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On the Schnirelman density of the K-free integers

M.V. Subbarao
University of Alberta
Edmonton, Canada T6G 2G1

1. Introduction and notation. In this paper we state several results and some problems concerning the K-free integers and some related sets of integers. Our interest here is mainly about the Schnirelman density of these sets of integers, and where it is attained, and to estimate, where possible, the difference between this density and the corresponding asymptotic density of the set under consideration.

Let K be an integer > 1 and Q_K the set of K-free integers, that is, integers not divisible by the Kth power of any prime. Note that unity belongs to Q_K. We write $Q_K(x)$ for the number of K-free integers not exceeding x.

The K-free integers are generalized in several ways. For integers r, K for which $1 < r < K$, we have the (K,r)-integers extensively studied, for example in ([7],[8],[10],[19],[20],[21],[22]). By definition, a (K,r)-integer is one whose Kth power-free part is also rth power free. The set of such integers will be denoted by $Q_{K,r}$, and the number of such integers $\leq x$ by $Q_{K,r}(x)$.

An integer is said to be semi K-free if no Kth power of any prime unitarily divides n [23]. p^K unitarily divides n precisely when n is divisible by p^K, but not by p^{K+1}. The K-free integers as well as the semi K-free integers are special cases of (K,r)-free integers [11] defined thus: for $1 < r < K \leq \infty$, n is said to be (K,r)-free provided in the prime-power decomposition of n, the exponent of every prime is either < r or $\geq$ K. Let $Q_{(K,r)}$ be the set of all such integers. These are extensively studied in [11, 12, 13].

We limit ourselves in this paper only to the sets Q_K, $Q_{K,r}$ and $Q_{(K,r)}$, and consider their Schnirelman and asymptotic densities.

Throughout what follows, $\zeta(s)$ denotes the Riemann zeta function. We denote the asymptotic and Schnirelman desities of Q_K by D_K and d_K respectively. Similar notation will be used for the corresponding densities of the sets $Q_{K,r}$ and $Q_{(K,r)}$. Thus we have

$$D_K = \lim_{x\to\infty} \frac{Q_K(x)}{x}$$

and

$$d_K = \inf_{n>0} \frac{Q_K(n)}{n} .$$

In view of the simple estimate

$$Q_K(x) = \frac{x}{\zeta(K)} + O(x^{1/K}) ,$$

we see that D_K exists and equals $\frac{1}{\zeta(K)}$. No such simple result exists for

d_K . Obviously, $d_K \leq D_K$ for all $K > 1$.

Our interest here is to give more information about d_K and its location relative to D_K; and similarly for the Schnirelman densities $d_{K,r}$ and $d_{(K,r)}$ of the sets $Q_{K,r}$ and $Q_{(K,r)}$ respectively. We write

$$Q_K(x) = \frac{x}{\zeta(K)} + E_K(x).$$

While the exact order of $E_K(x)$ is still not known, it is conjectured that

$$E_K(x) = 0\left(x^{\frac{1}{2K}+\varepsilon}\right)$$

for every positive ε.

The best result known so far is due to Walfisz [24]:

$$E_K(x) = 0(x^{1/k}\exp\{-cK^{-8/5}(\log x)^{3/5}(\log \log x)^{-1/5}\})$$

where c is an absolute constant.

If the truth of the Riemann hypothesis is assumed, better order results can be obtained. For example, Montgomery and Vaughn [15] proved in 1976 that

$$E_K(x) = 0\left(x^{\frac{1}{K+1}+\varepsilon}\right) \text{ for every } K > 2,$$

and every positive ε, and

$$E_2(x) = O\left(x^{\frac{9}{28}+\varepsilon}\right).$$

The last result has been improved by S.W. Graham [9] to

$$E_2(x) = O\left(x^{\frac{8}{25}+\varepsilon}\right),$$

again assuming the truth of the Riemann hypothesis.

For our discussion of d_K, it is the information about the changes of sign of $E_K(x)$ that is useful, not its order results. We consider this in the next section.

2. $\underline{d_K < D_K}$ <u>and related results</u>. As already stated, it follows from the definition of d_K and D_K that $d_K \leq D_K$. The question arises whether $d_K < D_K$ for all K. In 1964, K. Rogers, [17] showed that

$$d_2 = \frac{53}{88} < \frac{6}{\pi^2} = D_2.$$

He also showed that d_2 is attained at (and only at) 176, that is, $\frac{Q_2(176)}{176} = d_2 = \frac{53}{88}$. In 1966, H.M. Stark [18] showed, by utilizing a Tauberian theorem of Ingham, that for all integers $K > 1$, we have

(2.1) $\quad d_K < D_K.$

It should be noted that (2.1) implies that d_K is attained for some value (or values) of n.

Entirely elementary proofs of (2.1) were given by R.C. Orr [16] and G.E. Hardy [11]. Orr also gave the values of d_3, d_4, d_5, d_6 and the values of n where they are attained. In [6], we extended these by tabulating the values of d_K for all $K \leq 75$ and the corresponding values of n_K, where n_K is the value of n for which $Q(n_K)/n_K = d_K$; if there exist more than one such n_K, the largest such alone in tabulated.

Note that (2.1) follows if we show that $E_K(n)$ is negative for atleast one value of n, say $n = n_0$. This is because we then have

$$\frac{Q_K(n_0)}{n_0} = \frac{1}{\zeta(K)} + E_K(n_0)$$

$$< \frac{1}{\zeta(K)} ,$$

so that $d_K < D_K$.

In fact, $E_K(n)$ changes sign infinitely many times (and so also the corresponding error term $E_{K,r}(n)$ for the set $Q_{K,r}$). This is an immediate corollary of the following useful proposition [1] which itself follows easily from a well known theorem of Landau.

2.2. Proposition. Let $\phi(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ have a finite abscissa of absolute convergence. Supoose that all the singularities of $\phi(s)$ on the real axis are poles, and that ϕ has a non-real singularity. Let H be the upper bound of the real parts of the non-real singularities of ϕ, and let $S_0(x)$ be the sum of the residues of $(\frac{\phi(s)}{s})x^s$ at the real poles $\geq h$. Then we have

$$A_\lambda^0(x) - S_0(x) = \Omega_\pm(x^{h-\varepsilon}),$$

for every $\varepsilon > 0$, where $S_0(x)$ is the sum of the residues of $(\phi(s)/s)x^s$ at the real poles $\geq h$, and

$$A_\lambda^0(x) = \sum\nolimits'_{\lambda_n \leq x} a_n,$$

the dash denoting that the last term has to be multiplied by $1/2$ if $x = \lambda_n$.

On taking $\phi(s) = \zeta(s)/\zeta(ks)$, which is the generating function for the K-free integers, and applying (2.2), we see that $E_K(x)$ changes sign infinitely often, and in particular that $E_K(x) < 0$ for infinitely many x. This gives $d_K < D_K$.

Since

$$(2.3) \qquad Q_2 \subset Q_3 \subset \cdots \subset Q_K \subset, \ldots,$$

we have

$$(2.4) \qquad d_2 \leq d_3 \leq \cdots \leq d_K \leq \cdots \leq 1.$$

As R.L. Duncan observed [4], the asymptotic and Schnirelman densities interlace:

(2.5) $$d_K < D_K < d_{K+1} < D_{K+1}.$$

Further, a simple estimate gives

(2.6) $$D_K > 1 - \sum_{p \text{ prime}} p^{-K} .$$

This is also due to Duncan [5] who also proved that d_{K+1} is closer to D_{K+1} than to D_K, and in fact,

(2.7) $$\frac{D_{K+1} - d_{K+1}}{D_{K+1} - D_K} < \frac{1}{2^K} .$$

3. Better bounds for d_K and n_K. Let n_K denote any of the values of n where $\frac{Q_K(n)}{n}$ attains the value d_K. In 1969, R.C. Orr [16] proved the important result that for $K \geq 5$,

(3.1) $$5^K \leq n_K < 6^K$$

with the help of this result, P.H. Diananda and M.V. Subbarao [4] vastly imporved the lower bound (2.6) for d_K given by Duncan by showing, among others, that

(3.2) $\qquad d_K > 1 - 2^{-K} - 3^{-K} - 5^{-K}$;

and

(3.3) $\qquad$ for $K \geq 5$, $d_K \geq 1 - 2^{-K} - 3^{-K} - 5^{-K} + \left(\frac{3^{-K}+2 \cdot 5^{-K}}{6^K-3^K+1}\right)$.

They also improved (3.1) by showing that the largest n_K for any given K satisfies

(3.4) $\qquad \frac{1}{2} 6^K \leq n_K < 6^K$.

Further, they proved the

3.5 <u>Theorem</u>. For $K \geq 5$, there is an n_K so that

(i) $3^K | n_K$ or $5^K | n_K$, or

(ii) $2^K | n_K$ and between $n_K - 2^K$ and n_K there is a multiple of 3^K or 5^K.

It is with the help of the results (3.4) and (3.5) and the computer that we could find the values of d_K and n_K for $K \leq 75$. These are listed in [6], as stated earlier.

The values of n_K, $Q(n_K)$, and d_K, D_K (each correct to ten decimal places) are given iat the end of paper [4]. They show how rapidly d_K approaches Q_K, while each $\rightarrow 1$ as $K \rightarrow \infty$: For instance,

$$D_{12} - d_{12} \approx .0000000004.$$

But in view of the result (3.2), one would be interested to know if d_K is closer to $1 - 2^{-K} - 3^{-K} - 5^{-K}$ than to D_K.

That this is indeed so is proved in [6]. Actually, more than this is shown, namely,

<u>Theorem</u> $$\frac{d_K - (1-2^{-K}-3^{-K}-5^{-K})}{D_K - d_K} = o\left(\frac{2}{3}\right)^K \to 0 \text{ as } K \to \infty.$$

4. <u>A conjecture concerning d_K and n_K.</u> Theorem 3.5 shows that the largest n_K where d_K is attained, besides lying in the interval $[\frac{1}{2} 6^K, 6^K)$ must satisfy one of the following possibilities:

(4.1) n_K = a multiple of 2^K following a multiple of 3^K that follows a multiple of 5^K as shown below

$$\underset{5^K}{}\overline{\underset{3^K}{\overset{a+m\cdot 3^K}{}}}\underset{2^K}{}$$

(4.2) n_K = a multiple of 3^K. This may follow a multiple of 2^K or 5^K.

(4.3) n_K = a multiple of 2^K following a multiple of 5^K.

(4.4) n_K = a multiple of 5^K. This may follow a multiple of 2^K or 3^K.

Thus, there are atmost $2^K + (\frac{6}{5})^K$ values for n_K, of which $\frac{1}{2}(\frac{6}{5})^K$ arise from the situation (3.1), 2^K from (3.2) $\frac{1}{2}(\frac{6}{5})^K$ from (3.3) and (3.4).

Using the values of d_K for $K \leq 75$, that we have in [6], we find that n_K satisfies the situation described in (3.1) with $m = 0$ in the case $K = 5$ and all K in the range $13 \leq K \leq 75$.

Hence we are led to make the following

4.5 Conjecture. For all sufficiently large K, we have $d_K = Q_K(n_0)/n_0$ for some integer n_0 which is the first multiple of 2^K following the first multiple of 3^K that follows some multiple of 5^K in $[\frac{1}{2}6^K, 6^K)$.

5. The Schnirelman denisites $d_{K,r}$ and $d_{(K,r)}$. We consider this only very briefly. Utilizing the geneating function $\frac{\zeta(Ks)\zeta(s)}{\zeta(r,s)}$ for the (K,r)-integers, we showed in [20] that for $1 < r < K$,

$$(5.1) \qquad d_{K,r} < D_{K,r} = \frac{\zeta(K)}{\zeta(r)} .$$

This can also be obtained from the fact that the associated error function $E_{K,r}(x)$ changes sign infinitely often - a consequence of Theorem 2.2. For an entirely elementary proof, see [14].

Corresponding to (2.2) we can prove that for all $0 < r < K$, we have

$$d_{K,r} > \zeta(K)(1-2^{-r}-3^{-r}-5^{-r}) - \frac{1}{K-1}\{1 - \frac{1}{K}(2^{-r}+3^{-4}+5^{-r})\}^K \tag{5.2}$$

The details of proof will appear elsewhere [21]. It can also be shown [11] that with the possible exception of the case when $r = 2$, $K = 3$, we have for $2 \leq r < K \leq \infty$,

$$d_{(K,r)} < D_{(K,r)} = \prod_{p \text{ prime}}(1-p^{-r}+p^{-K}).$$

It is also possible to show ([11]) that for $3 \leq r < K < \infty$, both the densities $d_{K,r}$ and $d_{(K,r)}$ are attained somewhere on the interval $[2^r,8^r)$ (see [11]). This can be improved for special values of K. Also for most values of K and r (except for $(K,r) = (2,3)$), we have $d_{K,r} = d_{(K,r)}$ and are achieved at the same point. As samples, we mention only three results:

(5.3) If $2^K < \dfrac{15^r - 9^r}{3^r + 5^r}$,

then

$$n_{K,r} = n_{(K,r)} = 2^r \, ;$$

(5.4) If $\dfrac{15^r - 9^r}{3^r + 5^r} < 2^r < \dfrac{15^r}{3^r + 5^r}$, then

$$n_{(K,r)} = 2^r \quad \text{and} \quad n_{K,r} \in [2^r, 6^r).$$

(5.5) If $3^r < 2^K < 3^r + 2^r$,

then

$$n_{K,r} = n_{(K,r)} \in [3^r, 6^r).$$

For proofs of these and for other results, we refer to [11], which also gives extensive tables of $n_{K,r}$ and $n_{(K,r)}$. For other results concerning $d_{K,r}$ we refer to [3].

6. Some open problems

(I). Let c be a fixed number > 2 and let $p_1 < p_2 < \ldots$ be the sequence of consecutive primes greater than c. Let $S(p_1^K, p_2^K, \ldots)$ denote the set of integers no one of which is divisible by any p_i^K for a fixed $K \geq 2$. Then its asymptotic density is $\prod_{p_i > c} (1 - \frac{1}{p_i^K})$.

Is its Schnirelman density less than its asymptotic density?

What happens if $p_1 < p_2 < \ldots$ is any sequence of primes $> c$, not necessarily consecutive?

(II) Are there infinitely many integers K for which the corresponding Schnirelman density d_K is attained at more than one point n_K. What is the asymptotic density of the set of K for which n_K is unique?

(III) Analogous to the interlacing property of D_K and d_K given in (2.5), do there exist results for $d_{K,r}$ and $D_{K,r}$; and $d_{(K,r)}$, $D_{(K,r)}$?

(IV) We stated in (2.5) that

$$D_{K+1} - d_{K+1} > 0.$$

What about the second and higher differences?

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