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On Convergence of Some Infinite Products

Takeshi Kano (Okayama Univ.)

§1. In [2] Hardy showed among other things that if $a_n \downarrow 0$ and

$$\sum_{n=0}^{\infty} a_n^k = +\infty$$

for all integers $k \geq 1$, then the infinite product

$$(1) \qquad \prod_{n=0}^{\infty} (1 + a_n e^{2\pi i n\theta})$$

diverges for all $\theta \in \mathbb{Q}$.

He then raised the problem to settle the case of convergence. It was soon after answered (partially) by Littlewood [4], who proved that (1) converges for all algebraic irrational values of θ if $a_n \downarrow 0$.

First we remark that his argument in fact yields the following more general assertion.

<u>Theorem</u> 1. Let $a_n \to 0$ be of bounded variation, i.e.

$$\sum_{n=0}^{\infty} |a_n - a_{n+1}| < +\infty .$$

Then (1) converges for all $\theta \notin \mathbb{Q}$, with the possible exception of Liouville numbers.

Furthermore, the following result is implicit in his proof.

Theorem 2. For any $a_n \to 0$ of bounded variation there exist Liouville numbers θ such that (1) converges.

It seems however still unknown whether there exists a Liouville number θ such that (1) diverges, e.g. when $a_n \downarrow 0$.

Does (1) converge for all $\theta \notin \mathbb{Q}$ if $a_n \downarrow 0$?

§2. We consider the real products

$$\prod_{n=0}^{\infty}(1 + a_n \cos 2\pi n\theta) \tag{2}$$

$$\prod_{n=0}^{\infty}(1 + a_n \sin 2\pi n\theta), \tag{3}$$

where a_n is not necessarily of bounded variation. First we note the following simple fact.

Theorem 3. If (1) diverges to 0, so do both (2) and (3), too.

This is immediate from the obvious lemma below.

Lemma. Let $\{z_n\}$ $(n = 1,2,\dots,N)$ be a sequence of complex numbers. Then

$$\left|\prod_{n=1}^{N}(1+z_n)\right| \geq \left|\prod_{n=1}^{N}\left(1 + {}^{\mathrm{Re}}_{\mathrm{Im}}\, z_n\right)\right|.$$

Proof.

$$\left|\prod_{n=1}^{N}(1+z_n)\right| = \prod_{n=1}^{N}|1+z_n| \geq \prod_{n=1}^{N}\left|{}^{\mathrm{Re}}_{\mathrm{Im}}(1+z_n)\right|$$

$$= \prod_{n=1}^{N} |1 + {}^{\mathrm{Re}}_{\mathrm{Im}} z_n| = |\prod_{n=1}^{N} (1 + {}^{\mathrm{Re}}_{\mathrm{Im}} z_n)| .$$

Next we show

<u>Theorem</u> 4. If

$$(4) \qquad \sum_{n=0}^{\infty} a_n^2 < +\infty,$$

then all of (1) - (3) converge for almost all θ.

Proof. Since (4) implies $|a_n| < 1$, we have

$$\sum_{n=0}^{N} \log(1 + a_n e^{2\pi i n\theta}) = \sum_{n=0}^{N} a_n e^{2\pi i n\theta} + O(\sum_{n=0}^{N} a_n^2) .$$

According to Carleson's L^2- theorem [5], it follows that

$$\sum_{n=0}^{\infty} a_n e^{2\pi i n\theta}$$

converges for almost all θ if (4) holds. Hence (4) implies the almost convergence of

$$\sum_{n=0}^{\infty} \log(1 + a_n e^{2\pi i n\theta}) .$$

§3. Let us now suppose $a_n \to 0$ to be of bounded variation. Then the situation in (2) and (3) is quite different from that in (1).

Theorem 5. Let $a_n \to 0$ be of bounded variation. Then, if (4) holds, both of (2) and (3) converge for all $\theta \in (0,1)$.

If on the contrary

$$\sum_{n=0}^{\infty} a_n^2 = +\infty, \tag{5}$$

then both of (2) and (3) diverge to 0 for all $\theta \in (0,1)$.

Proof. If (4) holds, the procedure of proof is the same as in Theorem 4. Indeed, it suffices to note the fact that if $a_n \to 0$ is of bounded variation, then

$$\sum_{n=0}^{\infty} a_n \begin{matrix}\sin\\ \cos\end{matrix} 2\pi n\theta \tag{6}$$

converges for all $\theta \in (0,1)$ (cf.[1] Chap.I, §30).

Suppose (5) holds, and $|a_n| < 1$. Since for $x \in (-1,1)$

$$x - \log(1 + x) \geq \frac{1}{4} x^2,$$

we have

$$\sum_{n=0}^{N} a_n \cos 2\pi n\theta - \sum_{n=0}^{N} \log(1 + a_n \cos 2\pi n\theta) \geq \frac{1}{4} \sum_{n=0}^{N} a_n^2 \cos^2 2\pi n\theta \tag{7}$$

$$= \frac{1}{8} \sum_{n=0}^{N} a_n^2 + \frac{1}{8} \sum_{n=0}^{N} a_n^2 \cos 4\pi n\theta .$$

We observe that

$$\sum_{n=0}^{\infty} |a_n^2 - a_{n+1}^2| = \sum_{n=0}^{\infty} |a_n + a_{n+1}| \cdot |a_n - a_{n+1}| \leq 2 \sum_{n=0}^{\infty} |a_n - a_{n+1}| < +\infty,$$

and hence both of $\sum a_n \cos 2\pi n\theta$ and $\sum a_n^2 \cos 4\pi n\theta$ converge for all $\theta \in (0,1)$, $\theta \notin 1/2$. Therefore it follows from (7) that (2) diverges to 0 for all $\theta \in (0,1)$, $\theta \neq 1/2$. Also the case $\theta = 1/2$ is obvious from (7).

Corollary.

$$\prod_{n=1}^{\infty} \left(1 + \frac{\cos 2\pi n\theta}{\sqrt{n}}\right) \qquad \text{diverge for all } \theta .$$

$$\prod_{n=2}^{\infty} \left(1 + \frac{\sin 2\pi n\theta}{\sqrt{n}\log n}\right) \qquad \text{converge for all } \theta .$$

Our argument in the preceding theorem also shows the following metric result, where a_n is not assumed to be of bounded variation.

Theorem 6. If (6) converges almost everywhere to an integrable function which is not in L^2, then (2) and (3) diverge to 0 almost everywhere.

We only remark for the proof that (5) implies

$$\sum_{n=0}^{\infty} a_n \left({\cos \atop \sin} 2\pi n\theta\right) = +\infty$$

for almost all θ (cf. [3] Chap. XIV).

References

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