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Reidemeister Torsion of a Homology Lens Space

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In my talk, I announced a formula giving the Reidemeister torsion of a homology lens space obtained by Dehn surgery on a knot in the three sphere associated to the universal abelian covering. And we applied it to get a generalization of Fukuhara's result in [2] that is a generalization of the classification of lens spaces.

After that, I found that essentially same result is already obtained by Turaev[4] in a more general setting. Here I will give the precise statement of result. For a proof of Theorem 1, the reader can refer the paper of Turaev[4] or a self-contained treatment in [3].

Statement of Results

Let k be a knot in the three sphere S^3. For coprime integers $p > 0$ and q, let $K = L(p,q:k)$ denote a 3-manifold obtained from S^3 by Dehn surgery on k with coefficient p/q. Then $H_1(K,Z)$ is isomorphic to the cyclic group of order p generated by a meridian loop of k. Let $\tilde{K}$ denote the universal abelian covering of K with the covering transformation group π and let T denote a generator of π corresponding to the meridian loop of k. We assume that K is triangulated. Then the integral cellular chain group $C_q(\tilde{K};Z)$ can be considered as a $Z\pi$-free module with the standard basis determined by K which is well defined up to sign, and up to multiplication, by elements of π.

Suppose a homomorphism h from π to the field of complex numbers F is given that takes T into a p-th root of unity $\tau(\neq 1)$. Using h, we can form the chain complex

$$C_* = F \otimes_\pi C_*(\tilde{K};Z)$$

over F. Then C_q is a finite dimensional vector space over F with the standard basis determined by the basis for $C_q(\tilde{K};Z)$ above(thus determined by K). For each q, let v_q denote the volume in C_q determined by this basis. Then

Theorem 1. Suppose that the Alexander polynomial of k is A(t). Then C_* is acyclic if and only if $A(\tau) \neq 0$. Therefore, if $A(\tau) \neq 0$, the Reidemeister torsion $\Delta_h(\tilde{K})$ is defined as $\pm h(\pi) v_0 v_1^{-1} v_2 v_3^{-1}$ and is equal to

$$\pm h(\pi) A(\tau)(\tau^r - 1)^{-1}(\tau - 1)^{-1}$$

where r is determined by the congruence $qr \equiv 1 (\mathrm{mod}\ p)$.

The following generalizes Theorem 2 in Fukuhara[2], which he proved by using EA-matrix, an invariant for closed orientable 3-manifold defined by Fukuhara and Kanno[1].

Theorem 2. Let k and k' be knots in S^3 with trivial Alexander polynomials. Then L(p,q:k) is homeomorphic to L(p,q:k') only if $\pm qq' \equiv 1 (\mathrm{mod}\ p)$ or $\pm q \equiv q' (\mathrm{mod}\ p)$.

The classification of lens spaces is the case that k and k' are trivial and Fukuhara's result is the case that k has the trivial Alexander polynomial and k' is trivial.

References

[1] Fukuhara, S. and Kanno, J.: Extended Alexander matices of 3-manifolds. Preprint. Tsuda College. Tokyo, Japan(1983).

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[4] Turaev, V.G.: Reidemeister torsion and the Alexander polynomial. Math. USSR Sbornik 30, (1976), 221-237.