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Paths and Edge-Connectivity in Graphs

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1. INTRODUCTION

We consider finite undirected graphs possibly with multiple edges but without loops. Let \( G \) be a graph and let \( V(G) \) and \( E(G) \) be the sets of vertices and edges of \( G \) respectively. For two distinct vertices \( x \) and \( y \), let \( \lambda_q(x,y) \) be the maximal number of edge-disjoint paths between \( x \) and \( y \), and let \( \lambda_q(x,x) = \infty \). For an integer \( k \geq 1 \), let \( \Gamma(G,k) \) be \( \{ X \subseteq V(G) \mid \text{For each } x,y \in X, \lambda_q(x,y) \geq k \} \).

Let \( (s_1, t_1), \ldots, (s_k, t_k) \) be pairs of vertices of \( G \). When is the following statement true?

\[(1.1) \text{There exist edge-disjoint paths } P_1, \ldots, P_k \text{ such that } P_i \text{ has ends } s_i, t_i (1 \leq i \leq k).\]

Seymour [8] and Thomassen [9] characterised such graphs when \( k = 2 \), and Seymour [8] when \( |(s_1, \ldots, s_k, t_1, \ldots, t_k)| = 3 \).

For integers \( k \geq 1 \) and \( n \geq 2 \), set

\[ g(k) = \min \{ m \mid \text{If } G \text{ is } m \text{-edge-connected, then (1.1) holds} \}, \]

\[ \lambda'(k,n) = \min \left\{ m \left| \begin{array}{l}
\text{If } |(s_1, \ldots, s, t, \ldots, t)| \leq n \text{ and } \{s_1, \ldots, s_k, t_1, \ldots, t_k\} \in \Gamma(G,m), \text{ then (1.1) holds.}
\end{array} \right. \right\}, \]
\( \lambda(k,n) = \min \left\{ m \mid \text{If } 1(s_1, \ldots, s_k, t_1, \ldots, t_k) \leq n \text{ and } \lambda_q(s_i, t_i) \geq m \ (1 \leq i \leq k), \text{ then (1.1) holds} \right\} \)

and set

\( \lambda'(k) = \lambda'(k, 2k) = \lambda'(k, m) \ (m > 2k) \) and \( \lambda(k) = \lambda(k, 2k) \).

Then for each \( k \geq 1 \),

\( \lambda'(k, 3) = \lambda(k, 3) \) and \( \lambda(k) \geq \lambda'(k) \geq g(k) \geq k \).

For \( n \geq 4 \) and even integer \( k \geq 2 \),

\( g(k) > k \) and \( \lambda(k) \geq \lambda(k, n) \geq \lambda'(k, n) \geq k \)

(see Figure 1 in which \( k/2 \) represents the number of parallel edges).

Figure 1.

Thomassen [9] gave following Conjecture 1, and we give following Conjecture 2 slightly stronger than Conjecture 1.

**CONJECTURE 1.** For each integer \( k \geq 1 \),

\( g(k) = \begin{cases} 
  k & \text{if } k \text{ is odd} \\
  k+1 & \text{if } k \text{ is even}
\end{cases} \)

**CONJECTURE 2.** For each integer \( k \geq 1 \),

\( \lambda(k) = \begin{cases} 
  k & \text{if } k \text{ is odd} \\
  k+1 & \text{if } k \text{ is even}
\end{cases} \).
It easily follows from Menger's theorem that $\lambda(k) \leq 2k-1$; thus $\lambda(1)=1$ and $\lambda(2)=3$. Cypher [1] proved $\lambda(4) \leq 6$ and $\lambda(5) \leq 7$, and $\lambda(3)=3$ was announced in [5] and proved in [6] by the author. Enomoto and Saito [2] proved $g(4)=5$, and independently Hirata, Kubota and Saito [3] proved $\lambda(k) \leq 2k-3$ ($k \geq 4$).

Our main results are the following.

THEOREM 1. Suppose that $k \geq 2$ is an integer, $G$ is a graph, $(a_1,a_2) \subseteq T \subseteq V(G)$, $|T| \leq 3$ and $T \in \Gamma(G,k)$. Then there exists a path $P$ between $a_1$ and $a_2$ such that $T \in \Gamma(G-E(P),k-1)$.

THEOREM 2. Suppose that $k \geq 5$ is an odd integer, $G$ is a graph, $(a_1,a_2,a_3) \subseteq T \subseteq V(G)$, $a_i \neq a_j$ ($1 \leq i < j \leq 3$), $|T| \leq 5$ and $T \in \Gamma(G,k)$. Then

(1) If $|T| \leq 4$, then there exists a path $P$ between $a_1$ and $a_2$ such that $T \in \Gamma(G-E(P),k-1)$.

(2) For $m=2,3$ if $|T| \leq 4$ and for $m=3$ if $|T| = 5$, there exist edge-disjoint paths $P_i$ between $a_1$ and $a_2$ and $P_2$ between $a_1$ and $a_m$ such that $T \in \Gamma(G- \bigcup_{i=1}^{2} E(P_i),k-2)$.

THEOREM 3. For each integer $k \geq 1$,

$$\lambda(k,3)=k \text{ and } \lambda(k,4)=\lambda(k,5)=\begin{cases} k & \text{if } k \text{ is odd} \\ k+1 & \text{if } k \text{ is even.} \end{cases}$$

In Theorem 2(2) if $m=2$ and $|T|=5$, then the conclusion does not always hold. Figure 2 gives a counterexample with $k=7$. 
Figure 2.

When \( k \) is odd and \(|s_1, \ldots, s_k, t_1, \ldots, t_k| \geq 4\), if for some \( 1 \leq i \leq k \),

\[ \mathcal{A}_G(s_i, t_i) < k, \]

then (1.1) does not always hold. Figure 3 gives a counterexample.

Figure 3.

Notations and Definitions. Let \( X, Y \subseteq V(G) \), \( F \subseteq E(G) \), \( (x, y) \subseteq V(G) \) and \( e \in E(G) \). We often denote \( \langle x \rangle \) by \( x \) and \( \langle e \rangle \) by \( e \). The subgraph of \( G \) induced by \( X \) is denoted by \( \langle X \rangle_G \) and the subgraph obtained from \( G \) by deleting \( X \) (\( F \)) is denoted by \( G-X \) (\( G-F \)). \( \mathcal{V}_G(X, Y) \) denotes the set of edges with one end in \( X \) and the other in \( Y \), and \( \mathcal{V}_G(X) \) denotes \( \mathcal{V}_G(X, V(G)-X) \). \( \mathcal{A}_G(X, Y) \) denotes the maximal number of edge-disjoint paths with one end in \( X \) and the other in \( Y \). \( \mathcal{V}_G(X) \)
is called an n-cut if \(|\partial_Q(X)|=n\) and \(\langle X \rangle_Q\) and \(\langle V(G)-X \rangle_Q\) are both connected. An n-cut \(\partial_Q(X)\) is called nontrivial if \(|X| \geq 2\) and \(|V(G)-X| \geq 2\), trivial otherwise. \(d_Q(x)\) denotes the degree of \(x\) and \(N_Q(x)\) denotes the set of vertices adjacent to \(x\). We regard a path and a cycle as subgraphs of \(G\). A path \(P=PX, y]\) denotes a path between \(x\) and \(y\), and for \(x', y' \in V(P)\), \(P(x', y')\) denotes a subpath of \(P\) between \(x'\) and \(y'\).

2. PROOF OF THEOREM 1

For a vertex \(w \in V(G)\) and \(b, c \in N_Q(w)\), we let \(G_{\partial_Q}^{b,c}\) be the graph \((V(G), E(G)\cup e-(f,g))\), where \(e\) is a new edge with ends \(b\) and \(c\), \(f \in \partial_Q(w, b)\) and \(g \in \partial_Q(w, c)\). We require the following lemmas.

LEMMA 2.1 (Mader [4]). Suppose that \(w\) is a non-separating vertex of a graph \(G\) with \(d_Q(w) \geq 4\) and with \(|N_Q(w)| \geq 2\). Then there exist \(b, c \in N_Q(w)\) such that for each \(x, y \in V(G)-w\),

\[\lambda_{G_{\partial_Q}^{b,c}}(x,y) = \lambda_Q(x,y).\]

Now we prove Theorem 1 by induction on \(|E(G)|\). We may assume that \(a_1\neq a_2\) and \(|T|=3\). If \(G\) has a nontrivial k-cut \(\partial_G(X) (X \subseteq V(G))\) separating \(T\), then let \(H (K)\) be the graph obtained from \(G\) by contracting \(V(G)-X (X)\) to a new vertex \(u (v)\). Set \(T_H=(X \cap T) \cup u\) and \(T_K=(T-X) \cup v\). We may
let $|T \cap X| = 2$. By induction for $H$ and $(T \cap X) \cup u$ instead of for $G$ and $T$, the result holds. Thus the result follows. Hence we may assume that each edge is incident to a vertex of $T$.

**Case 1.** There exists $x \in V(G) - T$.

If $d_G(x) \geq 4$, then by Lemma 2.1 there exists $b, c \in N_G(x)$ such that for each $y, z \in V(G) - x$,

$$\lambda_{G^b}^{bc}(y, z) = \lambda_G(y, z).$$

By induction the result holds in $G_x$. Thus we may assume that $d_G(x) = 3$ and clearly that $N_G(x) = T$. Now the path $P[a_1, a_2]$ with $E(P) \subseteq \partial G(x)$ is a required path.

**Case 2.** $V(G) = T$.

The result easily follows.

3. **PROOF OF THEOREM 2.**

We call a graph $G$ is elemental for $V_1 \subseteq V(G)$ if $V(G) = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$ and for each $x \in V_2$, $d_G(x) = 3$, $|N_G(x)| = 3$ and $N_G(x) \subseteq V_1$. We call a graph $G$ is elemental for $V_1 \subseteq V(G)$ and an integer $k \geq 1$ if $G$ is elemental for $V_1$ and for each $x \in V_1$, $d_G(x) = k$. For integers $p \geq 0$ and $q \geq 0$, we call a graph $G$ is $G(p, q)$ if $G$ is elemental for some $V_1 = (x_1, x_2, x_3) \subseteq V(G)$, $|V(G) - V_1| = q$ and $|\partial_G(x_i, x_j)| = p$ ($1 \leq i < j \leq 3$). Let $G$ be an elemental graph for $V_1 \subseteq V(G)$. We call a subgraph $S$ an elemental star if $V(S) \subseteq V_1$, $|V(S)| = 2$ and $|E(S)| = 1$, or if for some $x \in V(G) - V_1$, $V(S) = N_G(x) \cup x$ and $E(S) = \partial_G(x)$. 

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We require the following lemmas.

**LEMMA 3.1 (Okamura [7]).** Suppose that $k \geq 4$ is an integer, $G$ is a graph, $(s, t) \subseteq T \subseteq V(G)$ and $T \in \Gamma(G, k)$. Then

1. For each non-separating edge $e$ incident to $s$, there exists a path $P$ between $s$ and $t$ passing through $e$ such that $T \in \Gamma(G-E(P), k-2)$ and $(s, t) \in \Gamma(G-E(P), k-1)$.

2. For each vertex $a$ of $T-(s, t)$ with fewer degree than $2k$ and for each edge $f$ incident to $a$, there exists a path $P$ between $s$ and $t$ not passing through a such that $T \in \Gamma(G-E(P), k-2)$, $(s, t, a) \in \Gamma(G-E(P), k-1)$, and $(s, a)$ or $(t, a) \in \Gamma(G-E(P)-f, k-1)$.

3. For each non-separating edges $e$ and $e'$ incident to $s$, there exists a cycle $C$ passing through $e$ and $e'$ such that $T \in \Gamma(G-E(C), k-2)$.

**LEMMA 3.2 (Okamura [7]).** Suppose that $n \geq 4$ is an integer and $k \geq 3$ is an odd integer. If for each odd integer $1 \leq m \leq k$, 

$$\lambda'(m, n) = m,$$

then

$$\lambda(k, n) = k \quad \text{and} \quad \lambda(k+1, n) = k+2.$$

**LEMMA 3.3.** Suppose that $k \geq 3$ is an integer, $G$ is an elemental graph for $T \subseteq V(G)$ and $k$, $T \in \Gamma(G, k)$, $G$ has no nontrivial $k$-cut separating $T$, and that $S_1, S_2, S_3$ are elemental stars of $G$. If $V(S_1) \cap V(S_2) \cap V(S_3) = \emptyset$, then
\( T \in \gamma(G - \bigcup_{i \neq 1} E(S_i), k - 2) \).

Proof. Assume that \( X \subseteq V(G) \), \( |X| \leq |V(G) - X| \) and \( X \) separates \( T \). Set \( G' = G - \bigcup_{i \neq 1} E(S_i) \). If \( |X| = 1 \), then let \( X = \{x\} \).

Since \( d_{G'}(x) \geq d_G(x) - 2 = k - 2 \), we have \( |\partial_{G'}(x)| \geq k - 2 \). If \( |X| \geq 2 \), then \( |\partial_G(X)| \geq k + 1 \), and so \( |\partial_{G'}(X)| \geq k - 2 \). Now Lemma 3.3 is proved.

Lemma 3.4. Suppose that \( k \geq 2 \) is an integer, \( G \) is an elemental graph for \( T = \{x_1, x_2, x_3, x_4\} \) \( V(G) \) and \( k \), \( |T| = 4 \) and \( T \in \gamma(G, k) \). Then

1. One of the following holds.
   (i) \( \partial_G(x_1, x_2) \neq \emptyset, \partial_G(x_1, x_3) \neq \emptyset \), or for some \( y \in V(G) - T \), \( N_G(y) = \{x_1, x_2, x_3\} \).
   (ii) \( k \) is even, \( |\partial_G(x_2, x_3)| = k/2 \), and
   \( |\{y \in V(G) - T \mid N_G(y) = \{x_1, x_1, x_4\}\}| = k/2 \) (\( i = 2, 3 \)).

2. One of the following holds.
   (i) For each \( 1 \leq i < j \leq k \), \( G \) has an elemental star \( S \) containing \( x_i \) and \( x_j \).
   (ii) \( k \) is even and \( G \) is the graph obtained from four cycle by replacing each edge by \( k/2 \) parallel edges.

3. If \( G \) has no nontrivial \( k \)-cut separating \( T \), then
   (i) \( \partial_G(x_1, x_2) \neq \emptyset \) or \( G \) has two elemental stars containing \( x_1 \) and \( x_2 \).
   (ii) One of the following holds.
   (a) \( G \) has edge-disjoint paths \( P_1[x_1, x_2] \) and \( P_2[x_1, x_3] \) such that for \( i = 2 \) or \( 4 \),

\[ 8 \]
$(x_i, x_3) \in \Gamma(G - \bigcup_{i=1}^{2} E(P_i), k-1)$ and $T \in \Gamma(G - \bigcup_{i=1}^{2} E(P_i), k-2)$.

(b) For each $e \in \partial G(x_3) - \partial G(x_3, x_2)$, $G$ has edge-disjoint paths $P_1[x_1, x_2]$ and $P_2[x_1, x_3]$ such that $e \in E(P_2)$ and $T \in \Gamma(G - \bigcup_{i=1}^{2} E(P_i), k-2)$.

Proof. For $1 \leq i, j \leq 4$, set

$p_{i, j} = |\partial G(x_i, x_j)|$,  

$R_i = \{y \in V(G) - T \mid N_G(y) = T - x_i\}$,  

$r_i = |R_i|$.  

(1) Assume $p_{1, 2} = p_{1, 3} = r_4 = 0$. Then

$d_G(x_1) = k = p_{1, 4} + r_2 + r_3$,  

$d_G(x_4) = k = p_{1, 4} + p_{2, 4} + p_{3, 4} + r_1 + r_2 + r_3$  

Thus

$p_{2, 4} = p_{3, 4} = r_1 = 0$.  

Since $T \in \Gamma(G, k)$, we have

$|\partial G(x_2, x_3)| = r_2 + r_3 \geq k$.  

Thus

$p_{1, 4} = 0$.  

By comparing $d_G(x_i)$ with $d_G(x_j)$ for $1 \leq i < j \leq 3$, we have

$r_2 = r_3 = p_{2, 3}$.  

Now (ii) follows.

(2) Assume $p_{1, 2} = r_3 = r_4 = 0$. Then by comparing $d_G(x_1) + d_G(x_2)$ with $d_G(x_3) + d_G(x_4)$, we have

$r_1 = r_2 = p_{3, 4} = 0$.  

Now by comparing $d_G(x_3) = k = p_{1, 3} + p_{2, 3}$ with $d_G(x_i)$  

for $i = 1, 2$, we have
\[ p_{1,4} = p_{2,3} \text{ and } p_{2,4} = p_{1,3}. \]

Moreover
\[ 1 \mathcal{G}(\langle x_1, x_4 \rangle) = p_{1,3} + p_{2,4} = 2p_{1,3} \geq k, \]
\[ 1 \mathcal{G}(\langle x_1, x_3 \rangle) = p_{1,4} + p_{2,3} = 2p_{1,4} \geq k. \]
Thus
\[ p_{1,3} = p_{2,3} = p_{2,4} = p_{1,4}, \]
and so (ii) follows.

(iii) We assume \( p_{1,2} = r_4 = 0 \), and then prove \( r_3 \geq 2 \). Since any cut separating \( (x_1, x_3) \) and \( (x_2, x_4) \) or separating \( (x_1, x_4) \) and \( (x_2, x_3) \) has more than \( k \) edges we have

\[ (3.1) \quad p_{1,4} + p_{2,3} + p_{3,4} + r_1 + r_2 + r_3 \geq k + 1, \]
and

\[ (3.2) \quad p_{1,3} + p_{2,4} + p_{3,4} + r_1 + r_2 + r_3 \geq k + 1. \]

By comparing \( d_G(x_3) + d_G(x_4) \) with (3.1)+(3.2), we have \( r_3 \geq 2 \).

(ii) If there exists \( f \in \mathcal{G}(x_1, x_3) \), then by Lemma 2.1 \( G \) has a path \( P[x_3, x_2] \) such that \( f \in E(P) \), \( (x_3, x_2) \in \Gamma(G - E(P), k - 1) \) and \( T \in \Gamma(G - E(P), k - 2) \), and so (a) follows. Thus we may let
\[ p_{1,3} = p_{1,2} = 0, \]
then by (1)
\[ r_4 > 0. \]

If \( r_3 > 0 \), then for \( y_1 \in R_4 \) and \( y_2 \in R_3 \),
\[ (x_3, x_4) \in \Gamma(G - \bigcup_{i=1}^{2} \mathcal{G}(y_i), k - 1) \text{ and } T \in \Gamma(G - \bigcup_{i=1}^{2} \mathcal{G}(y_i), k - 2), \]
and so (a) follows. Thus we may let
\[ r_3 = 0. \]
Then by (1) and (3)
\[ p_1,4 > 0 \text{ and } r_4 \geq 2. \]
Let \( y \) be another end of \( e \), then \( y = x_4 \) or \( y \notin R_i \) (i=1,2 or 4).
In each case (b) easily follows.

**LEMMA 3.5.** Suppose that \( k \geq 3 \) is an odd integer, \( G \) is a graph, \( (x_1,x_2,x_3) \subseteq T \subseteq V(G) \), \( x_i \neq x_j \) (1 \leq i < j \leq 3),
\( T \in \Gamma(G,k) \) and \( e \in E(G) \). If following (i) or (ii) holds, then
for \( m=2,3 \), \( G \) has edge-disjoint paths \( P_1[x_1,x_2] \) and \( P_2[x_1,x_m] \) such that \( e \in E(P_1) \cup E(P_2) \) and
\( T \in \Gamma(G- \bigcup_{i=1}^{2} E(P_i), k-2) \).

(i) \( e \in \partial_G(x_1,x_2) \),

(ii) \( e \in \partial_G(x_1,y) \) for some \( y \in V(G)-T \) with \( d_G(y)=3 \) and with \( N_G(y)=(x_1,x_2,x_3) \).

Proof. Assume that (i) holds. By Theorem 1 if \( m=2 \), then
\( G \) has a cycle \( C \) such that \( e \in E(C) \) and \( T \in \Gamma(G-E(C), k-2) \), and
if \( m=3 \), then \( G \) has a path \( P[x_2,x_3] \) such that \( e \in E(P) \) and
\( T \in \Gamma(G-E(P), k-2) \).

Assume that (ii) holds. We may assume that \( G \) is
2-connected. If \( d_G(x_3)=d > k \), then we replace \( x_3 \) by \( d \)
vertices of degree \( k \) (Figure 4 gives an example with \( d=8 \) and
\( k=5 \)), producing a new graph \( G' \). In \( G' \) we assign \( x_3 \) on
\( N_G'(y)-(x_1,x_2) \). If the result holds in \( G' \), then
clearly the result holds in \( G \), and so we may assume that
\( d_G(x_3)=k \). Let \( f \in \partial_G(x_3) \cap \partial_G(y,x_3) \). By Lemma 3.1
Figure 4.

G has a path $P[x_1,x_2]$ such that $x_3 \not\in V(P), T \in \Gamma(G \setminus E(P), k-2)$, $(x_1,x_2,x_3) \in \Gamma(G \setminus E(P), k-1)$ and $(x_i,x_3) \in \Gamma(G \setminus E(P) \setminus f, k-1)$ $(i=1 \text{ or } 2)$. Then $y \not\in V(P)$, because $d_G(x_3)=k$ and $d_G(y)=3$. Moreover $T \in \Gamma(G \setminus E(P) \setminus y, k-2)$. Thus the result follows.

Now we prove Theorem 2. We may assume that $G$ is 2-connected, $d_G(x) = k$ for each $x \in T$ (see the proof of Lemma 3.5 and Figure 4, in this case we can assign $x$ on any vertex of new $d_G(x)$ vertices of degree $k$) and that $d_G(y) = 3$ for each $y \in V(G) \setminus T$ (see Case 1 in the proof of Theorem 1). We proceed by induction on $|E(G)|$. If $|T| \leq 3$, then the result follows from Theorem 1. Thus let $|T| \geq 4$.

Case 1. G has a nontrivial $k$-cut $\partial_G(X) = (e_1, \ldots, e_k)$ $(X \subseteq V(G))$ separating $T$.

We define $H,K,u,v,T_H$ and $T_K$ similarly as in the proof of Theorem 1. If $|X \cap T| = 1$, then the results hold in $K$, and so in $G$. Thus let $|X \cap T| \geq 2$ and $|T \setminus X| \geq 2$.

We require the following.

(3.3) If $G$ has a nontrivial $k$-cut $\partial_G(Y) = (f_1, \ldots, f_k)$
(Y \subseteq X) separating T, then we may assume that \((X-Y) \cap T \neq \emptyset\).

Proof. Assume \((X-Y) \cap T = \emptyset\). Let \(b_1, c_1\) be the end of \(e_i, f_i\) in \(Y \setminus V(G) - X\) \((1 \leq i \leq k)\). We may assume that the graph obtained from \((X-Y)_G\) by adding \(b_1, \ldots, b_k, c_1, \ldots, c_k, e_i, \ldots, e_k, f_i, \ldots, f_k\) has edge-disjoint paths \(P_1[b_1, c_1], \ldots, P_k[b_k, c_k]\). Let \(G'\) be the graph obtained from \(G-(X-Y)\) by adding new edges \(g_1, \ldots, g_k\), where \(g_i\) has ends \(b_i\) and \(c_i\) \((1 \leq i \leq k)\). Then \(|E(G')| < |E(G)|\), and the results of Theorem 2 hold in \(G'\), and so in \(G\). Now (3.3) is proved.

(3.4) If \(|X-T| = 2\) \((|T-X| = 2)\), then we may assume that \(H\) \((K)\) is \(G(p, q)\) \((G(p', q'))\) for some integers \(p\) and \(q\) \((p',and q')\).

Proof. Assume \(|X \cap T| = 2\). If \(H\) has a nontrivial \(k\)-cut \(\partial_H(Y) \subseteq V(H) - u\) separating \(T_H\), then by (3.3) \((X-Y) \cap T \neq \emptyset\), and so \(|T \cap Y| = 1\). Then by taking \(Y\) instead of \(X\) the results of Theorem 2 hold. Thus we may assume that an end of each edge of \(H\) is in \(T_H\). Hence the result easily follows.

We return to the proof of Theorem 2. By Lemma 3.5 we may assume the following.

(3.5) \(\partial_G(a_1, a_i) = \emptyset\) \((i = 2, m)\) and for each \(y \in V(G) - T\), \((a_1, a_2, a_m) \notin N_G(y)\).
Let \( a_1 \in X \).

(1) Now \( |X-T|=|T-X|=2 \). If \( a_2 \in X \), then by (3.4) the result easily follows. Thus let \( a_2 \in V(G)-X \). Since
\[
p+q \geq (k+1)/2 \quad \text{and} \quad p'+q' \geq (k+1)/2,
\]
for some \( 1 \leq i \leq k \), \( H \) has an elemental star \( S_1 \) containing \( a_1 \) and \( e_i \) and \( K \) has an elemental star \( S_2 \) containing \( a_2 \) and \( e_i \). Then \( T \in \Gamma(G-\bigcup_{i=1}^{2} E(S_i),k-1) \).

(2) Subcase 1-1. \( (a_2,a_m) \subseteq X \).

\( H \) has required paths. If one of them passes through \( u \), then we can deduce the result by using Lemma 3.1(3) on \( K \).

Subcase 1-2. \( (a_2,a_m) \subseteq V(G)-X \) and \( |X \cap T|=2 \).

Set \( X \cap T=\{a_1,a_5\} \). By (3.4) \( H \) is \( G(p,q) \). Thus if following (3.6) or (3.7) holds, then the result follows.

(3.6) For some \( e_i \in \partial H(u,a_1) \), \( K \) has edge-disjoint paths \( P_1[v,a_2] \) and \( P_2[v,a_m] \) such that \( e_i \in E(P_1) \cup E(P_2) \) and \( T \in \Gamma(K-\bigcup_{i=1}^{2} E(P_i),k-2) \).

(3.7) For some \( e_i,e_j \in \partial H(u)-\partial H(u,a_5) \), \( K \) has edge-disjoint paths \( P_1[v,a_2] \) and \( P_2[v,a_m] \) such that \( \{e_i,e_j\} \subseteq E(P_1) \cup E(P_2) \) and \( T \in \Gamma(K-\bigcup_{i=1}^{2} E(P_i),k-2) \).

If \( p=0 \), then \( \partial_H(u,a_5)=\emptyset \), and so (3.7) follows. Thus let \( p>0 \). If \( |T-X|=2 \), then by (3.4) \( K \) is \( G(p',q') \), and so (3.6) follows. Thus let \( |T-X|=3 \) and \( m=3 \). Set \( T-X=\{a_2,a_3,a_4\} \).

Subcase 1-2-1. \( K \) has nontrivial \( k \)-cut \( \partial_K(Y) \) \( (Y \subseteq V(K)-v) \) separating \( T \).

By (3.3) We may let \( |Y \cap T|=|T-Y|=2 \). Let \( K_1 \) and \( K_2 \) be the graphs obtained from \( K \) by contracting \( Y \) and \( V(K)-Y \) to a vertex respectively. Then similarly as (3.4)
$K_i$ is $G(p_i, q_i)$ for some integers $p_i$ and $q_i$ ($i=1,2$)

Let $M$ be

$\{ (x_1, x_2) \in V(K) - T_K \mid \partial K(x_1, x_2) \not\subseteq \emptyset \}$,

and let $M'$ be

$\{ x \mid \text{For some } N \subseteq M, x \in N \}$.

For each $N \subseteq M$, $N \cap V(K_i) \not= \emptyset$ ($i=1,2$),

distance $d_{K-N}(a_j) = d_{K-N}(u_j) = k-1$ $(j=2,3,4)$ and $T_K \subseteq \Gamma(K-N, k-1)$.

If $k \not= 1M_1$, then $p_1 = p_2 = 0$ and the result easily follows,

and so let $k \not= 1M_1$. $K-M'$ is elemental for $T_K$ and $K-1M_1$.

Assume that $k-1M_1$ is even and $K-M'$ is the graph obtained from four cycle by replacing each edge by $(k-1M_1)/2$ parallel edges. For each cycle $C$ of $K-M'$ such that $|V(C)| = |E(C)| = 4$,

we have $T_K \subseteq \Gamma(G-E(C), k-2)$. If $\partial G(a_1, a_4) \not= \emptyset$, then

$3.6$ follows, and if not, then by (3.5) $a_1$ is adjacent to

$p$ vertices of $M'$. If $1M_1 \not= 2$, then (3.6) follows. Thus

assume $1 \geq 1M_1 \geq p \geq 1$. Since $(k-1M_1)/2 \geq (5-1)/2 = 2$, for some

$1 \leq i < j \leq k$,

$$(e_i, e_j) \subseteq \partial_H(u) - \partial_H(u, a_5),$$

and $K$ has a four cycle $C$ such that $|V(C)| = |E(C)| = 4$ and

$(e_i, e_j) \subseteq E(C)$. Hence (3.7) follows.

By Lemma 3.4(2) we may assume that for each two vertices of $T_K$, $K-M'$ has an elemental star containing them. Set

$a_0 = u$, and for $i, j = 0, 2, 3, 4$, set

$p_i, j = |\partial K(a_i, a_j)|$,

$r_i = |\{ x \in V(K) - T_K \mid N_K(x) = T_K - a_0 \}|$.

For $i, j = 0, 2, 3, 4$, since $|\partial K(a_i, a_j)| \geq k$,
If $a_1$ is adjacent to a vertex of $M'$ in $G$, then (3.6) follows. If for some $x \in V(G)-T$, $N_G(x)=(a_1,a_i,a_4)$ $(i=2$ or $3)$, then (3.6) follows. Thus and by (3.5) we may assume that

$|\partial G(a_1,a_4)|=p$.

If $a_4 \in Y$, then (3.6) easily follows, and thus let $T_{H-Y}=(a_0,a_4)$. Since $p_0,4 \geq |\partial G(a_1,a_4)|=p>0$, by Lemma 3.4(1) we have

$p_4,2>0$, $p_4,3>0$, or $r_0>0$,

and

$p_0,2>0$, $p_0,3>0$, or $r_4>0$.

If $r_0>0$, $r_4>0$, $p_0,2+p_3,4>0$, or $p_0,3+p_2,4>0$,
then (3.6) follows (note that $K_i$ is $G(p_i,q_i)$ for $i=1,2$)

Thus we may assume that

(3.8) $p_0,2>0$, $p_2,4>0$ and $r_0=r_4=p_0,3=p_3,4=0$.

Assume $|M|=0$. Then

$d_G(a_3)=p_2,3+r_2$ and $p_2,3 \leq (k-1)/2$,

and so

(3.9) $r_2 \geq (k+1)/2 \geq p+1$.

By comparing $d_G(a_2)$ with $d_G(a_4)$ we have

$p_0,2+p_2,3=p_0,4+r_2$.

Thus

(3.10) $p_0,2 \geq p_0,4 \geq p$.

From (3.9) and (3.10), (3.7) follows.

Now we may let $|M|>0$. Since $(a_2,a_3) \subseteq Y$, we have
\[ l \mathcal{E}_K(Y) = k = d_K(a_2) + d_K(a_3) - 2p_2,3 - |M| \]

\[ = 2k - 2p_2,3 - |M|, \]

and so

\[ 2p_2,3 + |M| = k. \]

Since \( d_G(a_3) = k = p_2,3 + r_2 + |M|, \)

\[ r_2 = p_2,3, \]

Since \( d_G(a_3) = 2r_2 + |M|, \) \( d_G(a_4) = p_0,4 + p_2,4 + r_2 + r_3 + |M|, \)

and \( p_2,4 > 0 \) (by (3.8)), we have

\[ (3.11) \quad r_2 > a_0,4 + 1 \geq p + 1. \]

By comparing \( d_G(a_2) \) with \( d_G(a_4) \), we have

\[ p_0,2 = p_0,4. \]

Thus

\[ (3.12) \quad p_0,2 + |M| \geq p + 1. \]

From (3.11) and (3.12), (3.7) follows.

Subcase 1-2-2. \( K \) has no nontrivial \( k \)-cut separating \( T_K \).

We may assume that an end of each edge of \( K \) in \( T_K \) and \( K \) is elemental for \( T_K \). The proof is similar as the case \( |M| = 0 \) in the proof of Subcase 1-2-1.

Subcase 1-3. \( (a_2, a_m) \subseteq V(G) - X \) and \( |X \cap M| = 3 \).

Now \( m = 3 \). By (3.4) \( K \) is \( G(p', q') \). Set \( X \cap T = \{ a_2, a_4, a_5 \} \)

If \( H \) has nontrivial \( k \)-cut \( \mathcal{E}_H(Y) \) (\( Y \subseteq V(H) - u \)) separating \( T_H \), then we may let \( |Y \cap T_H| = 2 \). Then for \( Y \) or \( V(G) - Y \) instead of \( X \) Subcase 1-1 or Subcase 1-2 occurs. Thus we may assume that this is not the case and \( H \) is elemental for \( T_H \).

If following (3.13) or (3.14) holds, then the result follows.

\[ (3.13) \quad \text{For some } e_i \in \partial K(v) - \bigcup_{i=1}^3 \partial K(v, a_i), \text{ H has edge-disjoint paths } P_1[a_1, u] \text{ and } P_2[a_1, u] \text{ such that} \]
\[ e_i \in E(P_i) \cup E(P_2) \quad \text{and} \quad T_{H,e} \Gamma \left( H \cup \bigcup_{i=1}^{2} E(P_i), k-2 \right). \]

(3.14) For \( l=2 \) or \( 3 \) and for some \( e_i \in \partial K(v, x_1) \) and \( e_j \in \partial K(v) - \partial K(v, x_1) \), \( H \) has edge-disjoint paths \( P_1[a_1, u] \) and \( P_2[a_1, u] \) such that \( \langle e_i, e_j \rangle \subseteq E(P_1) \cup E(P_2) \) and \( T_{H,e} \Gamma \left( H \cup \bigcup_{i=1}^{2} E(P_i), k-2 \right) \).

Set \( a_0 = u \) and for \( i, j = 0, 1, 4, 5 \) set
\[
\begin{aligned}
P_i, j &= \partial H(a_i, a_j), \\
R_i &= \{ x \in V(H) - T_H \mid N_H(x) = T_{H-a_i} \}, \\
r_i &= 1R_i. 
\end{aligned}
\]

By (3.5) \( p_0, 1 = 0 \).

Assume \( p_1, 4 = p_1, 5 = 0 \). If \( r_0 \leq (k-1)/2 \), then
\[
r_4 + r_5 = d_G(a_1) - r_0 \geq (k+1)/2 \geq p' + 1,
\]
and so (3.13) or (3.14) follows. Thus let \( r_0 \geq (k+1)/2 \).

Since \( d_G(a_0) = p_0, 4 + p_0, 5 + r_1 + r_4 + r_5 \) and
\[
d_G(a_5) = p_0, 5 + p_4, 5 + r_0 + r_1 + r_4,
\]
we have
\[
p_0, 4 + r_5 = p_4, 5 + r_0.
\]
Hence
\[
d_G(a_4) = k \geq p_0, 4 + r_0 + r_5 \geq 2r_0 \geq k,
\]
a contradiction.

Now we may let \( p_{1, i} > 0 \) for \( i = 4 \) or \( 5 \), say \( i = 4 \).

Since \( p_0, 1 = 0 \) and by Lemma 3.4(3), we have
\[
r_4 + r_5 \geq 2.
\]

For each \( x \in R_4 \cup R_5 \), if \( x \) is adjacent to a vertex of \( V(K) - T_K \) in \( G \), then (3.13) follows, thus assume that \( \partial G(x, a_1) \neq \emptyset \) (i=2 or 3). For each \( x, y \in R_4 \cup R_5 \), if \( \partial G(x, a_2) \neq \emptyset \) and \( \partial G(y, a_3) \neq \emptyset \), then (3.14) follows, thus assume that for \( i = 2 \) or \( 3 \), \( \partial G(x, a_i) = \partial G(y, a_i) = \emptyset \), say \( i = 3 \),

\[ \]
and that \( r_4 + r_5 \leq p' \).

Assume \( r_4 > 0 \). For some \( e_1 \in \partial K(u) - \partial K(u, a_2) \), \( e_1 \) is incident to \( a_4 \) or a vertex of \( R_1 \) in \( G \), because
\[
p' + q' \geq (k+1)/2 > p_0, 5.
\]
Thus (3.14) follows.

Now we may assume that \( r_4 = 0 \), \( r_5 > 0 \) and \( p_1, 5 = 0 \).

Thus \( p_0, 1 = p_1, 5 = r_4 = 0 \), contrary to Lemma 3.4(1). Subcase 1-4. \( a_2 \in X \) and \( a_m \in V(G) - X \). now \( m = 3 \).

Subcase 1-4-1. \( |X \cap T| = 2 \).

By (3.4) \( H = G(p, q) \), and by (3.5) \( p = 0 \). Since \( |T_K| \leq 4 \), by induction \( K \) has a path \( p[v, a_3] \) such that
\( T_k \in \overrightarrow{P}(K-E(P), k-1) \). Let \( e_1 \in E(P) \). \( H \) has an elemental star \( S_1 \) containing \( a_1 \) and \( e_1 \). Let \( S_2 \) be another elemental star of \( H \). Then \( T_{H} \in \overrightarrow{P}(H - \bigcup_{i=1}^{2} E(S_i), k-2) \), and so the result follows.

Subcase 1-4-2. \( |X \cap T| = 3 \) and \( |T - X| = 2 \).

Assume that \( H \) has a nontrivial \( k \)-cut \( \partial_H(Y) = (f_1, \ldots, f_k) \) (\( Y \subseteq V(H) - u \)) separating \( T_H \). Then we may assume that \( |Y \cap T_H| = 2 \), \( a_2 \in Y \) and \( a_1 \in X - Y \). Let \( H_1 \) (\( H_2 \)) be the graph obtained from \( H \) by contracting \( V(H) - Y \) (\( Y \)) to a new new vertex \( u_1 \) (\( u_2 \)). Then similarly as (3.4) \( H_1 \) is \( G(p_i, q_i) \) for some integers \( p_i \) and \( q_i \) (\( i = 1, 2 \)). If \( p_2 = 0 \), then the result easily follows. If \( p_2 > 0 \), then we may let \( (f_1, e_1) \subseteq \partial g(a_1) \) and we can easily deduce the result.

Now we may assume that \( H \) has no nontrivial \( k \)-cut
Let $F$ be a cut of $G$ separating $(a_1, a_5)$ and $(a_2, a_3)$.

Since $p_{i,j,k}(k-1)/2$ for each $i$, $j$, $k$, we have $(1, 1, 1) > 0$.

Proof. Assume that each elemental star of $G$ does not contain $(a_1, a_2)$ nor $(a_1, a_3)$. Then $d_G(a_1) = p_{i,4} + p_{i,5} + r(1, 1, 1) = 1$.

By Lemma 3.3, the result follows. Thus let $|T| = 5$ and $m = 3$.

We may assume that $G$ is elemental for $T$. If $|T| = 4$, then $\varnothing(\mu) \neq \varnothing(\mu, u, v, w)$ and $e_i \in \varnothing(\mu, u, v, w, x, y, z)$, for some $x, y, z$. Instead of $x, y, z$, assume that $u, v, w$.

Case 2. $G$ has no nontrivial $k$-cut separating $T$. Then the result follows.

So we require the following.

For each distinct $i, j, k, 1 \\leq i, j, k \\leq 5$, $G$ has an elemental star containing $(a_1, a_2)$ or $(a_1, a_3)$.
\[ |F| = d_\mathcal{G}(a_4) + d_\mathcal{G}(a_5) - (p_{1,4} + p_{1,5} + 2r(1,4,5)) < k, \]
a contradiction. Now (3.15) is proved.

We return to the proof of Theorem 2. By (3.5)

\[ p_{1,2} = p_{1,3} = r(1,2,3) = 0. \]

If \( r(1,2,i) > 0 \) and \( r(1,3,j) > 0 \) (\( i,j = 4 \) or \( 5 \)), then the result follows. Thus and by (3.15) we may assume that

\[ r(1,2,4) > 0 \text{ and } r(1,3,i) = 0 \quad (i = 4,5). \]

By (3.15)

\[ p_{1,5} + r(i,5,2) + r(i,5,4) > 0 \quad (i = 1,3). \]

If \( p_{1,5} > 0 \), \( p_{3,5} > 0 \), \( r(1,5,2) \cdot r(3,5,4) > 0 \), or

\[ r(1,5,4) \cdot r(3,5,2) > 0, \]
then by Lemma 3.3 the result follows.

Thus we may assume that for \( (i,j) = (2,4) \) or \( (4,2) \),

\[ p_{1,5} = p_{3,5} = 0, \ r(1,5,i) = r(3,5,i) = 0, \]

and

\[ r(1,5,i) \cdot r(3,5,j) > 0. \]

Assume \( r(1,5,2) = r(3,5,2) = 0. \) Then

\[ d_\mathcal{G}(x_1) = p_{1,4} + r(1,2,4) + r(1,4,5), \]

and

\[ d_\mathcal{G}(x_4) \geq p_{1,4} + r(1,2,4) + r(1,4,5) + r(3,4,5) > k, \]
a contradiction. Thus

\[ r(1,5,4) = r(3,5,4) = 0. \]

Since \( r(1,2,5) > 0 \), by the same argument we have

\[ p_{1,4} = p_{3,4} = 0. \]

Thus

\[ d_\mathcal{G}(x_1) = r(1,2,4) + r(1,2,5) \]

and
\[ d_G(x_2) \geq r(1,2,4) + r(1,2,5) + r(2,3,5) > k, \]
a contradiction.

4. PROOF OF THEOREM 3.

Suppose that \( k \geq 1 \) is an integer, \( G \) is a graph, \( T = (s_1, \ldots, s_k, t_1, \ldots, t_k) \subseteq V(G) \) and \( T \in \Gamma(G,k) \). We prove that if \( |T| = 3 \), or if \( k \) is odd and \( |T| = 4 \) or \( 5 \), then \((1,1)\) holds by induction on \( k \).

Assume \( |T| = 3 \). By Theorem 1 \( G \) has a path \( p s_k s_k \) such that \( T \in \Gamma(G-E(P),k-1) \). By induction for \( k-1 \), \((1.1)\) holds in \( G - E(P) \), and so for \( k \), \((1.1)\) holds.

Assume that \( k \geq 5 \) is odd and \( |T| = 4 \) or \( 5 \). For some \( 1 \leq i < j \leq k \), if \( |T| = 4 \), then
\[ s_i = s_j \text{ or } t_i, \]
and if \( |T| = 5 \), then
\[ s_i = s_j \text{ or } t_j \text{ and } (s_i, t_i) \notin (s_j, t_j), \]
say for \( i = k-1 \) and \( j = k \). By Theorem 2 \( G \) has edge-disjoint paths \( P_i [s_{k-1}, t_{k-1}] \) and \( P_j [s_k, t_k] \) such that \( T \in \Gamma(G - \bigcup_{i=1}^2 E(P_i),k-2) \). By induction for \( k-2 \), \((1.1)\) holds in \( G - \bigcup_{i=1}^2 E(P_i) \), and so for \( k \), \((1.1)\) holds in \( G \).

Thus for integer \( k \geq 1 \),
\[ \lambda'(k,3) = \lambda(k,3) = k, \]
and for odd integer \( k \geq 1 \),
\[ \lambda'(k,4) = \lambda'(k,5) = k. \]
By Lemma 3.2 for odd integer \( k \geq 1 \),
\[ \lambda(k,4) = \lambda(k,5) = k \text{ and } \lambda(k+1,4) = \lambda(k+1,5) = k+2. \]
Now Theorem 3 is proved.
REFERENCES


