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# On the indices and integral bases of abelian biquadratic fields

By

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1. Introduction. Let  $K$  be an algebraic number field over the rationals  $Q$  with finite degree.  $Z$  and  $\mathcal{O}_K$  denote the ring of rational integers and the integer ring of  $K$  respectively. If  $\mathcal{O}_K = Z[\alpha]$  for some number  $\alpha$  in  $K$ , it is called that  $\mathcal{O}_K$  has a power basis. For a number  $\xi$  in  $\mathcal{O}_K$  we denote by  $\text{Ind } \xi$  the group index  $(\mathcal{O}_K : Z[\xi])$  if  $\xi$  is a primitive element of  $K$  and 0 otherwise. Then the index  $m(K)$  of any field  $K$  is defined by  $\text{g.c.d.} \{ \text{Ind } \xi ; \xi \in \mathcal{O}_K \}$ . The minimum index  $\tilde{m}(K)$  of any  $K$  is defined by  $\min \{ \text{Ind } \eta ; \eta \in \mathcal{O}_K, Q(\eta) = K \}$ . In §2 we shall give an estimate of the index  $m(K)$  without using the decomposition theory of primes when  $K$  is any abelian biquadratic field. In §3 we shall investigate some relations between  $m(K)$  and an integral basis related to a problem of Hasse and construct such a field  $K$  that the minimum index  $\tilde{m}(K)$  is greater than any given integer  $N$  applying a method of M. Hall[2].

2. An estimate of the indices. By [8] it is well known that if a prime  $p$  divides the index  $m(K)$ , then  $p$  is smaller than the degree  $[K : Q]$ .

In our situation we obtain more precisely the next lemma.

Lemma 1. For any abelian biquadratic field  $K$  over  $Q$  it holds that if the number  $2^e 3^{e'}$  exactly divides the index  $m(K)$ ,

then  $e \leq 2$  and  $e' \leq 1$ . Especially if the discriminant  $d(K)$  of a field  $K$  is even, then  $e = 0$ .

Proof. i) The cyclic cases. Let  $\chi$  be a biquadratic character with odd conductor  $n$  determined by the biquadratic residue symbol.  $k_n$  denotes the  $n$ -th cyclotomic field  $Q(\zeta_n)$ , herein  $\zeta_n = \exp(2\pi i/n)$ . Let  $G$  be the Galois group of  $k_n/Q$ . The group  $\langle \chi \rangle$  is a cyclic subgroup with order 4 of the character group of  $G$ . Let  $K$  denote the subfield of  $k_n$  corresponding to the kernel  $H$  of  $\chi$ . Then we have  $K = Q(\eta)$  with the Gauss period  $\eta = \sum_{x \in H} \zeta_n^x$ . We fix an element  $\sigma$  in  $G$  such that

$\chi(\sigma) = i$ , and denote  $\sigma(\xi)$ ,  $\sigma^2(\xi)$ ,  $\sigma^3(\xi)$  by  $\xi'$ ,  $\xi''$ ,  $\xi'''$  respectively for  $\xi$  in  $K$ .

First we consider the case of odd conductor  $n$ . Since the set  $\{1, \eta, \eta', \eta''\}$  makes an integral basis of  $K$ , it is enough for computation of the  $\text{Ind } \xi$  to choose  $\xi = x\eta + y\eta' + z\eta''$  for  $\xi$  in  $K$ . Let  $n = \ell m$  be square-free for odd integers  $\ell = a^2 + 4b^2$ ,  $m$  where any prime factor of  $\ell$  is congruent to 1 modulo 4 and  $\lambda = a + 2bi \equiv 1 \pmod{2(1-i)}$ . Then by using the Gauss sum

$$\tau(\chi) = \sum_{x \in G} \chi(x) \zeta_n^x \text{ attached to } \chi \text{ and the Jacobi sum } \tau(\chi)^2 / \tau(\chi^2)$$

we obtain  $\text{Ind } \xi = \sqrt{|d(\xi)/d(K)|} = |cN\alpha_n|$ , where  $\alpha_n = (cm + d\sqrt{\ell})/2$ ,  $c = ((x-z)^2 - y^2)b - (x-z)ya$ ,  $d = ((x-y+z)^2 - \chi(-1) \times ((x-z)^2 + y^2)m)/2$ . Herein  $d(\xi)$ ,  $N$  mean the discriminant of a number  $\xi$ , the norm with respect to  $Q(\sqrt{\ell})/Q$  respectively.

i)<sub>1</sub> If  $\ell \equiv 1 \pmod{8}$  and  $b \equiv 0, 4 \pmod{8}$  (resp.  $b \equiv \pm 2 \pmod{8}$ ), then for  $\xi = 2\eta + \eta' - \eta''$  (resp.  $2\eta \pm \eta'$  from  $\chi(-1) = \begin{cases} 1, \\ -1, \end{cases}$

$m \equiv 1 \pmod{4}$   
 $m \equiv -1 \pmod{4}$  ) we have  $\text{Ind } \xi \equiv 4 \pmod{8}$ . i)<sub>2</sub> If  $\ell \equiv 5 \pmod{8}$ ,

then for  $\xi = \eta + \eta' - \eta''$  we get  $\text{Ind } \xi \equiv 1 \pmod{2}$ . i)<sub>3</sub> If  $m \equiv 0 \pmod{3}$  and  $a \equiv 0 \pmod{3}$  (resp.  $a \not\equiv 0 \pmod{3}$ ), then we choose  $\xi = \eta$  (resp.  $\eta + \eta'$ ). Thus we have  $\text{Ind } \xi \not\equiv 0 \pmod{3}$ . i)<sub>4</sub> If  $m \not\equiv 0 \pmod{3}$  and  $a \equiv 0 \pmod{3}$ , then  $b \not\equiv 0 \pmod{3}$  and for  $\xi = \eta$   $4N\alpha_n \equiv b^2 |1 - 2\chi(-1)m| \pmod{9}$  holds. When  $1 - 2\chi(-1)m \equiv 0 \pmod{9}$ , we reset  $\xi = 2\eta + 3\eta'$ . Then  $4N\alpha_n \equiv b^2 |1 - 8\chi(-1)m| \not\equiv 0 \pmod{9}$ . i)<sub>5</sub> The case of  $m \not\equiv 0$  and  $a \not\equiv 0 \pmod{3}$ . For  $\xi = \eta$  if  $4^2 N\alpha_n \equiv |4b^2 m^2 - (1 - \chi(-1)m)^2 \ell| \equiv 0 \pmod{9}$  holds, then we reset  $\xi = 2\eta + \eta''$ . If  $4^2 N\alpha_n \equiv |4b^2 m^2 - (-\chi(-1)m)^2 \ell| \equiv 0 \pmod{9}$ , then we have  $1 - 2\chi(-1)m \equiv 0 \pmod{9}$  from  $(\ell, 3) = 1$ . Then we take again  $\xi = 3\eta + 2\eta''$ . If  $4^2 N\alpha_n \equiv |4b^2 m^2 - (7 - \chi(-1)m)^2 \ell| \equiv 0 \pmod{9}$ , then  $1 + \chi(-1)m \equiv 0 \pmod{9}$  must hold. This is a contradiction. By i)<sub>1-5</sub> we have  $e \leq 2$  and  $e' \leq 1$ .

Next we estimate the case of even conductor. At first we consider the case of  $\chi = \chi_0^{(\nu)} \chi_\ell \psi_m$ ,  $n = 16\ell m$ ,  $\ell m \equiv 1 \pmod{2}$ , where  $\chi_0^{(\nu)}(x) = (-1)^{\nu(x-1)/2} i^{(x^2-1)/8}$  are the even and the odd biquadratic characters with conductor 16 for  $\nu = 0$  and 1 respectively, and  $\chi_\ell, \psi_m$  are the biquadratic, the quadratic characters with conductors  $\ell, m$  respectively. From  $\chi((n/2) + 1) = -1$ , it follows that  $\eta'' = \sigma^2(\eta) = \sum_{x \in H} \zeta_n^{((n/2)+1)x} = - \sum_{x \in H} \zeta_n^x = -\eta$ . However it is known that  $\{1, \eta, \eta', \sqrt{f}/2\}$  is an integral basis of  $K$ , where  $d(K) = fn^2$  and  $f = 8\ell$  is the conductor of  $Q(\sqrt{8\ell})$  [3]. Then for  $\xi = x\eta + y\eta' + z(\sqrt{f}/2)$  we obtain  $\text{Ind } \xi = |cN\alpha_f|$ , where  $\alpha_f = cm + d(\sqrt{f}/2)$ ,  $c = -2xy(a-b) + (x^2 - y^2) \times (a+2b)$ ,  $d = 2z^2 - \chi(-1)(x^2 + y^2)m$ . For  $\xi_1 = \eta$ , we have

$$\text{Ind } \xi_1 = |a + 2b|m^2 |(a + 2b)^2 - 2\ell| \equiv 1 \pmod{2}. \quad \text{If } a + 2b \equiv \pm 3 \pmod{9}$$

and  $m \not\equiv 0 \pmod{3}$ , then  $\text{Ind } \xi_1 \equiv \pm 3 \pmod{9}$ . If  $a + 2b \equiv \pm 3 \pmod{9}$  (resp.  $\not\equiv 0 \pmod{3}$ ) and  $m \equiv 0 \pmod{3}$ , then for  $\xi_2 = \eta + (\sqrt{f}/2)$  we get  $\text{Ind } \xi_2 = |a + 2b| |(a + 2b)^2 m^2 - 8\ell| \equiv \pm 3 \pmod{9}$  (resp.  $\not\equiv 0 \pmod{3}$ ). If  $a + 2b \equiv 0 \pmod{9}$  and  $m \equiv 0 \pmod{3}$ , then  $a - 2b \not\equiv 0 \pmod{3}$  holds. Thus for  $\xi_3 = \eta + \eta' + (\sqrt{f}/2)$  we have  $\text{Ind } \xi_3 \equiv 2|a - 2b|\ell \not\equiv 0 \pmod{3}$ . In the case of  $(a + 2b)m \not\equiv 0 \pmod{3}$ , we have  $\text{Ind } \xi_1 \not\equiv 0 \pmod{3}$  for  $a \not\equiv -b \pmod{3}$ , and put  $\xi_4 = \eta + \eta'$ , then  $\text{Ind } \xi_4 \equiv \pm 3 \pmod{9}$  for  $a \equiv -b \pmod{3}$ . If  $a + 2b \equiv 0 \pmod{9}$  and  $m \not\equiv 0 \pmod{3}$ , then we have  $\text{Ind } \xi_4 \not\equiv 0 \pmod{9}$ . Therefore we obtain  $e = 0$  and  $e' \leq 1$ . Secondly we treat the case of  $\mathcal{X} = \mathcal{X}_\ell \psi_m$ ,  $n = \ell m$ ,  $m \equiv 0 \pmod{2}$ . In this case the set  $\{1, \eta, \eta', \eta''\}$  is also not an integral basis of  $K$ . But  $\{1, \eta, \eta', (1 + \sqrt{\ell})/2\}$  is an integral basis, where  $d(K) = \ell n^2$ ,  $f = \ell[3]$ . Then for  $\xi = x\eta + y\eta' + z(1 + \sqrt{\ell})/2$  we have  $\text{Ind } \xi = |cN\alpha_f|$ , where  $\alpha_f = cm + d\sqrt{f}$ ,  $c = -xya + (x^2 - y^2)2b$ ,  $d = (x^2 + y^2)(m/2) - \mathcal{X}(-1)z^2$ . For  $\xi_1 = \eta + \eta' + (1 + \sqrt{\ell})/2$  we get  $\text{Ind } \xi_1 \equiv 1 \pmod{2}$ . Put  $\xi_2 = \eta + \eta'$ . If  $abm \not\equiv 0 \pmod{3}$ , then  $\text{Ind } \xi_2 \not\equiv 0 \pmod{3}$ . We choose  $\xi_3 = \eta$ ,  $\xi_4 = \eta + (1 + \sqrt{\ell})/2$ . If  $a \equiv 0 \pmod{3}$ , then from  $b \not\equiv 0 \pmod{3}$  we have  $\text{Ind } \xi_3 \equiv \pm 3 \pmod{9}$  for  $m \not\equiv 0 \pmod{3}$  and  $\text{Ind } \xi_4 \not\equiv 0 \pmod{3}$  for  $m \equiv 0 \pmod{3}$ . If  $b \equiv 0 \pmod{3}$ , then from  $a \not\equiv 0 \pmod{3}$  we obtain  $\text{Ind } \xi_1 \not\equiv 0 \pmod{3}$  for  $m \not\equiv 0 \pmod{3}$  and  $\text{Ind } \xi_1 \equiv \pm 3 \pmod{9}$  for  $m \equiv 0 \pmod{3}$ . Thus we have  $e = 0$  and  $e' \leq 1$ .

ii) The non-cyclic cases. Without loss of generality we can set  $K = \mathbb{Q}(\sqrt{\ell m_1}, \sqrt{\ell m_2})$ , where  $\ell m_1, m_2$  is a square-free integer and  $\ell > 0$ . For brevity we denote  $(1 + \sqrt{\ell m_1})/2$ ,  $(1 + \sqrt{\ell m_2})/2$ ,  $(\sqrt{\ell m_2} + \sqrt{m_1 m_2})/2$  by  $\alpha$ ,  $\beta$ ,  $\gamma$  respectively. ii)<sub>1</sub> If  $\ell m_1 \equiv 1$ ,  $\ell m_2 \equiv 2, 3 \pmod{4}$ , then  $\{1, \alpha, 2\beta - 1, \gamma\}$  is an integral basis

of  $K$  and the field discriminant  $d(K) = 16\ell^2 m_1^2 m_2^2$  holds. For  $\xi_1 = \alpha - (2\beta - 1) + 2\gamma$  we can compute  $\text{Ind } \xi_1 \equiv \ell m_1^2 \equiv 1 \pmod{2}$ . If  $(\ell - m_2)m_1 \not\equiv 0 \pmod{3}$ , then  $\text{Ind } \xi_1 \not\equiv 0 \pmod{3}$  follows. If  $m_1 \equiv 0 \pmod{3}$ , then for  $\xi_2 = \alpha + (2\beta - 1)$  we have  $\text{Ind } \xi_2 \equiv \ell^2 m_2 \not\equiv 0 \pmod{3}$ . If  $\ell - m_2 \equiv 0 \pmod{3}$ , then  $\ell m_2 \not\equiv 0 \pmod{3}$ . We can restrict  $m_1 \not\equiv 0 \pmod{3}$ . In the case of  $\ell - m_2 \equiv 0 \pmod{9}$ , we have  $\ell - 4m_2 \not\equiv 0 \pmod{9}$ . Then  $\text{Ind } \xi_1 \not\equiv 0 \pmod{9}$ . In the case of  $\ell - m_2 \equiv \pm 3 \pmod{9}$ , for  $\xi_3 = \alpha + (2\beta - 1) + \gamma$  we get  $\text{Ind } \xi_3 \equiv |(\ell - m_2)m_1^2| \equiv \pm 3 \pmod{9}$ . Thus we obtain  $e = 0$  and  $e' \leq 1$ .

ii)<sub>2</sub> if  $\ell m_1 \equiv 3$ ,  $\ell m_2 \equiv 2 \pmod{4}$ , then  $\{1, 2\alpha - 1, 2\beta - 1, \gamma\}$  is an integral basis and  $d(K) = 64\ell^2 m_1^2 m_2^2$ . By  $\ell \equiv 1 \pmod{2}$ ,  $\ell - m_1 \equiv 2 \pmod{4}$  for  $\xi_1 = \gamma$  we have  $\text{Ind } \xi_1 \equiv 1 \pmod{2}$ . Next if  $(\ell - m_1)m_2 \not\equiv 0 \pmod{3}$ , then  $\text{Ind } \xi_1 \not\equiv 0 \pmod{3}$  holds. If  $m_2 \equiv 0 \pmod{3}$ , then for  $\xi_2 = (2\alpha - 1) + (2\beta - 1)$  we have  $\text{Ind } \xi_2 = |(4\ell)^2 (4m_2 - m_1)| \not\equiv 0 \pmod{3}$ . If  $\ell - m_1 \equiv 0 \pmod{9}$ , then  $\ell m_1 \not\equiv 0 \pmod{3}$ . We can restrict  $m_2 \not\equiv 0 \pmod{3}$ . For  $\xi_3 = 2(2\beta - 1) + \gamma$   $\text{Ind } \xi_3 \equiv |(-m_2)(25\ell - m_1)(25m_2)| \equiv \pm 3 \pmod{9}$  holds. If  $\ell - m_1 \equiv \pm 3 \pmod{9}$ , then  $\text{Ind } \xi_1 \equiv \pm 3 \pmod{9}$ . Therefore we have  $e = 0$  and  $e' \leq 1$ .

ii)<sub>3</sub> If  $\ell m_1 \equiv \ell m_2 \equiv 1 \pmod{4}$ , then  $\{1, \alpha, \beta, \alpha\beta \pm ((\ell - 1)/4)(2\gamma - 2\beta + 1)\}$  for  $\ell \equiv m_1 \equiv m_2 \equiv 1 \pmod{4}$  and  $\{1, \alpha, \beta, \alpha\beta + (1/2) \mp ((\ell - 1)/4)(2\gamma - 2\beta + 1)\}$  for  $\ell \equiv m_1 \equiv m_2 \equiv 3 \pmod{4}$  are integral bases, where the sign is positive if and only if  $m_1 < 0$  and  $m_2 < 0$ . For any integer  $\xi$  in  $K$  we have  $\text{Ind } \xi \equiv 0 \pmod{2}$ . Moreover in the case of  $m_1 - m_2 \equiv 4 \pmod{8}$  (resp.  $\equiv 0 \pmod{8}$ ), for  $\xi_1 = \alpha + \beta$  (resp.  $2\alpha + \beta$ ) we get  $\text{Ind } \xi_1 \equiv 4 \pmod{8}$ . If  $\ell(m_1 - m_2) \not\equiv 0 \pmod{3}$ , then  $\text{Ind } \xi_1 \not\equiv 0 \pmod{3}$ . We denote by  $\delta$  the fourth numbers of the integral bases. If  $\ell \equiv 0$  and  $m_1 - m_2 \not\equiv 0 \pmod{3}$ , then for  $\xi_2 = \alpha + \beta + 2\delta$

we have  $\text{Ind } \xi_2 \equiv |m_1 m_2 (m_1 - m_2)| \not\equiv 0 \pmod{3}$ . If  $\ell \neq 0$  and  $m_1 - m_2 \equiv 0 \pmod{9}$ , then  $m_1 m_2 \not\equiv 0 \pmod{3}$  holds. If for  $\xi_3 = 2\alpha + \beta$   $\text{Ind } \xi_3 = |4\ell^2(m_2 - 4m_1)| \equiv 0 \pmod{9}$ , then we have  $3m_1 \equiv 0 \pmod{9}$ . This is a contradiction. If  $\ell \not\equiv 0 \pmod{3}$  and  $m_1 - m_2 \equiv \pm 3 \pmod{9}$ , then  $\text{Ind } \xi_1 \equiv \pm 3 \pmod{9}$ . Next if  $\ell \equiv 0 \pmod{3}$  and  $m_1 - m_2 \equiv \pm 3 \pmod{9}$ , then  $\text{Ind } \xi_2 \equiv \pm 3 \pmod{9}$ . Finally if  $\ell \equiv 0 \pmod{3}$  and  $m_1 - m_2 \equiv 0 \pmod{9}$ , then for  $\xi_4 = \beta + 2\delta$  we have  $\text{Ind } \xi_4 \equiv \pm 3 \pmod{9}$ . The estimates of  $ii)_{1, \sim 3}$  imply  $1 \leq e \leq 2$  and  $e' \leq 1$ . Therefore we have proved Lemma 1.

3. Results. Works related to the problem of Hasse are found in [1], [4], [5] and the references mentioned in [7]. From [6] and [7] we have

**Theorem 1.** There exist infinitely many non-cyclic but abelian (resp. exist cyclic) biquadratic fields over  $\mathbb{Q}$  whose integer rings have a power basis.

In our case by Lemma 1 the index  $m(K)$  is not larger than 12. In fact it follows

**Theorem 2.** There exist infinitely many such abelian biquadratic fields  $K$  over  $\mathbb{Q}$  that the index is equal to 12 (resp. 6) and that neither  $\{1, \alpha, \alpha^2, \beta\}$  nor  $\{1, \alpha, \beta, \alpha^3\}$  (resp.  $\{1, \alpha, \beta, \alpha^3\}$ ) for any  $\alpha, \beta$  in  $K$  forms (resp. does not form) an integral basis of  $K$ .

The method of a proof of this theorem is the same as in [6].

i) The cyclic case. Let  $n$  be the conductor of the field  $K$ .

We choose  $n = a^2 + 72^2$ ,  $a \equiv 5 \pmod{12}$  (resp.  $n = a^2 + 12^2$ ,  $a \equiv 1 \pmod{12}$ ). Since the set  $\{1, \eta, \eta', \eta''\}$  with the Gauss period  $\eta$  makes an integral basis of  $K$ , we may put  $\xi = x\eta + y\eta' + z\eta''$  for any integer  $\xi$  in  $K$ . Then we obtain  $\text{Ind } \xi = |cN\alpha_n|$  where  $\alpha_n$  is the same number of  $Q(\sqrt{n})$  as in the previous section. By virtue of  $\lambda = a + 72i$  and

$$N\alpha_n \equiv \left\{ \begin{array}{l} (x-z)^2 y^2 - (xy + yz + zx)^2 \pmod{3} \\ 2(x+y+z)(x+z)y - (x-y)(z-y)xz \pmod{4} \end{array} \right\} \quad (\text{resp.}$$

$$\lambda = a + 12i \quad \text{and} \quad N\alpha_n \equiv \left\{ \begin{array}{l} (x-z)^2 y^2 - (xy + yz + zx)^2 \pmod{3} \\ 0 \pmod{2} \end{array} \right\} )$$

we have  $\text{Ind } \xi \equiv 0 \pmod{12}$  (resp.  $\text{Ind } \xi \equiv 0 \pmod{6}$  and  $\text{Ind } \eta \equiv 2 \pmod{4}$ ). Then by Lemma 1 we get  $m(K) = 12$  (resp.  $m(K) = 6$ ).

Moreover by  $\chi(2) = 1$  (resp.  $\left\{ \begin{array}{l} \chi(2) = -1 \\ \chi(3) = 1 \end{array} \right\}$ ) we can see

$$\sigma^j(\eta)^2 \equiv \sigma^j(\eta) \pmod{2} \quad (\text{resp.} \quad \left\{ \begin{array}{l} \sigma^j(\eta)^2 \equiv \sigma^{j+2}(\eta) \pmod{2} \\ \sigma^j(\eta)^3 \equiv \sigma^j(\eta) \pmod{3} \end{array} \right\} ). \quad \text{Since}$$

$\text{Ind } \xi$  is equal to the absolute value of the determinant of the transformation matrix for  $\{1, \xi, \xi^2, \xi^3\}$  with respect to an integral basis  $\{1, \eta, \eta', \eta''\}$ , we can see that any three rows in the matrix are linearly dependent modulo 2 (resp. the second and the fourth rows are so modulo 3). Then none of  $\{1, \alpha, \alpha^2, \beta\}$  nor  $\{1, \alpha, \beta, \alpha^3\}$  (resp.  $\{1, \alpha, \beta, \alpha^3\}$ ) for all integers  $\alpha, \beta$  can make (resp. can not make) a  $\mathbb{Z}$ -basis of  $\mathcal{O}_K$ . Finally our parametrization satisfies the next lemma.

Lemma 2[6]. For  $a > 0, b, c \in \mathbb{Z}$ ,  $a \equiv b, c \equiv 1 \pmod{2}$ , set

$$n(t) = at^2 + bt + c.$$

Let the congruences  $n(t) \equiv 0 \pmod{q^2}$  have at most two solutions



for every prime  $q$  within  $1 \leq t \leq q^2$ . Then the number  $n(t)$  is square-free for infinitely many  $t \in \mathbb{Z}$ .

ii) The non-cyclic case. For a field  $K = \mathbb{Q}(\sqrt{\ell m_1}, \sqrt{\ell m_2})$  assume  $\ell m_1 \equiv \ell m_2 \equiv 1 \pmod{24}$ . Then using Lemma 1 we have  $m(K) = 12$ . Next we choose  $\ell m_1 \equiv \ell m_2 \equiv 1 \pmod{3}$ ,  $\ell \equiv 5 \pmod{8}$  and  $m_1 \equiv m_2 \equiv 1 \pmod{16}$ . Then we can see  $\text{Ind} \xi \equiv 0 \pmod{6}$  for any integer  $\xi$  in  $K$ . Also for  $\xi_0 = ((1 + \sqrt{\ell m_1})/2) + ((1 + \sqrt{\ell m_1}) \times (1 + \sqrt{\ell m_2})/4) + ((\ell - 1)/4)\sqrt{m_1 m_2}$  it follows  $\text{Ind} \xi_0 \equiv 2 \pmod{4}$ . Thus we obtain  $m(K) = 6$ . Under this parametrization we can perform the same argument as in the case i). Therefore we obtain Theorem 2.

Remark 1. Among the fields  $K$  with even conductor there does not exist any  $K$  which satisfies the properties in Theorem 2.

Theorem 3. There exist infinitely many non-cyclic but abelian biquadratic fields  $K$  which have the index 1 and still whose minimum indices are greater than  $N$  for any given integer  $N$ . Consequently the integer rings  $\mathcal{O}_K$  have not a power basis.

Proof. We consider the field  $K_\ell = \mathbb{Q}(\sqrt{\ell m_1}, \sqrt{\ell m_2})$  with  $\ell m_1 \equiv 1, \ell m_2 \equiv -1 \pmod{12}$ . Then from Lemma 1 the index  $m(K_\ell)$  is odd. Under the same notations as in the proof ii), of Lemma 1 for a number  $\xi = x\alpha + y\beta + z\gamma$  we obtain

$$\text{Ind} \xi = |(x^2 \ell - z^2 m_2)(z^2 m_1 - (2y + z)^2 \ell)(x^2 m_1 - (2y + z)^2 m_2)|/4.$$

Thus it holds that  $\text{Ind}(\alpha + \beta) \not\equiv 0 \pmod{3}$ . Then  $m(K_\ell) = 1$  holds. In an imaginary case we select  $0 > m_1 \equiv 1, 0 < -m_2 \equiv 1, 0 < \ell \equiv 1 \pmod{12}$ . Then  $\text{Ind} \xi > \ell$  holds for any primitive element  $\xi$  in  $\mathcal{O}_{K_\ell}$ .

In a real case set  $0 < \ell \equiv -1 \pmod{12}$ . We estimate the factor  $I = x^2 m_1 - (2y + z)^2 m_2$  of  $\text{Ind} \xi$ . For any integer  $N > 0$  we can find the following primes  $p_i \equiv -1, q_i \equiv 1 \pmod{12}$  and  $p_i \neq \ell$  for  $1 \leq i \leq N$ . Put  $m_1 = p_1$  and  $m_2 = q_1$  such that

$$\left(\frac{x^2 p_1}{q_1}\right) = \left(\frac{p_1}{q_1}\right) \neq \left(\frac{1}{q_1}\right), \text{ where } \left(\frac{*}{p}\right) \text{ denotes the Legendre symbol.}$$

Then  $I \neq \pm 1$ . Next for a prime  $q_2 > q_1$ , there exists an integer

$$a_2 \text{ with } \left(\frac{a_2}{q_2}\right) \neq \left(\frac{2}{q_2}\right). \text{ We select } p_2 \text{ such that } p_2 \equiv \begin{cases} p_1 \pmod{q_1} \\ a_2 \pmod{q_2} \end{cases}.$$

Reset  $m_1 = p_2, m_2 = q_1 q_2$ , then  $I = \pm 1, \pm 2$ . Successively we

$$\text{can choose primes } p_N, q_N \text{ such that } p_N \equiv \begin{cases} p_{N-1} \pmod{q_1 \dots q_{N-1}} \\ a_N \pmod{q_N} \end{cases}$$

$$\text{with } q_N > q_{N-1} \text{ and } \left(\frac{a_N}{q_N}\right) \neq \left(\frac{N}{q_N}\right). \text{ For } m_1 = p_N, m_2 = q_1 \dots q_N$$

define the biquadratic field  $K_N = Q(\sqrt{\ell m_1}, \sqrt{\ell m_2})$ , then it holds that  $\mathfrak{M}(K_N) > N$ . Therefore we have proved Theorem 3.

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