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Finite Biautomata on Two-way Infinite Words

(Preliminary Report)

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Abstract

The classes of two-way infinite languages accepted by finite biautomata through several acceptance conditions are studied. A two-way infinite word is a two-way infinite sequence of symbols of finite kinds. A finite biautomaton is a pair of finite automata, one of which runs leftward infinitely while the other runs rightward infinitely starting at some point of a two-way infinite word. This paper deals with the classes characterized by finite biautomata under four types of acceptance conditions which have been used to study behaviours of finite automata on ω-words.

1. Introduction

A two-way infinite word is a two-way infinite sequence of symbols of finite kinds whose left/right shift denotes the same two-way infinite word. Finite biautomata on this kind of words was firstly investigated by Nivat and Perrin [4]. A biautomaton is a pair of finite automata, one of which runs leftward infinitely while the other runs rightward infinitely starting at some point of a two-way infinite word.

Nivat and Perrin defined both nondeterministic and deterministic biautomata through Büchi type acceptance condition [1]. The class of two-way infinite languages accepted by nondeterministic biautomata can be considered as an extension of ω-regular languages [3] to two-way infinite case. They have shown that nondeterministic models define a larger class than deterministic ones. It has been also shown that nondeterministic class is the Boolean closure of deterministic one. These results are regarded as extensions of the corresponding results for ω-languages and require more difficult arguments.

In this paper we consider nondeterministic and deterministic biautomata on two-way infinite words through other acceptance conditions which have been used to study behaviours of finite automata on ω-words [5,6] and characterize classes they define.

2. Preliminaries

Definition: Let A be a finite alphabet, and A^ω denote the set of mapping $x : \{0,1,2,\ldots\} \to A$. We call the mapping x an ω-word, and write $x=a_0a_1a_2\ldots$ where $x(n)=a_n$ $(n=0,1,2,\ldots)$.

Let $A^\infty=A^* \cup A^\omega$ where A^* stands for the set of finite words over A. We call the members of A^∞ ∞-words(infinitary words). For an ∞-word w and $n \geq 0$, we write

$$w[n]=\begin{cases} w(0)w(1)\ldots w(n-1) & \text{if } w \text{ is in } A^\omega, \\ a_0a_1\ldots a_{i-1} & \text{if } w=a_0a_1\ldots a_{m-1} \text{ and } i=\min\{n,m\} \end{cases}$$

where $a_0,a_1,\ldots,a_{m-1}$ are in A. In this paper unless otherwise stated, we assume that u,v,w stand for arbitrary ∞-words, x,y,z for ω-words, f,g,h for finite words, W,U,V for ∞-languages (subset of A^∞),X,Y,Z for languages(subset of A^*),a,b for symbols in A, and n,m,i for natural numbers(≥ 0).

We define a partial order $\leq$ in A^∞ by

$w \leq v$ iff $w=v$ or $w=v[n]$ for some n,

and write

$\downarrow w=\{f \text{ in } A^* \mid f \leq w\}=\{w[n] \mid n\geq 0\}$.

For an ∞-language W, we write

$\downarrow W=\cup\{\downarrow w \mid w \in W\}=\{w[n] \mid w \in W, n\geq 0\}$.

For an increasing sequence

$w_0 \leq w_1 \leq w_2 \leq w_3 \leq \ldots$

of elements w_i in A^∞, the supremum of (w_i) is denoted by $\sup(w_i)$. Given an ∞-language W, sup(W) denotes the set of supremums of increasing sequences whose elements are in W.

Now we extend the regular operations to ∞-languages. First we extend the concatenation operation in A^* to A^∞ by

$$wv=\begin{cases} wv(0)v(1)v(2)\ldots & \text{if } w\in A^* \text{ and } v\in A^\omega, \\ w & \text{if } w\in A^\omega \text{ and } v\in A^\infty, \end{cases}$$

and define WV and W^* as usual. That is,

$WV = \{wv \mid w \in W, v \in V\}$,

$W^* = \{e\}\cup W\cup WW\cup WWW\cup\ldots$

Here e stands for the empty word. We define the ω-power of an ∞-language W as

$W^{\omega} = \{w_0 w_1 w_2 \ldots \mid w_0, w_1, w_2, \ldots \in W - \{e\}\}$

where $w_0 w_1 w_2 \ldots$ means the ω-word w such that $w_0 w_1 \ldots w_n \leq w$ for all n. For an ∞-language W,

$W^{a} = \{w \text{ in } A^{\omega} \mid \downarrow w \subset \downarrow W\}$.

We define another operation L^e for a language $L \subset A^*$ by

$L^e = \{w \text{ in } A^{\omega} \mid \downarrow w \cap L \text{ is infinite}\} = \sup(L) \cap A^{\omega}$.

We call (u,v) in $A^{\infty} \times A^{\infty}$ a <u>bi-word</u>. Over the set of bi-words $A^{\infty} \times A^{\infty}$, we define an equivalence relation denoted by ~ as

(u,v) ~ (u',v') iff there exists f in A^* such that v=fv' and $u' = f^R u$ or v'=fv and $u = f^R u'$, where R is the reverse operator of A^*.

We say an equivalence class of bi-words under ~ a <u>bilateral word</u>. The set of bilateral words is denoted by ${}^{\infty}A^{\infty}$ and the canonical surjection from $A^{\infty} \times A^{\infty}$ onto ${}^{\infty}A^{\infty}$ is denoted by ρ.

A bi-word (u,v) is said to be

finite if u, $v \in A^*$,

right-infinite if $u \in A^*$, $v \in A^{\omega}$,

left-infinite if $u \in A^{\omega}$, $v \in A^*$,

two-way infinite (biinfinite) if u, $v \in A^{\omega}$.

We can identify the set of finite bilateral words with A^*. If $(f,g) \in A^* \times A^*$, we can make correspondence with $f^R g$. And we have (f,g) ~ (f',g') iff $f^R g = f'^R g'$. Therefore we are allowed to denote $f^R g$ the class $\rho(f,g)$.

In the same way the set of right-infinite bilateral words can be identified with A^{ω} by making correspondence of $(f,y) \in A^* \times A^{\omega}$ with the ω-word $f^R y \in A^{\omega}$.

We denote the set of left-infinite bilateral words by ${}^{\omega}A$. One can define a bijection

$x \in {}^{\omega}A \to x^R \in A^{\omega}$

in association with the identification of a bi-word $(u,g) \in A^{\omega} \times A^*$ with the ω-word $g^R u \in A^{\omega}$.

We can also define the product of a word in ${}^{\omega}A$ with a word in A^{∞} by associating for $x \in {}^{\omega}A$, $v \in A^{\infty}$ with the equivalence class of (x^R, v). The corresponding element in ${}^{\infty}A^{\infty}$ is denoted by xv. We also use the following notation:

For an ∞-language W,

${}^{a}W = \{w \text{ in } {}^{\omega}A \mid \downarrow(w^R) \subset \downarrow(W^R)\}$.

For a language $L \subset A^*$,

${}^{e}L = \{w \text{ in } {}^{\omega}A \mid \downarrow(w^R) \cap L^R \text{ is infinite}\}$.

For a language $L \subset A^*$,

$^{\omega}L$ is a subset of $^{\omega}A$ defined by: $(^{\omega}L)^R = (L^R)^{\omega}$.

If the word is two-way infinite bilateral, then we call it simply biinfinite. The set of biinfinite words over A is denoted by $^{\omega}A^{\omega}(=A^{\omega}\times A^{\omega}/\sim)$.

The set $A^{\infty}\times A^{\infty}$ of bi-words is naturally ordered by

$(u,v) \le (u',v')$ iff $u \le u'$ and $v \le v'$

where $\le$ is the relation already defined on A^{∞}.

For an increasing sequence (u_n,v_n) of bi-words, the supremum denoted by $\sup(u_n,v_n)$ equals to $(\sup(u_n),\sup(v_n))$.

For a language $W \subset A^*$, we associate a biinfinite language $^eW^e=\{\rho(\sup(f_n,g_n)) \in {}^{\omega}A^{\omega} \mid (f_n, g_n)$ are strictly increasing sequences of bi-words such that $f_n^R g_n \in W$ for all n $\}$.

For a language $W \subset A^*$, we associate a biinfinite language $^aW^a=\{(x,y) \in A^{\omega}\times A^{\omega} \mid x[n]^R y[m] \in C(W)$ for all $n,m\ge 0\}/\sim$, where C(W) denotes the set of subwords occurring in W, that is $C(W)=\{g \in A^* \mid fgh \in W$ for some $f,h \in A^*\}$.

Definition: A finite automaton is a 5-tuple M=(Q,A,T,D,F), where

(1) Q is a finite set of states.

(2) A is a finite alphabet.

(3) T is a subset of $Q\times A\times Q$ such that the set $T(q,a)=\{p \mid (q,a,p)$ is in T$\}$ is not empty for q in Q and a in A. Elements in T are called transitions.

(4) D is a subset of Q called the set of initial states.

(5) F is a subset of Q called the set of final states.

A finite automaton said to be deterministic if $|D|=1$ and $|T(q,a)|=1$ for all q in Q and a in A.

For a finite automaton M, a finite word $f=a_0a_1\ldots a_{n-1}$, and two states p and q in Q, we write

$p \xrightarrow{f} q$ in M

if there exists a finite consecutive sequence of transitions $(q_0,a_0,q_1), (q_1,a_1,q_2),\ldots,(q_{n-1}, a_{n-1},q_n)$ such that $q_0=p$ and $q_n=q$.

For a finite automaton M, $L_*(M)$ denotes the language accepted by M, i.e.

$L_*(M)=\{f\in A^* \mid$ There exist $p\in D$ and $q\in F$ such that $p \xrightarrow{f} q$ in M $\}$.

Given an infinite word x and an automaton M, a computation α is an infinite consecutive sequence of transitions $(q_0,a_0,q_1), (q_1,a_1,q_2),\ldots, (q_{n-1},a_{n-1},q_n),\ldots$, where q_0 is in D and $x=a_0a_1a_2\ldots$ We denote the set of computations of M for x by R(M,x). For a computation α, we define

(1) $I(\alpha)=\{q \mid \text{state } q \text{ occurs in } \alpha \text{ infinitely many times}\}$,

(2) $O(\alpha)=\{q \mid \text{state } q \text{ occurs in } \alpha\}$.

For a finite automaton $M=(Q,A,T,D,F)$ and x in A^ω, we say that M accepts x in the sense of C_i $(i=1,\ldots,4)$ if there exists a computation α in $R(M,x)$ satisfying the condition C_i, where

(C_1) $I(\alpha)\cap F\neq\emptyset$.

(C_2) $I(\alpha)\subset F$.

(C_3) $O(\alpha)\cap F\neq\emptyset$.

(C_4) $O(\alpha)\subset F$.

We call α an <u>accepting computation</u> of M on x in the sense of C_i, respectively. For $i=1,\ldots,4$, we denote by $L_i(M)$ the set of ω-words accepted by M in the sense of C_i, further to clarify the initial state of α, we use the notation $L_i(M;d)$ for the set of ω-words accepted by M in the sense of C_i with accepting computations beginning at the initial state d. Therefore $L_i(M)=\cup_{d\in D} L_i(M;d)$. We say that M recognizes an ω-language L in the sense of C_i if $L=L_i(M)$.

<u>Definition</u>: For $i=1,\ldots,4$, we define

(1) $\mathbf{N}_i = \{L_i(M) \mid M \text{ is a nondeterministic finite automaton}\}$,

(2) $\mathbf{D}_i = \{L_i(M) \mid M \text{ is a deterministic finite automaton}\}$.

The classes $\mathbf{D}_i$ and $\mathbf{N}_i$ $(i=1,\ldots,4)$ have been characterized in terms of general topology and the representations of the ω-languages in these classes have been obtained by applying several operations to regular languages [5,6]. Table I summarizes the known results on deterministic and nondeterministic finite automata on ω-words. The classes concerned, denoted by ω-R, $\mathbf{G}^R$, $\mathbf{F}^R$, $\mathbf{G}^R_\delta$ and $\mathbf{F}^R_\sigma$, are defined as follows:

(1) ω-R: An ω-language in ω-R is of the form $\cup_{i=1}^n X_i Y_i^\omega$ for some regular languages X_i and Y_i. ω-R is called the class of ω-regular languages [3].

(2) $\mathbf{G}^R$: An ω-language in $\mathbf{G}^R$ is of the form XA^ω, where X is a regular language.

(3) $\mathbf{F}^R$: An ω-language in $\mathbf{F}^R$ is described as X^a for some regular language X.

(4) $\mathbf{G}^R_\delta$: An ω-language in $\mathbf{G}^R_\delta$ is written as X^e for some regular language X.

(5) $\mathbf{F}^R_\sigma$: An ω-language in $\mathbf{F}^R_\sigma$ is of the form $\cup_{i=1}^n X_i Y_i^a$ for some regular languages X_i and Y_i.

TABLE I

	C_1	C_2	C_3	C_4
$\mathbf{D}_i$	$\mathbf{G}^R_\delta$	$\mathbf{F}^R_\sigma$	$\mathbf{G}^R$	$\mathbf{F}^R$
$\mathbf{N}_i$	ω-R	$\mathbf{F}^R_\sigma$	$\mathbf{G}^R$	$\mathbf{F}^R$

<u>Definition</u>: A finite biautomaton is a 3-tuple $M=(M_-,M_+,S)$, where

(1) M_- and M_+ are finite automata with

$M_-=(Q_-,A,T_-,D_-,F_-)$ and $M_+=(Q_+,A,T_+,D_+,F_+)$.

(2) S is a subset of $D_-\times D_+$.

For a biautomaton $M=(M_-,M_+,S)$, the set of bi-words accepted by M in the sense of C_i, denoted $L_i(M)$ is defined as

$L_i(M)=\{(x,y)\in A^\omega\times A^\omega \mid$ There exists (d_-,d_+) in S such that x is in $L_i(M_-;d_-)$ and y is in $L_i(M_+;d_+)\}$.

For $i=1,\ldots,4$, we define

$\mathbf{N}_i(A^\omega\times A^\omega)=\{L_i(M) \mid M$ is a biautomaton$\}$.

A biautomaton M is said to be <u>bilateral</u> in the sense of C_i if $L_i(M)$ is closed under the relation $\sim$. If M is bilateral in the sense of C_i, the set of biinfinite words recognized by M, denoted $B_i(M)$ is defined as

$B_i(M)=\rho(L_i(M))$.

For a biinfinite language $L\subset {}^\omega A^\omega$, L is said to be C_i-recognizable if there exists a bilateral biautomaton M such that $L=B_i(M)$. The classes of C_i-recognizable biinfinite languages are denoted by $\mathbf{BN}_i$, respectively for $i=1,\ldots,4$.

A biautomaton $M=(M_-,M_+,S)$ is said to be <u>strictly</u> <u>deterministic</u>, if it satisfies:

(1) Both M_- and M_+ are deterministic finite automata.

(2) $S=\{(d_-,d_+)\}$, where d_- is the unique initial state of M_- and d_+ is the unique initial state of M_+.

A <u>deterministic</u> <u>biautomaton</u> is a finite union of strictly deterministic biautomata. For a deterministic biautomaton $M=\{M_1, \ldots,M_n\}$, the set of bi-words accepted by M in the sense of C_i, denoted $L_i(M)$ is defined as

$L_i(M)=\cup_{j=1}^{n}L_i(M_j)$.

For $i=1,\ldots,4$, we define

$D_i(A^\omega\times A^\omega)=\{L_i(M) \mid M$ is a deterministic biautomaton$\}$.

A deterministic biautomaton M is said to be bilateral in the sense of C_i if $L_i(M)$ is closed under the relation $\sim$. For $i=1,\ldots,4$, $\mathbf{BD}_i$ are the classes of biinfinite languages recognizable by deterministic bilateral biautomata in the sense of C_i, that is,

$BD_i=\{L \subset {}^{\omega}A^{\omega} \mid L=\rho(L_i(M))$ for some deterministic bilateral biautomaton $M\}$.

3. Characterization of BN_i's

Theorem 1 (Nivat and Perrin [4]). For $L \subset {}^{\omega}A^{\omega}$, the following conditions are equivalent.

(1) $L \in \mathbf{BN}_1$.

(2) L is a finite union of sets of the form ${}^{\omega}XYZ^{\omega}$ where $X,Y,Z \in$ R(the class of regular sets).

(3) There exists L' in $\mathbf{N}_1(A^{\omega}\times A^{\omega})$ such that $L=\rho(L')$.

(4) $\rho^{-1}(L) \in \mathbf{N}_1(A^{\omega}\times A^{\omega})$.

Proof. (1) => (2). Let $L=B_1(M)$. Then $L=\rho(L_1(M))$ and $L_1(M)$ is a finite union of the sets of the form $(UV^{\omega},WZ^{\omega})$ with $U,V,W,Z \in \mathbf{R}$. If we set $X=V^R$ and $Y=U^RW$, then $\rho(UV^{\omega},WZ^{\omega})={}^{\omega}XYZ^{\omega}$.

(2) => (3). Since $\mathbf{N}_1(A^{\omega}\times A^{\omega})$ is clearly closed under union, it sufficies to show that (2) implies (3) for a set of the form $L={}^{\omega}XYZ^{\omega}$ with $X,Y,Z \in$ R. Let M_+ be an automaton such that $L_1(M_+)=YZ^{\omega}$ and M_- be an automaton such that $L_1(M_-)=(X^R)^{\omega}$. If we choose $S=D_-\times D_+$, then the biautomaton $M=(M_-,M_+,S)$ has the property that $L_1(M)=((X^R)^{\omega},YZ^{\omega})$ and $L=\rho(L_1(M))$.

(3) => (1). Let $M=(M_-,M_+,S)$ be a biautomaton such that $L=\rho(L_1(M))$. We define a biautomaton M' as follows:

$Q'_-=Q'_+=D'_-=D'_+=Q_-\cup Q_+\cup\{\$\}$, where \$ is a new state,

$F'_-=F_-$, $F'_+=F_+$,

$S'=S\cup\{(q,q) \mid q \in Q_-\cup Q_+\}$;

For $p,q \in Q_-\cup Q_+$ and $a \in A$, $(p,a,q) \in T'_-$ if $(p,a,q) \in T_-$ or $(q,a,p) \in T_+$ or there exists $r \in Q_-$ such that $(r,a,q) \in T_-$ and $(r,p) \in S$. For $p \in Q_-\cup Q_+$ and $a \in A$, if such q does not exist in $Q_-\cup Q_+$, then $(p,a,\$)$ is in T'_-. And $(\$,a,\$)$ is in T'_- for each a in A. In the same way $(p,a,q) \in T'_+$ if $(p,a,q) \in T_+$ or $(q,a,p) \in T_-$ or there exists $r \in Q_+$ such that $(r,a,q) \in T_+$ and $(p,r) \in S$. If such q does not exist, then $(p,a,\$)$ is in T'_+. And $(\$,a,\$)$ is in T'_+.

It is easily seen that if $Q_+\cap Q_-=\emptyset$, the automaton M' is bilateral and $L=B_1(M')$. (This process of making biautomaton bilateral is called **bilateralization.**) Therefore L is in $\mathbf{BN}_1$.

(1) <=> (4). Evident. □

Theorem 2. For $L \subset {}^{\omega}A^{\omega}$, the following conditions are equivalent.

(1) $L \in \mathbf{BN}_2$.

(2) L is a finite union of sets of the form ${}^{a}XYZ^{a}$ where $X,Y,Z \in \mathbf{R}$.

(3) There exists L' in $\mathbf{N}_2(A^{\omega} \times A^{\omega})$ such that $L=\rho(L')$.

(4) $\rho^{-1}(L) \in \mathbf{N}_2(A^{\omega} \times A^{\omega})$.

Proof. (1) => (2). Let $L=B_2(M)$. Then $L=\rho(L_2(M))$ and $L_2(M)$ is a finite union of the sets of the form (UV^a,WZ^a) with $U,V,W,Z \in \mathbf{R}$. If we set $X=V^R$ and $Y=U^RW$, then $\rho(UV^a,WZ^a)={}^{a}XYZ^{a}$.

(2) => (3). Since $\mathbf{N}_2(A^{\omega} \times A^{\omega})$ is clearly closed under union, it sufficies to show that (2) implies (3) for a set of the form $L={}^{a}XYZ^{a}$ with $X,Y,Z \in \mathbf{R}$. Let M_+ be an automaton such that $L_2(M_+)=YZ^a$ and M_- be an automaton such that $L_2(M_-)=(X^R)^a$. If we choose $S=D_- \times D_+$, then the biautomaton $M=(M_-,M_+,S)$ has the property that $L_2(M)=((X^R)^a,YZ^a)$ and $L=\rho(L_2(M))$.

(3) => (1). By bilateralization which preserves the condition C_2.

(1) <=> (4). Evident. □

Theorem 3. For $L \subset {}^{\omega}A^{\omega}$, the following conditions are equivalent.

(1) $L \in \mathbf{BN}_3$.

(2) L is of the form ${}^{\omega}AXA^{\omega}$ where $X \in \mathbf{R}$.

(3) There exists L' in $\mathbf{N}_3(A^{\omega} \times A^{\omega})$ such that $L=\rho(L')$.

(4) $\rho^{-1}(L) \in \mathbf{N}_3(A^{\omega} \times A^{\omega})$.

Proof. (1) => (2). Let $L=B_3(M)$. Then $L=\rho(L_3(M))$ and $L_3(M)$ is of the form $(UA^{\omega},VA^{\omega})$ with $U,V \in \mathbf{R}$. If we set $X=U^RV$,then $\rho(UA^{\omega},VA^{\omega})= {}^{\omega}AXA^{\omega}$.

(2) => (1). Let $L={}^{\omega}AXA^{\omega}$ with $X \in \mathbf{R}$. Let $M=(Q,A,T,D,F)$ be a finite automaton such that $X=L_*(M)$. Then evidently we have $XA^{\omega}=L_3(M)$. From M we can easily construct three types of biautomata M_1, M_2 and M_3 such that:

$L_3(M_1)=(A^{\omega},A^*XA^{\omega})$,

$L_3(M_2)=(A^*X^RA^{\omega},A^{\omega})$,

$L_3(M_3)=\{(fx,gy) \mid f^Rg \in X,\ x \text{ and } y \in A^{\omega}\}$.

We can construct a biautomaton M' from M_1, M_2 and M_3 such that $L_3(M')=\cup_{i=1}^{3} L_3(M_i)$. It is easily seen that M' is bilateral and $L=\rho(L_3(M'))$.

(2) => (3). Let $L={}^{\omega}AXA^{\omega}$ with $X \in \mathbf{R}$. Let M_+ be an automaton such that $L_3(M_+)=XA^{\omega}$ and M_- be an automaton such that $L_3(M_-)=A^{\omega}$. If we choose $S=D_- \times D_+$, then the biautomaton $M=(M_-,M_+,S)$ has the property that $L_3(M)=(A^{\omega}, XA^{\omega})$ and $L=\rho(L_3(M))$.

(3) => (2). Along the same line as done in (1) => (2).

(1) <=> (4). Evident. □

Theorem 4. For $L \subset {}^{\omega}A^{\omega}$, the following conditions are equivalent.

(1) $L \in \mathbf{BN}_4$.

(2) L is of the form ${}^aX^a$ where $X \in \mathbf{R}$.

Proof. (1) => (2). Let $L=B_4(M)$ and $M=(M_-,M_+,S)$. Then $L=\rho(L_4(M))$. Put

$$Y=(A^*-L_*(M_-))^R(A^*-L_*(M_+))\cup(L_*(M_-))^R(A^*-L_*(M_+))\cup(A^*-L_*(M_-))^R L_*(M_+)$$

and $X=A^*-A^*YA^*$. We will prove that $L={}^aX^a$. Notice that $C(X)=X$ where $C(X)$ is the set of subwords occurring in X. Let $z \in L$, then z can be written as x^Ry such that $x \in L_4(M_-)$ and $y \in L_4(M_+)$. The acceptance condtion C_4 implies that $x[n] \in L_*(M_-)$ and $y[m] \in L_*(M_+)$ for all n and m. Therefore for all n and m, $x[n]^Ry[m] \in (L_*(M_-))^R(L_*(M_+))$. This implies that $x[n]^Ry[m] \notin A^*YA^*$ since M is bilateral. Thus we have $x[n]^Ry[m] \in X=C(X)$ for all n and m. Therefore $z=x^Ry$ is in ${}^aX^a$. Conversely, let z be in ${}^aX^a$. There exists $(x,y) \in A^{\omega}\times A^{\omega}$ such that $z=x^Ry$ and $x[n]^Ry[m] \in C(X)=X$ for all n and m. Since $x[n]^Ry[m] \notin A^*YA^*$, $x[n] \in L_*(M_-)$ and $y[m] \in L_*(M_+)$ for all n and m. Thus x is in $L_4(M_-)$ and y is in $L_4(M_+)$. Therefore z is in $B_4(M)$.

(2) => (1). Let X be in **R**, then there exists a (deterministic) finite automaton $M=(Q,A,T,\{d\},F)$ such that $C(X)=L_*(M)$. From M ,we can easily construct finite automata M_- and M_+ such that $(L_*(M_-))^RL_*(M_+)=C(X)$ with

$M_-=(Q_-,A,T_-,D_-,F_-)$ and

$M_+=(Q_+,A,T_+,D_+,F_+)$.

Set $M'=(M_-,M_+,D_-\times D_+)$. We will show that $L_4(M')=\rho^{-1}({}^aX^a)$.

Let (x,y) be in $L_4(M')$. Then x is in $L_4(M_-)$ and y is in $L_4(M_+)$. Therefore $x[n]$ is in $L_*(M_-)$ for all n and $y[m]$ is in $L_*(M_+)$ for all m. Thus we have $x[n]^Ry[m]$ is in $(L_*(M_-))^RL_*(M_+)=C(X)$ for all n and m. Therefore $\rho(x,y)$ is in ${}^aX^a$.

Conversely, if $\rho(x,y)$ is in ${}^aX^a$ and let (x',y') be a bi-word such that $(x,y)\sim(x',y')$ with $(x'[n])^Ry'[m] \in C(X)$ for all n and m. Suppose there there exists f in A^* such that $y\dot{=}fy'$ and $x'=f^Rx$. The other case is symmetric.

For all m large enough, we have $f < y[m]$. Put $y[m]=fy'[m-|f|]$. (For f in A^*, $|f|$ denotes the length of f.) Since $x[n]^Ry[m]=x[n]^Rfy'[m-|f|]=(f^Rx[n])^Ry'[m-|f|]=(x'[n+|f|])^Ry'[m-|f|] \in C(X)$ for all n and m large enough. This implies $x[n]^Ry[m]$ is in $C(X)$ for all n and m. Therefore $x[n]$ is in $L_*(M_-)$ for all n and $y[m]$ is in $L_*(M_+)$ for all m. Therefore (x,y) is in $L_4(M')$ and the inclusion $\rho^{-1}({}^aX^a) \subset L_4(M')$ has been demonstrated. □

Remark. The class $\mathbf{BN}_4$ has been studied under the name of sofic systems by Weiss et al. [2,7].

4. Characterization of BD_i's

Theorem 5 (Nivat and Perrin [4]). For $L \subset {}^{\omega}A^{\omega}$, the following conditions are equivalent.

(1) $L \in \mathbf{BD}_1$.

(2) L is of the form ${}^{e}X^{e}$ where $X \in \mathbf{R}$.

Proof. (2) => (1). Let X be in **R**, then there exists a deterministic finite automaton $M=(Q,A,T,\{d\},F)$ such that $X=L_*(M)$. Remark that if M is deterministic, then $(L_*(M))^e=L_1(M)$. For each q in Q, we can make a deterministic automaton $M_{q-}=(Q_{q-},A,T_{q-},\{d_{q-}\},F_{q-})$ such that

$L_*(M_{q-})=\{f^R \mid d \xrightarrow{f} q \text{ in } M\}$.

And we set $M_{q+}=(Q,A,T,\{q\},F)$. Then the biautomaton

$M_q=(M_{q-},M_{q+},\{(d_{q-},q)\})$

is strictly deterministic. The automaton $M'=\cup_{q\in Q} M_q$ is a deterministic biautomaton. We will show that $L_1(M')=\rho^{-1}({}^{e}X^{e})$.

Let (x,y) be in $L_1(M')$. There exists q in Q such that (x,y) is in $L_1(M_q)$. Then x is in $L_1(M_{q-})$ and y is in $L_1(M_{q+})$. Since these automata are deterministic, we have $L_1(M_{q-})=L_*(M_{q-})^e$ and $L_1(M_{q+}) = L_*(M_{q+})^e$. There also exist strictly increasing sequences (f_n) and (g_n) such that $x=\sup(f_n)$, $y=\sup(g_n)$ with

$d \xrightarrow{f_n^R} q$ and $q \xrightarrow{g_n} t_n$ for t_n in F.

Then we have $f_n^R g_n \in X$. Therefore $\rho(x,y)$ is in ${}^{e}X^{e}$.

Conversely, if $\rho(x,y)$ is in ${}^{e}X^{e}$, and let (f_n,g_n) be a strictly increasing sequence of bi-words such that $(x,y)\sim(x',y')$ with $x'=\sup(f_n)$, $y'=\sup(g_n)$, and $f_n^R g_n \in X$ for all n. Suppose there exists f in A^* such that $y=fy'$ and $x'=f^R x$. The other case is symmetric.

For all n large enough, we have $f < f_n$. Put $f_n=fh_n$. Since $h_n^R f^R g_n=f_n^R g_n \in X$ for all n, there exists a state q in Q such that there hold:

$d \xrightarrow{h_n^R} q$ and $q \xrightarrow{f^R g_n} t_n \in F$

for infinitely many n. Thus we have $x=\sup(h_n) \in L_*(M_{q-})^e=L_1(M_{q-})$ and $y=\sup(f^R g_n) \in L_*(M_{q+})^e=L_1(M_{q+})$. Therefore (x,y) is in $L_1(M_q)$ and the inclusion $\rho^{-1}({}^{e}X^{e}) \subset L_1(M')$ has been demonstrated.

(1) => (2). Let M be a bilateral deterministic automaton such that $L=B_1(M)$. Then $M=\cup_{i=1}^{n} M_i$ and each M_i is a strictly deteministic biautomaton. We will prove that $L={}^{e}X^{e}$ where

$X=\cup_{i=1}^{n} (L_*(M_{i-}))^R L_*(M_{i+})$.

In fact we have

$L=\cup_{i=1}^{n} (L_1(M_{i-}))^R L_1(M_{i+})$

$=\cup_{i=1}^{n} ((L_*(M_{i-})^e)^R (L_*(M_{i+}))^e$.

Thus $L \subset {}^eX^e$ holds. Conversely, let z be in ${}^eX^e$. There exists an increasing sequence of bi-words (f_n,g_n) such that $f_n^R g_n \in X$ and $z=\sup(f_n)^R \sup(g_n)$. By choosing an appropriate subsequence, we can assume $f_n^R g_n \in (L_*(M_{i-}))^R L_*(M_{i+})$. Then we have $\sup(f_n) \in L_1(M_{i-})$ and $\sup(g_n) \in L_1(M_{i+})$. Thus z is in $B_1(M)$. □

Theorem 6. For $L \subset {}^\omega A^\omega$, the following conditions are equivalent.

(1) $L \in \mathbf{BD}_2$.

(2) L is a finite union of sets of the form ${}^aXYZ^a$ where $X,Y,Z \in \mathbf{R}$.

(3) There exists L' in $\mathbf{D}_2(A^\omega \times A^\omega)$ such that $L=\rho(L')$.

(4) $\rho^{-1}(L) \in \mathbf{D}_2(A^\omega \times A^\omega)$.

Proof. (1) => (2). Let $L=B_2(M)$. Then $L=\rho(L_2(M))$ and M is a finite union of strictly deteministic biautomata. For each component automaton M_i, $L_2(M_i)$ is a finite union of the sets of the form (UV^a,WZ^a) with U,V,W,Z $\in \mathbf{R}$. If we set $X=V^R$ and $Y=U^R W$, then $\rho(UV^a,WZ^a)={}^aXYZ^a$.

(2) => (3). Since $\mathbf{D}_2(A^\omega \times A^\omega)$ is clearly closed under union, it sufficies to show that (2) implies (3) for a set of the form $L={}^aXYZ^a$ with $X,Y,Z \in \mathbf{R}$. Let M_+ be a deterministic automaton such that $L_2(M_+)=YZ^a$ and M_- be a deterministic automaton such that $L_2(M_-)=(X^R)^a$. If we choose $S=\{(d_-,d_+)\}$, then the biautomaton $M=(M_-,M_+,S)$ is strictly deterministic and has the property that

$L_2(M)=((X^R)^a,YZ^a)$ and $L=\rho(L_2(M))$.

(3) => (1). By modification of bilateralization for deterministic automata as shown below which preserves the condition C_2. Let $M=(M_-,M_+,S)$ be a strictly deterministic biautomaton such that $L=\rho(L_2(M))$. (Strictly speaking, we must assume that M is deterministic biautomaton which is a finite union of strict ones. But the proof is along the same line.) Without loss of generality we can also assume that the initial state of M_- has in-degree 0 when M_- is viewed as a directed graph because it can be easily transformed to satisfy without changing the accepting language if it does not. The same assumption is also made on M_+. For each q in $Q_- -\{d_-\}$, where d_- is the initial state of M_-, we define a biautomaton $M_q=(M_{q-},M_{q+},S_q)$ as follows:

$M_{q-}=(Q_-,A,T_-,\{q\},F_-)$.

To define M_{q+} to be deterministic, we make use of subset construction method to simulate finite behaviours of M_- backwardly. $Q_{q+}=Q_+ \cup \{ P \mid P$ is a subset of $Q_-.\}$,

$D_{q+}=\{q\}$,

$F_{q+}=F_+$,

4. Characterization of BD_1's

Theorem 5 (Nivat and Perrin [4]). For $L \subset {}^{\omega}A^{\omega}$, the following conditions are equivalent.

(1) $L \in \mathbf{BD}_1$.

(2) L is of the form ${}^eX^e$ where $X \in \mathbf{R}$.

Proof. (2) => (1). Let X be in **R**, then there exists a deterministic finite automaton $M=(Q,A,T,\{d\},F)$ such that $X=L_*(M)$. Remark that if M is deterministic, then $(L_*(M))^e=L_1(M)$. For each q in Q, we can make a deterministic automaton $M_{q-}=(Q_{q-},A,T_{q-},\{d_{q-}\},F_{q-})$ such that

$L_*(M_{q-})=\{f^R \mid d \xrightarrow{f} q \text{ in } M\}$.

And we set $M_{q+}=(Q,A,T,\{q\},F)$. Then the biautomaton

$M_q=(M_{q-},M_{q+},\{(d_{q-},q)\})$

is strictly deterministic. The automaton $M'=\cup_{q\in Q} M_q$ is a deterministic biautomaton. We will show that $L_1(M')=\rho^{-1}({}^eX^e)$.

Let (x,y) be in $L_1(M')$. There exists q in Q such that (x,y) is in $L_1(M_q)$. Then x is in $L_1(M_{q-})$ and y is in $L_1(M_{q+})$. Since these automata are deterministic, we have $L_1(M_{q-}) = L_*(M_{q-})^e$ and $L_1(M_{q+}) = L_*(M_{q+})^e$. There also exist strictly increasing sequences (f_n) and (g_n) such that $x=\sup(f_n)$, $y=\sup(g_n)$ with

$d \xrightarrow{f_n^R} q$ and $q \xrightarrow{g_n} t_n$ for t_n in F.

Then we have $f_n^R g_n \in X$. Therefore $\rho(x,y)$ is in ${}^eX^e$.

Conversely, if $\rho(x,y)$ is in ${}^eX^e$, and let (f_n,g_n) be a strictly increasing sequence of bi-words such that $(x,y)\sim(x',y')$ with $x'=\sup(f_n)$, $y'=\sup(g_n)$, and $f_n^R g_n \in X$ for all n. Suppose there exists f in A^* such that $y=fy'$ and $x'=f^R x$. The other case is symmetric.

For all n large enough, we have $f < f_n$. Put $f_n=fh_n$. Since $h_n^R f^R g_n=f_n^R g_n \in X$ for all n, there exists a state q in Q such that there hold:

$d \xrightarrow{h_n^R} q$ and $q \xrightarrow{f^R g_n} t_n \in F$

for infinitely many n. Thus we have $x=\sup(h_n) \in L_*(M_{q-})^e=L_1(M_{q-})$ and $y=\sup(f^R g_n) \in L_*(M_{q+})^e=L_1(M_{q+})$. Therefore (x,y) is in $L_1(M_q)$ and the inclusion $\rho^{-1}({}^eX^e) \subset L_1(M')$ has been demonstrated.

(1) => (2). Let M be a bilateral deterministic automaton such that $L=B_1(M)$. Then $M=\cup_{i=1}^n M_i$ and each M_i is a strictly deteministic biautomaton. We will prove that $L={}^eX^e$ where

$X=\cup_{i=1}^n (L_*(M_{i-}))^R L_*(M_{i+})$.

In fact we have

$L=\cup_{i=1}^n (L_1(M_{i-}))^R L_1(M_{i+})$

$=\cup_{i=1}^n ((L_*(M_{i-})^e)^R (L_*(M_{i+}))^e$.

$L=\rho(L_3(M))$.
(3) => (2). Along the same line as done in (1) => (2).
(1) <=> (4). Evident. □

Corollary 2. $\mathbf{BN}_3=\mathbf{BD}_3$.
Proof. From Theorem 3 and Theorem 7. □

Theorem 8. For $L \subset {}^{\omega}A^{\omega}$, the following conditions are equivalent.

(1) $L \in \mathbf{BD}_4$.

(2) L is of the form ${}^{a}X^{a}$ where $X \in \mathbf{R}$.

Proof. (1) => (2). Along the same line as in the proof of Theorem 4. Let $L=B_4(M)$. Then $L=\rho(L_4(M))$ and M is a finite union of strictly deterministic automata. For each component automaton M_i, let $Y_i=$

$$(A^*-L_*(M_{i-}))^R(A^*-L_*(M_{i+}))\cup(L_*(M_{i-}))^R(A^*-L_*(M_{i+}))\cup(A^*-L_*(M_{i-}))^R L_*(M_{i+}).$$

Let $Y=\cup_{i=1}^{n}Y_i$ and $X=A^*-A^*YA^*$.We can easily shown that $L={}^{a}X^{a}$.
(2) => (1). Along the same line as in the proof of Theorem 4. Let X be in R, then there exists a deterministic finite automaton $M=(Q,A,T,\{d\},F)$ such that $C(X)=L_*(M)$. For each q in Q, we can make a deterministic automaton $M_{q-}=(Q_{q-},A,T_{q-},\{d_{q-}\},F_{q-})$ such that

$$L_*(M_{q-})=\{f^R \mid d \xrightarrow{f} q \text{ in } M\}.$$

And we set $M_{q+}=(Q,A,T,\{q\},F)$. Then the biautomaton

$$M_q=(M_{q-},M_{q+},\{(d_{q-},q)\})$$

is strictly deterministic. The automaton $M'=\cup_{q\in Q} M_q$ is a deterministic biautomaton. We can easily show that $L_4(M')=\rho^{-1}({}^{a}X^{a})$ by using the fact that $C(X)=\cup_{q\in Q} (L_*(M_{q-}))^R L_*(M_{q+})$. □

Corollary 3. $\mathbf{BN}_4=\mathbf{BD}_4$.
Proof. From Theorem 4 and Theorem 8. □

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