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Paths and Edge-Connectivity in Graphs

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1. INTRODUCTION

We consider finite undirected graphs passably with multiple edges but without loops. Let $G$ be a graph and let $V(G)$ and $E(G)$ be the sets of vertices and edges of $G$ respectively. For two distinct vertices $x$ and $y$, let $\lambda_q(x,y)$ be the maximal number of edge-disjoint paths between $x$ and $y$, and let $\lambda_q(x,x) = \infty$. For an integer $k \geq 1$, let $\Gamma(G,k)$ be

$$(X \subseteq V(G) \mid \text{For each } x, y \in X, \lambda_q(x,y) \geq k).$$

Let $(s_1, t_1), \ldots, (s_k, t_k)$ be pairs of vertices of $G$. When is the following statement true?

(1.1) There exist edge-disjoint paths $P_1, \ldots, P_k$ such that $P_i$ has ends $s_i, t_i (1 \leq i \leq k)$.

Seymour [8] and Thomassen [9] characterised such graphs when $k=2$, and Seymour [8] when $|\{s_1, \ldots, s_k, t_1, \ldots, t_k\}|=3$.

For integers $k \geq 1$ and $n \geq 2$, set

$$g(k) = \min \{m \mid \text{If } G \text{ is } m\text{-edge-connected, then (1.1) holds}\},$$

$$\lambda^*(k,n) = \min \left\{ m \mid \begin{array}{l}
\text{If } |\{s_1, \ldots, s_k, t_1, \ldots, t_k\}| \leq n \text{ and } \\
(s_1, \ldots, s_k, t_1, \ldots, t_k) \in \Gamma(G,m), \text{ then (1.1) holds}
\end{array} \right\}.$$
\[ \lambda(k,n) = \min \left\{ m \left| \begin{array}{l} \text{If } l(s_1, \ldots, s_k, t_1, \ldots, t_k) \leq n \text{ and} \\ \lambda_q(s_i, t_i) \geq m \ (1 \leq i \leq k), \text{ then (1.1) holds} \end{array} \right. \right\}, \]

and set

\[ \lambda'(k) = \lambda'(k, 2k) = \lambda'(k, m) \ (m > 2k) \text{ and } \lambda(k) = \lambda(k, 2k). \]

Then for each \( k \geq 1, \)

\[ \lambda'(k, 3) = \lambda(k, 3) \text{ and } \lambda(k) \geq \lambda'(k) \geq g(k) \geq k. \]

For \( n \geq 4 \) and even integer \( k \geq 2, \)

\[ g(k) \geq k \text{ and } \lambda(k) \geq \lambda(k, n) \geq \lambda'(k, n) \geq k \]

(see Figure 1 in which \( k/2 \) represents the number of parallel edges).

**Figure 1.**

Thomassen [9] gave following Conjecture 1, and we give following Conjecture 2 slightly stronger than Conjecture 1.

**CONJECTURE 1.** For each integer \( k \geq 1, \)

\[ g(k) = \begin{cases} k & \text{if } k \text{ is odd} \\ k+1 & \text{if } k \text{ is even} \end{cases}. \]

**CONJECTURE 2.** For each integer \( k \geq 1, \)

\[ \lambda(k) = \begin{cases} k & \text{if } k \text{ is odd} \\ k+1 & \text{if } k \text{ is even} \end{cases}. \]
It easily follows from Menger's theorem that $\lambda(k) \leq 2k-1$; thus $\lambda(1)=1$ and $\lambda(2)=3$. Cypher [1] proved $\lambda(4) \leq 6$ and $\lambda(5) \leq 7$, and $\lambda(3)=3$ was announced in [5] and proved in [6] by the author. Enomoto and Saito [2] proved $g(4)=5$, and independently Hirata, Kubota and Saito [3] proved $\lambda(k) \leq 2k-3$ ($k \geq 4$).

Our main results are the following.

**THEOREM 1.** Suppose that $k \geq 2$ is an integer, $G$ is a graph, $(a_1,a_2) \subseteq T \subseteq V(G)$, $|T| \leq 3$ and $T \in \Gamma(G,k)$. Then there exists a path $P$ between $a_1$ and $a_2$ such that $T \in \Gamma(G-E(P),k-1)$.

**THEOREM 2.** Suppose that $k \geq 5$ is an odd integer, $G$ is a graph, $(a_1,a_2,a_3) \subseteq T \subseteq V(G)$, $a_i \neq a_j$ ($1 \leq i < j \leq 3$), $|T| \leq 5$ and $T \in \Gamma(G,k)$. Then

1. If $|T| \leq 4$, then there exists a path $P$ between $a_1$ and $a_2$ such that $T \in \Gamma(G-E(P),k-1)$.

2. For $m=2,3$ if $|T| \leq 4$ and for $m=3$ if $|T| = 5$, there exist edge-disjoint paths $P_i$ between $a_1$ and $a_2$ and $P_2$ between $a_1$ and $a_m$ such that $T \in \Gamma(G- \bigcup_{i=1}^{m-2} E(P_i),k-2)$.

**THEOREM 3.** For each integer $k \geq 1$,

$\lambda(k,3)=k$ and $\lambda(k,4)=\lambda(k,5)=\begin{cases} k & \text{if } k \text{ is odd} \\ k+1 & \text{if } k \text{ is even}. \end{cases}$

In Theorem 2(2) if $m=2$ and $|T|=5$, then the conclusion does not always hold. Figure 2 gives a countexample with $k=7$. 

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Figure 2.

When \( k \) is odd and \( \{s_1, \ldots, s_k, t_1, \ldots, t_k\} \geq 4 \), if for some \( 1 \leq i \leq k \),

\[ \lambda_G(s_i, t_i) \leq k, \]

then (1.1) does not always hold. Figure 3 gives a counterexample.

Figure 3.

Notations and Definitions. Let \( X, Y \subseteq V(G) \), \( F \subseteq E(G) \), \( (x, y) \subseteq V(G) \) and \( e \in E(G) \). We often denote \( (x) \) by \( x \) and \( (e) \) by \( e \). The subgraph of \( G \) induced by \( X \) is denoted by \( \langle X \rangle_G \) and the subgraph obtained from \( G \) by deleting \( X \) (\( F \)) is denoted by \( G-X \) (\( G-F \)). \( \partial_G(X, Y) \) denotes the set of edges with one end in \( X \) and the other in \( Y \), and \( \partial_G(X) \) denotes \( \partial_G(X, V(G)-X) \). \( \lambda_G(X, Y) \) denotes the maximal number of edge-disjoint paths with one end in \( X \) and the other in \( Y \). \( \partial_G(X) \)
is called an n-cut if $|\partial_G(X)|=n$ and $\langle X \rangle_G$ and $\langle V(G)-X \rangle_G$ are both connected. An n-cut $\partial_G(X)$ is called nontrivial if $|X| \geq 2$ and $|V(G)-X| \geq 2$, trivial otherwise. $d_G(x)$ denotes the degree of $x$ and $N_G(x)$ denotes the set of vertices adjacent to $x$. We regard a path and a cycle as subgraphs of $G$. A path $P=P[x,y]$ denotes a path between $x$ and $y$, and for $x',y' \in V(P)$, $P(x',y')$ denotes a subpath of $P$ between $x'$ and $y'$.

2. PROOF OF THEOREM 1

For a vertex $w \in V(G)$ and $b,c \in N_G(w)$, we let $G^b,c_w$ be the graph $(V(G), E(G) \cup e-(f,g))$, where $e$ is a new edge with ends $b$ and $c$, $f \in \partial_G(w,b)$ and $g \in \partial_G(w,c)$. We require the following lemmas.

LEMMA 2.1 (Mader [4]). Suppose that $w$ is a non-separating vertex of a graph $G$ with $d_G(w) \geq 4$ and with $|N_G(w)| \geq 2$. Then there exist $b,c \in N_G(w)$ such that for each $x,y \in V(G)-w$,

$$\lambda^b,c_{G_w}(x,y)=\lambda_G(x,y).$$

Now we prove Theorem 1 by induction on $|E(G)|$. We may assume that $a_1 \neq a_2$ and $|T|=3$. If $G$ has a nontrivial $k$-cut $\partial_G(X) (X \subseteq V(G))$ separating $T$, then let $H(K)$ be the graph obtained from $G$ by contracting $V(G)-X$ (X) to a new vertex $u$ (v). Set $T_H=(X \cap T) \cup u$ and $T_K=(T-X) \cup v$. We may
let \( |T \cap X| = 2 \). By induction for \( H \) and \((T \cap X) \cup \{u\}\) instead of for \( G \) and \( T \), the result holds. Thus the result follows. Hence we may assume that each edge is incident to a vertex of \( T \).

Case 1. There exists \( x \in V(G) - T \).

If \( d_G(x) \geq 4 \), then by Lemma 2.1 there exists \( b, c \in N_G(x) \) such that for each \( y, z \in V(G) - x \),

\[
\lambda_{G^b}^{x} (y, z) = \lambda_{G}^{y, z}.
\]

By induction the result holds in \( G_x \). Thus we may assume that \( d_G(x) = 3 \) and clearly that \( N_G(x) = T \). Now the path \( P[a_1,a_2] \) with \( E(P) \subseteq \partial G(x) \) is a required path.

Case 2. \( V(G) = T \).

The result easily follows.

3. PROOF OF THEOREM 2.

We call a graph \( G \) is elemental for \( V_1 \subseteq V(G) \) if \( V(G) = V_1 \cup V_2 \), \( V_1 \cap V_2 = \emptyset \) and for each \( x \in V_2 \), \( d_G(x) = 3 \), \( \{V_1 - \partial G(x) \subseteq V_1 \). We call a graph \( G \) is elemental for \( V_1 \subseteq V(G) \) and an integer \( k \geq 1 \) if \( G \) is elemental for \( V_1 \) and for each \( x \in V_1 \), \( d_G(x) = k \). For integers \( p \geq 0 \) and \( q \geq 0 \), we call a graph \( G \) is \( G(p,q) \) if \( G \) is elemental for some \( V_1 = (x_1,x_2,x_3) \subseteq V(G) \), \( |V(G) - V_1| = q \) and \( |\partial G(x_i,x_j)| = p \) \((1 \leq i < j \leq 3)\). Let \( G \) be an elemental graph for \( V_1 \subseteq V(G) \). We call a subgraph \( S \) an elemental star if \( V(S) \subseteq V_1 \), \( |V(S)| = 2 \) and \( |E(S)| = 1 \), or if for some \( x \in V(G) - V_1 \), \( V(S) = N_G(x) \cup x \) and \( E(S) \subseteq \partial G(x) \).
We require the following lemmas.

**LEMMA 3.1** (Okamura [7]). Suppose that \( k \geq 4 \) is an integer, \( G \) is a graph, \( (s,t) \subseteq T \subseteq V(G) \) and \( T \in \Gamma(G,k) \). Then

1. For each non-separating edge \( e \) incident to \( s \), there exists a path \( P \) between \( s \) and \( t \) passing through \( e \) such that \( T \in \Gamma(G-E(P),k-2) \) and \( (s,t) \in \Gamma(G-E(P),k-1) \).

2. For each vertex \( a \) of \( T-(s,t) \) with fewer degree than \( 2k \) and for each edge \( f \) incident to \( a \), there exists a path \( P \) between \( s \) and \( t \) not passing through \( a \) such that \( T \in \Gamma(G-E(P),k-2) \), \( (s,t,a) \in \Gamma(G-E(P),k-1) \), and

\[ (s,a) \text{ or } (t,a) \in \Gamma(G-E(P)-f,k-1). \]

3. For each non-separating edges \( e \) and \( e' \) incident to \( s \), there exists a cycle \( C \) passing through \( e \) and \( e' \) such that \( T \in \Gamma(G-E(C),k-2) \).

**LEMMA 3.2** (Okamura [7]). Suppose that \( n \geq 4 \) is an integer and \( k \geq 3 \) is an odd integer. If for each odd integer \( 1 \leq m \leq k \),

\[ \lambda'(m,n) = m, \]

then

\[ \lambda(k,n) = k \quad \text{and} \quad \lambda(k+1,n) = k+2. \]

**LEMMA 3.3.** Suppose that \( k \geq 3 \) is an integer, \( G \) is an elemental graph for \( T \subseteq V(G) \) and \( k \), \( T \in \Gamma(G,k) \), \( G \) has no nontrivial \( k \)-cut separating \( T \), and that \( S_1, S_2, S_3 \) are elemental stars of \( G \). If \( V(S_1) \cap V(S_2) \cap V(S_3) = \emptyset \), then
\[ T \in \Gamma(G - \bigcup_{i=1}^{3} E(S_i), k-2). \]

Proof. Assume that \( X \subseteq V(G) \), \( |X| \leq |V(G) - X| \) and \( X \) separates \( T \). Set \( G' = G - \bigcup_{i=1}^{3} E(S_i) \). If \( |X| = 1 \), then let \( X = \{x_0\} \).
Since \( d_{G'}(x) \geq d_G(x) - 2 = k - 2 \), we have \( |\partial G'(X) | \geq k - 2 \). If \( |X| \geq 2 \), then \( |\partial G(X) | \geq k + 1 \), and so \( |\partial G'(X) | \geq k - 2 \). Now Lemma 3.3 is proved.

**Lemma 3.4.** Suppose that \( k \geq 2 \) is an integer, \( G \) is an elemental graph for \( T = (x_1, x_2, x_3, x_4) \subset V(G) \) and \( k \), \( |T| = 4 \) and \( T \in \Gamma(G, k) \). Then

1. One of the following holds.
   1. \( \partial G(x_1, x_2) \neq \emptyset, \partial G(x_1, x_3) \neq \emptyset \), or for some \( y \in V(G) - T \), \( N_G(y) = (x_1, x_2, x_3) \).
   2. \( k \) is even, \( |\partial G(x_2, x_3) | = \frac{k}{2} \), and \( |\{y \in V(G) - T \mid N_G(y) = (x_i, x_1, x_4) \}| = \frac{k}{2} \) \((i=2, 3)\).

2. One of the following holds.
   1. For each \( 1 \leq i < j \leq k \), \( G \) has an elemental star \( S \) containing \( x_i \) and \( x_j \).
   2. \( k \) is even and \( G \) is the graph obtained from a four cycle by replacing each edge by \( k/2 \) parallel edges.

3. If \( G \) has no nontrivial \( k \)-cut separating \( T \), then
   1. \( \partial G(x_1, x_2) \neq \emptyset \) or \( G \) has two elemental stars containing \( x_1 \) and \( x_2 \).
   2. One of the following holds.
      1. \( G \) has edge-disjoint paths \( P_1[x_1, x_2] \) and \( P_2[x_1, x_3] \) such that for \( i = 2 \) or \( 4 \),
(x_i, x_3) \in \Gamma(G - \bigcup_{i=1}^{2} E(P_i), k-1) \) and \( T \in \Gamma(G - \bigcup_{i=1}^{3} E(P_i), k-2) \). 

(b) For each edge in \( \partial G(x_3) - \partial G(x_3, x_2) \), \( G \) has edge-disjoint paths \( P_1[x_1, x_2] \) and \( P_2[x_1, x_3] \) such that \( e \in E(P_2) \) and \( T \in \Gamma(G - \bigcup_{i=1}^{3} E(P_i), k-2) \).

Proof. For \( 1 \leq i, j \leq 4 \), set 
\[
\begin{align*}
p_i, j &= |\partial G(x_i, x_j)|, \\
R_i &= \{ y \in V(G) - T \mid N_G(y) = T - x_i \}, \\
r_i &= 1 \overline{R_i}.
\end{align*}
\]

(1) Assume \( p_1, 2 = p_1, 3 = r_4 = 0 \). Then 
\[
\begin{align*}
d_G(x_1) &= k = p_1, 4 + r_2 + r_3, \\
d_G(x_4) &= k = p_1, 4 + p_2, 4 + p_3, 4 + r_1 + r_2 + r_3
\end{align*}
\]

Thus 
\[
p_2, 4 = p_3, 4 = r_1 = 0.
\]

Since \( T \in \Gamma(G, k) \), we have 
\[
|\partial G((x_2, x_3))| = r_2 + r_3 \geq k.
\]

Thus 
\[
p_1, 4 = 0.
\]

By comparing \( d_G(x_i) \) with \( d_G(x_j) \) for \( 1 \leq i < j \leq 3 \), we have 
\[
r_2 = r_3 = p_2, 3.
\]

Now (ii) follows.

(2) Assume \( p_1, 2 = r_3 = r_4 = 0 \). Then by comparing 
\[
d_G(x_1) + d_G(x_2) \) with \( d_G(x_3) + d_G(x_4) \), we have 
\[
r_1 = r_2 = p_3, 4 = 0.
\]

Now by comparing \( d_G(x_3) = k = p_1, 3 + p_2, 3 \) with \( d_G(x_i) \) for \( i = 1, 2 \), we have
\[ p_{1,4} = p_{2,3} \text{ and } p_{2,4} = p_{1,3}. \]

Moreover
\[ 1 \partial G((x_1,x_4)) \leq p_{1,3} + p_{2,4} = 2p_{1,3} \geq k, \]
\[ 1 \partial G((x_1,x_3)) \leq p_{1,4} + p_{2,3} = 2p_{1,4} \geq k. \]

Thus
\[ p_{1,3} = p_{2,3} = p_{2,4} = p_{1,4}, \]

and so (ii) follows.

(3) (i) We assume \( p_{1,2} = r_4 = 0 \), and then prove \( r_3 \geq 2 \).

Since any cut separating \((x_1, x_3)\) and \((x_2, x_4)\) or separating \((x_1, x_4)\) and \((x_2, x_3)\) has more than \( k \) edges we have

\[ (3.1) \; p_{1,4} + p_{2,3} + p_{3,4} + r_1 + r_2 + r_3 \geq k + 1, \]

and

\[ (3.2) \; p_{1,3} + p_{2,4} + p_{3,4} + r_1 + r_2 + r_3 \geq k + 1. \]

By comparing \( d_G(x_3) + d_G(x_4) \) with \((3.1) + (3.2)\), we have

\[ r_3 \geq 2. \]

(ii) If there exists \( f \in \partial G(x_1, x_3) \), then by Lemma 2.1 \( G \) has a path \( P[x_3, x_2] \) such that \( f \in E(P) \), 
\((x_3, x_2) \in \Gamma(G - E(P), k - 1)\) and \( T \in \Gamma(G - E(P), k - 2) \), and so (a) follows. Thus we may let

\[ p_{1,3} = p_{1,2} = 0, \]

then by (1)

\[ r_4 > 0. \]

If \( r_3 > 0 \), then for \( y_1 \in R_4 \) and \( y_2 \in R_3 \),
\((x_3, x_4) \in \Gamma(G - \bigcup_{i=1}^{2} \partial G(y_i), k - 1)\) and \( T \in \Gamma(G - \bigcup_{i=1}^{2} \partial G(y_i), k - 2) \),

and so (a) follows. Thus we may let
\[ r_3 = 0. \]
Then by (1) and (3) 

$p_1,A>0$ and $r_A \geq 2$.

Let $y$ be another end of $e$, then $y=x_A$ or $y=R_i$ $(i=1,2 \text{ or } A)$. In each case (b) easily follows.

**Lemma 3.5.** Suppose that $k \geq 3$ is an odd integer, $G$ is a graph, $(x_1,x_2,x_3) \subseteq T \subseteq V(G)$, $x_i \neq x_j$ $(1 \leq i < j \leq 3)$, $T \in \Gamma(G,k)$ and $e \in E(G)$. If following (i) or (ii) holds, then for $m=2,3$, $G$ has edge-disjoint paths $P_1[x_1,x_2]$ and $P_2[x_1,x_m]$ such that $e \in E(P_1) \cup E(P_2)$ and $T \in \Gamma(G- \bigcup_{i \neq j} E(P_i),k-2)$.

(i) $e \in \partial_G(x_1,x_2)$,

(ii) $e \in \partial_G(x_1,y)$ for some $y \in V(G)-T$ with $d_G(y)=3$ and $N_G(y) = (x_1,x_2,x_3)$.

**Proof.** Assume that (i) holds. By Theorem 1 if $m=2$, then $G$ has a cycle $C$ such that $e \in E(C)$ and $T \in \Gamma(G-E(C),k-2)$, and if $m=3$, then $G$ has a path $P[x_2,x_3]$ such that $e \in E(P)$ and $T \in \Gamma(G-E(P), k-2)$.

Assume that (ii) holds. We may assume that $G$ is 2-connected. If $d_G(x_3)=d>k$, then we replace $x_3$ by $d$ vertices of degree $k$ (Figure 4 gives an example with $d=8$ and $k=5$), producing a new graph $G'$. In $G'$ we assign $x_3$ on $N_G'(y)-(x_1,x_2)$. If the result holds in $G'$, then clearly the result holds in $G$, and so we may assume that $d_G(x_3)=k$. Let $f \in \partial_G(x_3)-\partial_G(y,x_3)$. By Lemma 3.1
Figure 4.

G has a path $P[x_1, x_2]$ such that $x_3 \notin V(P)$, $T \in \Gamma(G-E(P), k-2)$, $(x_1, x_2, x_3) \in \Gamma(G-E(P), k-1)$ and $(x_i, x_j) \in \Gamma(G-E(P) - f, k-1)$ ($i=1$ or 2). Then $y \notin V(P)$, because $d_0(x_3) = k$ and $d_0(y) = 3$.

Moreover $T \in \Gamma(G-E(P) - y, k-2)$. Thus the result follows.

Now we prove Theorem 2. We may assume that G is 2-connected, $d_0(x) = k$ for each $x \in T$ (see the proof of Lemma 3.5 and Figure 4, in this case we can assign x on any vertex of new $d_0(x)$ vertices of degree k) and that $d_0(y) = 3$ for each $y \in V(G) - T$ (see Case 1 in the proof of Theorem 1). We proceed by induction on $|E(G)|$. If $|T| \leq 3$, then the result follows from Theorem 1. Thus let $|T| \geq 4$.

Case 1. G has a nontrivial k-cut $\partial G(X) = \{e_1, \ldots, e_k\}$ ($X \subseteq V(G)$) separating T.

We define $H, K, u, v, T_H$ and $T_K$ similarly as in the proof of Theorem 1. If $|X \cap T| = 1$, then the results hold in K, and so in G. Thus let $|X \cap T| \geq 2$ and $|T - X| \geq 2$.

We require the following.

(3.3) If G has a nontrivial k-cut $\partial G(Y) = \{f_1, \ldots, f_k\}$
(Y \subseteq X) separating T, then we may assume that \((X-Y) \cap T \neq \emptyset\).

Proof. Assume \((X-Y) \cap T = \emptyset\). Let \(b_1, c_i\) be the end of \(e_i\) (if \(i\)) in \(Y \subseteq V(G) - X\) \((1 \leq i \leq k)\). We may assume that the graph obtained from \(X-Y \subseteq G\) by adding \(b_1, \ldots, b_k, c_1, \ldots, c_k, e_1, \ldots, e_k, f_1, \ldots, f_k\) has edge-disjoint paths \(P_1[b_1, c_1], \ldots, P_k[b_k, c_k]\). Let \(G'\) be the graph obtained from \(G-(X-Y)\) by adding new edges \(g_1, \ldots, g_k\), where \(g_i\) has ends \(b_i\) and \(c_i\) \((1 \leq i \leq k)\). Then \(|E(G')| < |E(G)|\), and the results of Theorem 2 hold in \(G'\), and so in \(G\). Now (3.3) is proved.

(3.4) If \(|X-T| = 2\) \((|T-X| = 2)\), then we may assume that \(H (K) = G(p, q) (G(p', q'))\) for some integers \(p\) and \(q\) \((p'\) and \(q'\)).

Proof. Assume \(|X \cap T| = 2\). If \(H\) has a nontrivial \(k\)-cut \(\partial H (Y) (Y \subseteq V(H)-u)\) separating \(T_H\), then by (3.3) \((X-Y) \cap T \neq \emptyset\), and so \(|T \cap Y| = 1\). Then by taking \(Y\) instead of \(X\) the results of Theorem 2 hold. Thus we may assume that an end of each edge of \(H\) is in \(T_H\). Hence the result easily follows.

We return to the proof of Theorem 2. By Lemma 3.5 we may assume the following.

(3.5) \(\partial G(a_1, a_i) = \emptyset\) \((i = 2, m)\) and for each \(y \in V(G)-T\), \((a_1, a_2, a_m) \not\subseteq N_G(y)\).
Let \( a_1 \in X \).

(1) Now \( |X-T| = |T-X| = 2 \). If \( a_2 \in X \), then by (3.4) the result easily follows. Thus let \( a_2 \in V(G)-X \). Since
\[
p + q \geq (k+1)/2 \quad \text{and} \quad p' + q' \geq (k+1)/2,
\]
for some \( 1 \leq i \leq k \), \( H \) has an elemental star \( S_1 \) containing \( a_i \) and \( e_1 \) and \( K \) has an elemental star \( S_2 \) containing \( a_2 \) and \( e_1 \). Then \( T \in \Gamma(G - \bigcup_{i=1}^{k} E(S_i), k-1) \).

(2) Subcase 1-1. \( (a_2, a_m) \subseteq X \).

\( H \) has required paths. If one of them passes through \( u \), then we can deduce the result by using Lemma 3.1(3) on \( K \).

Subcase 1-2. \( (a_2, a_m) \subseteq V(G)-X \) and \( |X \cap T| = 2 \).

Set \( X \cap T = \{ a_1, a_5 \} \). By (3.4) \( H \) is \( G(p,q) \). Thus if following (3.6) or (3.7) holds, then the result follows.

(3.6) For some \( e_i \in \partial H(u,a_1) \), \( K \) has edge-disjoint paths \( P_1[v,a_2] \) and \( P_2[v,a_m] \) such that \( e_i \in E(P_1) \cup E(P_2) \) and \( T \subseteq \Gamma(K - \bigcup_{i=1}^{k} E(P_i), k-2) \).

(3.7) For some \( e_i, e_j \in \partial H(u)-\partial H(u,a_5) \), \( K \) has edge-disjoint paths \( P_1[v,a_2] \) and \( P_2[v,a_m] \) such that \( \langle e_i, e_j \rangle \subseteq E(P_1) \cup E(P_2) \) and \( T \subseteq \Gamma(K - \bigcup_{i=1}^{k} E(P_i), k-2) \).

If \( p = 0 \), then \( \partial H(u,a_5) = \emptyset \), and so (3.7) follows. Thus let \( p > 0 \). If \( |T-X| = 2 \), then by (3.4) \( K \) is \( G(p',q') \), and so (3.6) follows. Thus let \( |T-X| = 3 \) and \( m = 3 \). Set \( T \cap X = \{ a_2, a_3, a_4 \} \).

Subcase 1-2-1. \( K \) has nontrivial \( k \)-cut \( \partial K(Y) \) (\( Y \subseteq V(K)-v \)) separating \( T \).

By (3.3) we may let \( |Y \cap T| = |T \cap Y| = 2 \). Let \( K_1 \) and \( K_2 \) be the graphs obtained from \( K \) by contracting \( Y \) and \( V(K)-Y \) to a vertex respectively. Then similarly as (3.4)
$K_i$ is $G(p_i, q_i)$ for some integers $p_i$ and $q_i$ $(i=1, 2)$

Let $M$ be

$$\{ (x_1, x_2) \in V(K) - T_k \mid \partial K(x_1, x_2) \neq \emptyset \},$$

and let $M'$ be

$$\{ x \mid \text{For some } N \in M, \ x \in N \}.$$  

For each $N \in M$, $N \cap V(K_i) \neq \emptyset$ $(i=1, 2)$,

$$d_{K-N}(a_j) = d_{K-N}(v) = k-1 \ (j=2, 3, 4) \text{ and } T_k \in \Gamma(K-N, k-1).$$

If $k=1M1$, then $p_1 = p_2 = 0$ and the result easily follows,

and so let $k > 1M1$. $K-M'$ is elemental for $T_k$ and $K-1M1$.

Assume that $k-1M1$ is even and $K-M'$ is the graph obtained

from four cycle by replacing each edge by $(k-1M1)/2$ parallel

edges. For each cycle $C$ of $K-M'$ such that $|V(C)|=|E(C)|=4$,

we have $T_k \in \Gamma(G-E(C), k-2)$. If $\partial G(a_1, a_4) \neq \emptyset$, then

(3.6) follows, and if not, then by (3.5) $a_1$ is adjacent to

$p$ vertices of $M'$. If $1M1 \geq 2$, then (3.6) follows. Thus

assume $1 \geq 1M1 \geq p \geq 1$. Since $(k-1M1)/2 \geq (5-1)/2 = 2$, for some

$1 \leq i < j \leq k$,  

$$\{ e_i, e_j \} \subseteq \partial_H(u) - \partial_H(u, a_5),$$

and $K$ has a four cycle $C$ such that $|V(C)|=|E(C)|=4$ and

$$\{ e_i, e_j \} \subseteq E(C).$$

Hence (3.7) follows.

By Lemma 3.4(2) we may assume that for each two vertices

d of $T_k$, $K-M'$ has an elemental star containing them. Set

$a_0 = v$, and for $i, j = 0, 2, 3, 4$, set

$$p_i, j = \left| \partial K(a_i, a_j) \right|,$$

$$r_i = \left| \{ x \in V(K) - T_k \mid N_k(x) = T_k - a_j \} \right|.$$  

For $i, j = 0, 2, 3, 4$, since $\left| \partial K(a_i, a_j) \right| \geq k,$

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If \( a_1 \) is adjacent to a vertex of \( M' \) in \( G \), then (3.6) follows. If for some \( x \in V(G)-T, N_G(x) = (a_1, a_i, a_4) \) \( (i=2 \text{ or } 3) \), then (3.6) follows. Thus and by (3.5) we may assume that 

\[ |\partial G(a_1, a_4)| = p. \]

If \( a_4 \in Y \), then (3.6) easily follows, and thus let 

\( T_{H-Y} = (a_0, a_4) \). Since \( p_0, 4 \geq |\partial G(a_1, a_4)| = p > 0 \), by Lemma 3.4(1) we have 

\[ p_4, 2 > 0, p_4, 3 > 0, \text{ or } r_0 > 0, \]

and 

\[ p_0, 2 > 0, p_0, 3 > 0, \text{ or } r_4 > 0. \]

If \( r_0 > 0, r_4 > 0, p_0, 2 + p_3, 4 > 0 \), or \( p_0, 3 + p_2, 4 > 0 \), 

then (3.6) follows (note that \( K_i = G(p_i, q_i) \) for \( i=1, 2 \)).

Thus we may assume that 

\[ (3.8) \ p_0, 2 > 0, p_2, 4 > 0 \text{ and } r_0 = r_4 = p_0, 3 = p_3, 4 = 0. \]

Assume \( |M| = 0 \). Then 

\[ d_G(a_3) = p_2, 3 + r_2 \text{ and } p_2, 3 \leq (k-1)/2, \]

and so 

\[ (3.9) \ r_2 \geq (k+1)/2 \geq p+1. \]

By comparing \( d_G(a_2) \) with \( d_G(a_4) \) we have

\[ p_0, 2 + p_2, 3 = p_0, 4 + r_2. \]

Thus 

\[ (3.10) \ p_0, 2 > p_0, 4 \geq p. \]

From (3.9) and (3.10), (3.7) follows.

Now we may let \( |M| > 0 \). Since \((a_2, a_3) \subseteq Y\), we have
\[ I^2 K(Y) = k = d_k(a_2) + d_k(a_3) - 2p_2,3 - 1MI = 2k - 2p_2,3 - 1MI, \]

and so
\[ 2p_2,3 + 1MI = k. \]

Since \( d_G(a_3) = k = p_2,3 + r_2 + 1MI \),
\[ r_2 = p_2,3. \]

Since \( d_G(a_3) = 2r_2 + 1MI \), \( d_G(a_4) = p_0,4 + p_2,4 + r_2 + r_3 + 1MI \),
and \( p_2,4 > 0 \) (by (3.8)), we have
\[ (3.11) \quad r_2 \geq a_0,4 + 1 \geq p + 1. \]

By comparing \( d_G(a_2) \) with \( d_G(a_4) \), we have
\[ p_0,2 = p_0,4. \]

Thus
\[ (3.12) \quad p_0,2 + 1MI \geq p + 1. \]

From (3.11) and (3.12), (3.7) follows.

Subcase 1-2-2. \( K \) has no nontrivial \( k \)-cut separating \( T_K \).

We may assume that an end of each edge of \( K \) in \( T_K \) and \( K \) is elemental for \( T_K \). The proof is similar as the case \( 1MI = 0 \) in the proof of Subcase 1-2-1.

Subcase 1-3. \( (a_2, a_m) \subseteq V(G) - X \) and \( 1X \cap T = 3. \)

Now \( m = 3. \) By (3.4) \( K \) is \( G(p', q') \). Set \( X \cap T = \{a_3, a_4, a_5\} \)

If \( H \) has nontrivial \( k \)-cut \( \partial_H(Y) \) (\( Y \subseteq V(H) - u \)) separating \( T_H \), then we may let \( 1Y \cap T = 2. \) Then for \( Y \) or \( V(G) - Y \)
instead of \( X \) Subcase 1-1 or Subcase 1-2 occurs. Thus we may assume that this is not the case and \( H \) is elemental for \( T_H \).

If following (3.13) or (3.14) holds, then the result follows.

(3.13) For some \( e_i \in \partial K(v) - \bigcup_{j=1}^{3} \partial K(v, a_i) \), \( H \) has edge-disjoint paths \( P_1[a_1, u] \) and \( P_2[a_1, u] \) such that
\( \text{For } i=1 \text{ or } 2 \text{ or } 3 \text{ and for some } e_i \in \partial K(v,x) \) and \( e_j \in \partial K(v) - \bigcup_{i=1}^{2} \partial K(v,x) \), \( H \) has edge-disjoint paths 

\( P_1[a_1,u] \) and \( P_2[a_1,u] \) such that 

\( (e_i, e_j) \subseteq E(P_1) \cup E(P_2) \) and \( T_H \in \bigcup_{i=1}^{2} E(P_i), k-2 \). 

Set \( a_0 = u \) and for \( i,j = 0,1,4,5 \) set 

\[
P_i,j = l_{\partial H}(a_i,a_j), \
R_i = \{ x \in V(H) - T_H \mid N_H(x) = T_H - a_i \}, \]

\[
r_i = l_{R_i}.\]

By (3.5) \( p_{0,1} = 0 \).

Assume \( p_{1,4} = p_{1,5} = 0 \). If \( r_0 \leq (k-1)/2 \), then 

\[
r_4 + r_5 = d_G(a_1) - r_0 \geq (k+1)/2 \geq p' + 1,\]

and so (3.13) or (3.14) follows. Thus let \( r_0 \geq (k+1)/2 \).

Since \( d_G(a_0) = p_{0,4} + p_{0,5} + r_1 + r_4 + r_5 \) and 

\( d_G(a_5) = p_{0,5} + p_{4,5} + r_0 + r_1 + r_4 \), we have 

\( p_{0,4} + r_5 = p_{4,5} + r_0 \).

Hence 

\[
d_G(a_4) = k \geq p_{0,4} + r_0 + r_5 \geq 2r_0 \geq k,\]

a contradiction.

Now we may let \( p_{1,i} \geq 0 \) for \( i = 4 \) or \( 5 \), say \( i = 4 \).

Since \( p_{0,1} = 0 \) and by Lemma 3.4(3), we have 

\[
r_4 + r_5 \geq 2.\]

For each \( x \in R_4 \cup R_5 \), if \( x \) is adjacent to a vertex of 

\( V(K) - T_K \) in \( G \), then (3.13) follows, thus assume that 

\( \partial G(x,a_1) \neq \emptyset \) (\( i = 2 \) or \( 3 \)). For each \( x,y \in R_4 \cup R_5 \), if 

\( \partial G(x,a_2) \neq \emptyset \) and \( \partial G(y,a_3) \neq \emptyset \), then (3.14) follows, 

thus assume that for \( i = 2 \) or \( 3 \), \( \partial G(x,a_i) = \partial G(y,a_i) = \emptyset \), say \( i = 3 \),
and that $r_4+r_5 \leq p'$.

Assume $r_4 > 0$. For some $e_i \in \partial_K(v) - \partial_K(v, a_2)$, $e_i$ is incident to $a_4$ or a vertex of $R_1$ in $G$, because $p' + q' \geq (k+1)/2 > p_0, 5$.

Thus (3.14) follows.

Now we may assume that $r_4 = 0$, $r_5 > 0$ and $p_{1,5} = 0$.

Thus $p_{0,1} = p_{1,5} = r_4 = 0$, contrary to Lemma 3.4(1).

Subcase 1-4. $a_2 \in X$ and $a_m \in V(G) - X$.

Now $m = 3$.

Subcase 1-4-1. $|X \cap T| = 2$.

By (3.4) $H = G(p, q)$, and by (3.5) $p = 0$. Since $|T_K| \leq 4$, by induction $K$ has a path $p[v, a_3]$ such that $T_K \in \Gamma(K - E(p), k-1)$. Let $e_1 \in E(p)$. $H$ has an elemental star $S_1$ containing $a_1$ and $e_1$. Let $S_2$ be another elemental star of $H$. Then $T_H \in \Gamma(H - \bigcup_{i=1}^2 E(S_i), k-2)$, and so the result follows.

Subcase 1-4-2. $|X \cap T| = 3$ and $|T - X| = 2$.

Assume that $H$ has a nontrivial $k$-cut $\partial_H(Y) = (f_1, \ldots, f_k)$ ($Y \subseteq V(H) - u$) separating $T_H$. Then we may assume that $|Y \cap T_H| = 2$, $a_2 \in Y$ and $a_1 \in X - Y$. Let $H_1$ ($H_2$) be the graph obtained from $H$ by contracting $V(H) - Y$ ($Y$) to a new new vertex $u_1$ ($u_2$). Then similarly as (3.4) $H_i$ is $G(p_i, q_i)$ for some integers $p_i$ and $q_i$ ($i = 1, 2$). If $p_2 = 0$, then the result easily follows. If $p_2 > 0$, then we may let $(f_1, e_1) \subseteq \partial G(a_1)$ and we can easily deduce the result.

Now we may assume that $H$ has no nontrivial $k$-cut.
separating $T_H$ and $H$ is elemental for $T_H$. Set
$X \cap T = (a_1, a_2, u, a_4)$ and $T - X = (a_3, a_5)$. For $a_1, a_2,$
u, a_4$ instead of $x_1, x_2, x_3, x_4$, (a) or (b) of
Lemma 3.4(3) holds. If (a) holds, then the result easily
follows, thus assume that (b) holds. Since
$|\partial_H(u) - \partial_H(u, a_2)| \geq (k+1)/2$ and $p' + q' \geq (k+1)/2$, for some
$1 \leq i \leq k,$
e_i \in \partial_H(u) - \partial_H(u, a_2)$ and $e_i \in \partial_K(v) - \partial_K(v, a_5),$
and so the result follows.

Case 2. $G$ has no nontrivial $k$-cut separating $T$.

We may assume that $G$ is elemental for $T$. If $|T| = 4$, then
by Lemma 3.3 the result follows. Thus let $|T| = 5$ and $m = 3.$
Set $T = (a_1, a_2, a_3, a_4, a_5)$ and for $1 \leq i, j, l \leq 5,$ set
$p_i, j = |\partial G(a_i, a_j)|,$
$R(i, j, 1) = \{ x \in V(G) - T | Ng(x) = (a_1, a_j, a_4) \},$
$r(i, j, 1) = |R(i, j, 1)|.$
We require the following.

(3.15) For each distinct $1 \leq i, j, l \leq 5$, $G$ has an elemental
star containing $(a_1, a_j)$ or $(a_i, a_1)$.

Proof. Assume that each elemental star of $G$ does not
contain $(a_1, a_2)$ nor $(a_1, a_3)$. Then
$deg(a_1) = p_1, 4 + p_1, 5 + r(1, 4, 5).$
Since $p_i, j \leq (k-1)/2$ for each $i, j$, we have $r(1, 4, 5) > 0.$
Let $F$ be a cut of $G$ separating $(a_1, a_4, a_5)$ and $(a_2, a_3)$,
then
|F| = d_G(a_4) + d_G(a_5) - (p_{1,4} + p_{1,5} + 2r(1,4,5)) < k,

a contradiction. Now (3.15) is proved.

We return to the proof of Theorem 2. By (3.5)

\[ p_{1,2} = p_{1,3} = r(1,2,3) = 0. \]

If \( r(1,2,i) > 0 \) and \( r(1,3,j) > 0 \) \( (i,j=4 \text{ or } 5) \), then the result follows. Thus and by (3.15) we may assume that

\[ r(1,2,4) > 0 \text{ and } r(1,3,i) = 0 \ (i=4,5). \]

By (3.15)

\[ p_{1,5} + r(i,5,2) + r(i,5,4) > 0 \ (i=1,3). \]

If \( p_{1,5} > 0 \), \( p_{3,5} > 0 \), \( r(1,5,2) \cdot r(3,5,4) > 0 \), or

\( r(1,5,4) \cdot r(3,5,2) > 0 \), then by Lemma 3.3 the result follows.

Thus we may assume that for \( (i,j)=(2,4) \) or \( (4,2) \),

\[ p_{1,5} = p_{3,5} = 0, \ r(1,5,i) = r(3,5,i) = 0, \]

and

\[ r(1,5,j) \cdot r(3,5,j) > 0. \]

Assume \( r(1,5,2) = r(3,5,2) = 0 \). Then

\[ d_G(x_1) = p_{1,4} + r(1,2,4) + r(1,4,5), \]

and

\[ d_G(x_4) \geq p_{1,4} + r(1,2,4) + r(1,4,5) + r(3,4,5) > k, \]

a contradiction. Thus

\[ r(1,5,4) = r(3,5,4) = 0. \]

Since \( r(1,2,5) > 0 \), by the same argument we have

\[ p_{1,4} = p_{3,4} = 0. \]

Thus

\[ d_G(x_1) = r(1,2,4) + r(1,2,5) \]

and
\[d_G(x_2) \geq r(1,2,4)+r(1,2,5)+r(2,3,5) > k,\]
a contradiction.

4. PROOF OF THEOREM 3.

Suppose that \(k \geq 1\) is an integer, \(G\) is a graph, \(T=(s_1, \ldots, s_k, t_1, \ldots, t_k) \subseteq V(G)\) and \(T \in \Gamma_g(G,k)\). We prove that if \(|T|=3\), or if \(k\) is odd and \(|T|=4\) or \(5\), then (1.1) holds by induction on \(k\).

Assume \(|T|=3\). By Theorem 1 \(G\) has a path \([s_k, s_k]\)
such that \(T \in \Gamma_g(G-E(P),k-1)\). By induction for \(k-1\), (1.1)
holds in \(G-E(P)\), and so for \(k\), (1.1) holds.

Assume that \(k \geq 5\) is odd and \(|T|=4\) or \(5\). For some
\(1 \leq i < j \leq k\), if \(|T|=4\), then
\[s_i=s_j \text{ or } t_j,\]
and if \(|T|=5\), then
\[s_i=s_j \text{ or } t_j \text{ and } (s_i, t_i) \notin (s_j, t_j),\]
say for \(i=k-1\) and \(j=k\). By Theorem 2 \(G\) has edge-disjoint
paths \(P_1(s_{k-1}, t_{k-1})\) and \(P_2(s_k, t_k)\) such that
\(T \in \Gamma_g(G-\bigcup_{i=1}^{2} E(P_i),k-2)\). By induction for \(k-2\), (1.1)
holds in \(G-\bigcup_{i=1}^{2} E(P_i)\), and so for \(k\), (1.1) holds in \(G\).

Thus for integer \(k \geq 1\),
\[\lambda'(k,3)=\lambda(k,3)=k,\]
and for odd integer \(k \geq 1\),
\[\lambda'(k,4)=\lambda'(k,5)=k.\]
By Lemma 3.2 for odd integer \(k \geq 1\),
\[\lambda(k,4)=\lambda(k,5)=k \text{ and } \lambda(k+1,4)=\lambda(k+1,5)=k+2.\]

Now Theorem 3 is proved.
REFERENCES


