \subseteq \mathfrak{A} \mathfrak{U}$.

On the other hand, $\mathbb{P}_0(X) \subseteq \Delta_0$ by $X \subseteq \Delta_0$ and (i) of Lemma 3.

Hence, we have that $\mathfrak{A} \subseteq \mathbb{P}_0(X) \mathfrak{C}$. Therefore, we conclude that $\mathfrak{C}$ is an elementary extension of $\mathfrak{A}$ by Corollary D. (q.e.d.)
From Theorem E, we have:

**COROLLARY F.** Suppose that $X$ is a subset of $\Delta_0$ and $T$ is a theory in $L$ such that $T_W \subseteq T$ for some $W$ in $\mathcal{Z}(X)$. If $\mathfrak{a}$ and $\mathfrak{u}$ are models of $T$ such that $\mathfrak{u}$ is an $\mathfrak{E}_0(X)$-extension of $\mathfrak{a}$, then there is an elementary extension $\mathfrak{z}$ of $\mathfrak{a}$ such that $\mathfrak{a} \subseteq_e \mathfrak{z} \subseteq_e \mathfrak{u}$.

Let $T = PA$ and $X$ the set of open formulas. Then, we have the following theorem from Corollary F and Matijasevic's theorem.

**COROLLARY G (Gaifman's Splitting Theorem).** If $\mathfrak{a}$ and $\mathfrak{u}$ are models of $PA$ such that $\mathfrak{u}$ is an extension of $\mathfrak{a}$, then there is an elementary extension $\mathfrak{z}$ of $\mathfrak{a}$ such that $\mathfrak{a} \subseteq_e \mathfrak{z} \subseteq_e \mathfrak{u}$.
2. Some proofs. In this section, we shall construct two formula-mappings \( h \) and \( q_W \), and prove Lemma 1 and Lemma 2 in the introduction of this paper.

**Lemma 4.** \( \vdash_L \forall x A(x) \equiv \forall u \exists x A(u) \).

Lemma 4 is an obvious consequence of the definition of \( \forall^* \).

**Lemma 5.**

(i) \( \vdash_L \exists z \forall \bar{u} \exists \bar{v} \exists \bar{w} \forall A(\bar{u}, \bar{v}) \supset \forall \bar{u} \exists \bar{v} \exists \bar{w} A(\bar{u}, \bar{v}) \).

(ii) \( \vdash_L \forall \bar{u} \exists \bar{v} \exists \bar{w} A(\bar{u}, \bar{v}) \supset \exists \bar{v} \forall \bar{u} \exists \bar{w} A(\bar{u}, \bar{v}) \).

(Proof). Since (i) is obvious, we prove (ii) only. For the sake of simplicity, we assume that the lengths of \( \bar{x} \) and \( \bar{z} \) are the same number 2. Let \( B, C, D, E \) be the following formulas;

\[ B : \forall u_1 \leq x_1 \forall u_2 \leq x_2 \exists z_1 \exists z_2 \forall (u_1, u_2, z_1, z_2) \]

\[ C : \forall u_1 \leq x_1 \exists z_1 \exists z_2 \forall u_2 \leq x_2 \exists v_1 \exists v_2 (W(z_1, u_2, v_1) \wedge W(z_2, u_2, v_2) \wedge A(u_1, u_2, v_1, v_2)) \]

\[ D : \exists z_1 \exists z_2 \forall u_1 \leq x_1 \exists w_1 \exists w_2 \forall u_2 \leq x_2 \exists v_1 \exists v_2 (W(z_1, u_1, w_1) \wedge W(z_2, u_1, w_2) \wedge W(w_1, u_2, v_1) \wedge W(w_2, u_2, v_2) \wedge A(u_1, u_2, v_1, v_2)) \]

\[ E : \exists z_1 \exists z_2 \forall u_1 \leq x_1 \forall u_2 \leq x_2 \exists v_1 \exists v_2 \exists z_2 A(u_1, u_2, v_1, v_2) \]

It is sufficient to prove \( \vdash_L B \supset E \).

But, this is obvious because \( \vdash_L B \supset C \), \( \vdash_L C \supset D \), and \( \vdash_L D \supset E \). (q.e.d.)
LEMMA 6. Suppose that \( A \) is a formula;
\[
\forall u \exists x \forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \forall u \leq x \forall \bar{u} \leq \bar{x} \exists z_1 \ldots \exists z_n (W(y_1, u, z_1) \land W(y_2, u, z_2) \land \ldots \land W(y_n, u, z_n) \land C(\bar{u}, u, x_1, \ldots, x_n, z_1, \ldots, z_n)).
\]
and \( B \) is a formula;
\[
\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \forall u \leq x \forall \bar{u} \leq \bar{x} \exists z_1 \ldots \exists z_n (W(y_1, u, z_1) \land W(y_2, u, z_2) \land \ldots \land W(y_n, u, z_n) \land C(\bar{u}, u, x_1, \ldots, x_n, z_1, \ldots, z_n)).
\]
Then, \( \text{Col}(W) \models_L A \supset B \) and \( \text{Ex}(W), \text{Un}(W) \models_L B \supset A \).

(Proof). For each \( i = 1, 2, \ldots, n+1 \), let \( A_i \) be the formula;
\[
\forall x_1 \exists y_1 \ldots \forall x_{i-1} \exists y_{i-1} \forall u \leq x \forall x_i \exists y_i \ldots \forall x_n \exists y_n \forall \bar{u} \leq \bar{x} \exists z_1 \ldots \exists z_{i-1} (W(y_1, u, z_1) \land \ldots \land W(y_{i-1}, u, z_{i-1}) \land C(\bar{u}, u, x_1, \ldots, x_n, z_1, \ldots, z_{i-1}, y_i, \ldots, y_n)).
\]
Then, \( A_1 \) is \( A \), \( A_{n+1} \) is \( B \), \( \text{Col}(W) \models_L A_i \supset A_{i+1} \), and \( \text{Ex}(W), \text{Un}(W) \models_L A_{i+1} \supset A_i \), for each \( i = 1, 2, \ldots, n \). Therefore, we have that \( \text{Col}(W) \models_L A \supset B \) and \( \text{Ex}(W), \text{Un}(W) \models_L B \supset A \). (q.e.d.)

Now, we define the formula-mapping \( h \) and prove Lemma 1.

For each formula \( A \) of the form \( \forall x_1 \ldots \forall x_n \exists z B(x_1, \ldots, x_n, z) \), let \( B \in X \), let \( h(A) \) be the formula;
\[
\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \exists z \forall u \leq x \exists \bar{z} B(\bar{u}, \bar{z}).
\]
By Lemma 4, \( A \) is equivalent to the formula \( \forall x \forall u \leq x \exists \bar{z} B(\bar{u}, \bar{z}) \).

On the other hand, \( \text{Tr}, \text{Bn}(W), \text{Col}(W) \models_L \forall x \forall u \leq x \exists \bar{z} B(\bar{u}, \bar{z}) \supset h(A) \), and \( \models_L h(A) \supset \forall x \forall u \leq x \exists \bar{z} B(\bar{u}, \bar{z}) \) by Lemma 5.

Therefore, we have that \( T_W \models_L A \supset h(A) \) and \( \models_L h(A) \supset A \).

This completes our proof of Lemma 1.

In order to construct \( g_W \) and prove Lemma 2, we require some preliminaries.

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A quasi $\mathcal{V}(X)$-formula $A$ of degree $k$ ($k=0,1,2,\ldots$) is a formula of the form:
\[
\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \forall x_{n+1} \exists y_{n+1} \ldots \forall x_{n+k} \exists y_{n+k} \forall u_1 \leq x_1 \ldots \forall u_n \leq x_n \forall v_1 \leq y_1 \ldots \forall v_{n+k} \leq y_{n+k} \exists z_{n+1} \ldots \exists z_{n+k} \quad (W(u_1,\ldots,u_n,v_1,\ldots,v_{n+k}) \land B(u_1,\ldots,u_n,v_1,\ldots,v_{n+k})).
\]
where $B(x_1,\ldots,x_{n+k},y_1,\ldots,y_{n+k}) \in \Sigma(X)$.

If $A$ is a quasi $\mathcal{V}(X)$-formula of degree $k$ of the above form and $k$ is a positive integer, let $j_W(A)$ be the formula:
\[
\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \forall x_{n+1} \exists y_{n+1} \ldots \forall x_{n+k} \exists y_{n+k} \forall u_1 \leq x_1 \ldots \forall u_n \leq x_n \forall v_1 \leq y_1 \ldots \forall v_{n+k} \leq y_{n+k} \exists z_{n+1} \ldots \exists z_{n+k} \quad (W(u_1,\ldots,u_n,v_1,\ldots,v_{n+k}) \land B(u_1,\ldots,u_n,v_1,\ldots,v_{n+k})).
\]
Then, $j_W(A)$ is a quasi $\mathcal{V}(X)$-formula of degree $k-1$ because $\mathcal{V}(X)$ is closed under $\land$ (cf. introduction of this paper).

If $A$ is a quasi $\mathcal{V}(X)$-formula of degree 0, then $A$ has the form:
\[
\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n \forall u_1 \leq x_1 \ldots \forall u_n \leq x_n \exists z C(u,\bar{y},\bar{z}), \quad C(x,\bar{y},\bar{z}) \in X.
\]
Let $j_v(A)$ be the formula:
\[
\forall x_1 \exists y_1 \ldots \forall x_n \exists z \exists \bar{x} \exists \bar{y} \exists \bar{z} C(u,\bar{y},\bar{z}).
\]
Then, $j_W(A)$ is a formula in $\mathcal{V}(X)$.

From Lemma 4, Lemma 5, and Lemma 6, we can obtain the following lemma.

**Lemma 7.** $T_W \vdash L \ A \supset j_W(A)$ and $Tr,Ex(W),Un(W) \vdash L \ j_W(A) \supset A$.
Now, we can define $g_W$ by the following; For each formula $A$ in $L$, let $A^o$ be one of formulas which are equivalent to $A$ and have the form $\forall x_1 \exists y_1 \ldots \forall x_n \exists y_n B(x, y)$, where $B$ is an open formula. Then, $A^o$ is a quasi-$\mathcal{R}(X)$-formula of degree $n$. Let $g_W(A)$ be the formula $j_W^{n+1}(A^o)$, where $j_W^0(A^o)$ is $A^o$ and $j_W^{i+1}(A^o)$ is $j_W(j_W^i(A^o))$ for each $i=0,1,\ldots\ldots$. Then, clearly $g_W$ is a formula-mapping whose domain is the set of formulas in $L$ and whose range is a sub-set of $\mathcal{R}(X)$. Moreover, Lemma 2 holds by Lemma 7. This completes our proofs of Lemma 1 and Lemma 2.
References


