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STATE SPACE APPROACH TO SPECTRUM ESTIMATION

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1. INTRODUCTION

The estimation of the power spectrum is one of the central issues of the time series analysis. Recently, a new parametric method of spectrum estimation has received much attention, which is based on the optimum prediction of the observed time series. This is called an AR (auto-regressive) estimation or a ME (maximum entropy) estimation [1][2]. The AR estimation is of particular interest because it allows a special type of circuit realization called lattice filter, which is suitable for LSI implementation [3][4].

In linear system theory, parametric spectrum estimation is regarded as a partial stochastic realization, which is a generalization of stochastic realization of stationary time series. Though the theory of stochastic realization has been well-established [5]∿[7], relatively little is known about the partial stochastic realization [8][9]. An essential feature of stochastic realization is that it specifies a circuit structure of the spectrum estimator by the state space representation. Therefore, partial stochastic realization is particularly useful when the circuit implementation of a spectrum estimation is taken into account. Along this line, state space analysis of lattice filter has been discussed by Morf [10] and Kailath and Porat [11].

In this paper, we shall investigate the state space realization of spectrum estimator with the special emphasis on the circuit structure determined by its state space representation. Some state space properties of AR estimator are discussed and an important characteristic feature of lattice realization is established based on the properties of scaled Schwarz matrix. These results generalize those obtained in [10] and [11].

Throughout the paper, we use the following notations for the bi-directional power series $p(z) = \cdots + p_{-1}z^{-1} + p_0 + p_1 z + p_2 z^2 + \cdots$:

$$[p(z)]_0^n = p_0 + p_1 z + \cdots + p_n z^n.$$

$$[p(z)]_+ = [p(z)]_0^\infty \qquad \text{(polynomial part)}.$$

$$[p(z)]_0 = p_0.$$

$$[p(z)]_- = [p(z)]_{-\infty}^0.$$

2. PARTIAL STOCHASTIC REALIZATION

Consider a discrete-time stationary Gauss-Markov process $\{y(t)\}$ which is represented by a state space form

$$x(t + 1) = Fx(t) + gu(t) \tag{1a}$$

$$y(t) = hx(t) + \nu u(t), \tag{1b}$$

where $u(t)$ is a scaler zero-mean white Gaussian process with $E[u^2(t)] = 0$. If the first $n + 1$ covariances of $\{y(t)\}$ coincide with a prescribed sequence $C^{(n)} = \{c_0, c_1, \cdots, c_n\}$, i.e.,

$$E[y(t)y(t + i)] = c_i, \quad i = 0, 1, \cdots, n, \tag{2}$$

we call the state space form (1) a *partial stochastic realization* of $C^{(n)}$. Sometimes (1) is simply denoted by (h, F, g, ν). Let

$$r(z) = h(zI - F)^{-1}g + \nu \tag{3}$$

be the transfer function of (1). The power spectrum of $\{y(t)\}$ is given by

$$S_y(z) = r(z)r(z^{-1}) \tag{4}$$

If $C^{(n)}$ is an estimated covariance sequence of a stationary process $\{\eta(t)\}$, the consistency condition (2) allows us to regard $\{y(t)\}$ as a good model of $\{\eta(t)\}$ provided that n is large enough. In that case, (4) becomes a spectrum estimator of $\{\eta(t)\}$, which satisfies

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |r(e^{j\omega})|^2 e^{j\omega k} d\omega = c_k, \quad k = 0, \pm 1, \pm n \tag{5}$$

Construction of a partial stochastic realization is formulated through a function $\ell(z) = [r(z)r(z^{-1})]_+ - 1/2[r(z)r(z^{-1})]_0$. Obviously,

$$r(z)r(z^{-1}) = \ell(z) + \ell(z^{-1}). \tag{6}$$

Due to (5), $\ell(z)$ can be represented as

$$\ell(z^{-1}) = \frac{1}{2} C_0 + C_1 z^{-1} + \cdots + C_n z^{-n} + 0(z^{-(n+1)}). \tag{7}$$

If $\ell(z^{-1})$ is a rational function, $\ell(z^{-1})$ is usually referred to as a partial realization of the Markov sequence $\{c_0/2, c_1, \cdots, c_n\}$. In order to distinguish from the partial stochastic realization, we call $\ell(z)$ in (7) a *rational interpolation* of the covariance sequence $C^{(n)}$. If a rational interpolation is associated with a partial realization, it must satisfy

$$\ell(e^{j\omega}) + \ell(e^{-j\omega}) \geq 0, \qquad \omega \in [-\pi, \pi], \tag{8}$$

according to (6). This imposes a strong restriction on rational interpolation associated with a partial stochastic realization. A function $\ell(z)$ satisfying (8) is called *positive real*. It is well-known that any positive real $\ell(z)$ allows a factorization (6) for some $r(z)$. Therefore, the construction of a partial stochastic realization is equivalent to the construction of a positive real rational interpolation of $C^{(n)}$.

The state space characterization of positive realness has already been established [12] - [14]. We briefly sketch its outline for later use. The key role is played by the state convariance matrix $P = E[x(t)x(t)^T]$, which is the solution of the covariance equation

$$P - FPF^T = gg^T. \tag{9}$$

The matrix P depends on the particular realization and characterizes the state-space properties of the partial stochastic realization. Let

$$b = FPh^T + g\nu, \quad 2d = hPh^T + \nu^2. \tag{10}$$

A straightforward manipulation verifies that

$$\ell(z) = h(zI - F)^{-1}b + d \tag{11}$$

satisfies (6) and is a rational interpolation. For a specified $\ell(z)$, $r(z)$ satisfying (6) is called a *spectrum factor* of $\ell(z)$. If we define

$$M(P) = \begin{bmatrix} P - FPF^T & b - FPh^T \\ (b - FPh^T)^T & 2d - hPh^T \end{bmatrix}, \tag{12}$$

we can write (9) and (10) as

$$M(P) = \begin{bmatrix} g \\ \nu \end{bmatrix} [g^T \quad \nu]. \tag{13}$$

The state covariance matrix P satisfying (13) is a special solution of a more general linear matrix inequality (LMI)

$$M(P) \geq 0. \tag{14}$$

We call a solution P of (14) satisfying (13) for some g and ν a *covariance solution*. A fundamental result on LMI is summarized as follows:

Lemma 1 [13][14]. A rational function $\ell(z)$ in (11) is positive real if and only if (14) is satisfied for some $P \geq 0$. In that case, there exist a finite number of covariance solutions of (13), each of which corresponds to a spectrum factor $r(z)$ satisfying (6).

It should be noted that if a state form of a rational interpolation is given as in (11), the state-space realization of a particular spectrum factor is completely determined. The most interesting is the minimum phase spectrum factor.

Lemma 2 [13][14]. If (14) is solvable, then there exists the minimum solution P^* which is obtained as the limit P_∞ of the following recursion

$$P_i = FP_{i-1}F^T + (b - FP_{i-1}h^T)(b - FP_{i-1}h^T)^T/(2d - hP_{i-1}h^T)$$

$$P_0 = 0. \tag{15}$$

The spectrum factor corresponding to P^* is of minimum phase.

3. RATIONAL INTERPOLATION ASSOCIATED WITH AR-TYPE SPECTRUM FACTORS

The purpose of this section is to derive a rational interpolation which leads to the spectrum factor of AR (auto-regressive) type which is the most popular parametric estimator of power spectrum. In what follows, we assume $c_0 = 1$ without loss of generality.

The sequence $C^{(n)} = \{1, c_1, \cdots, c_n\}$ has a stationary partial stochastic realization, if and only if the associated Toeplitz matrix is positive definite, i.e.,

$$T_n = \begin{bmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_0 & \cdots & c_{n-1} \\ & \cdots\cdots & & \\ c_n & c_{n-1} & \cdots & c_0 \end{bmatrix} > 0. \tag{16}$$

The necessity of (16) is obvious from $E[(\sum_{i=0}^{n} a_i y(t+i))^2] = \sum_{i,j=0}^{n} c_{|i-j|} a_i a_j > 0$ for any $a = (a_0\ a_1\ \cdots\ a_n)^T \neq 0$. The sufficiency will be demonstrated by actually constructing a realization, which is the main subject of this section.

First, we shall give a brief account of the properties of the Szegö orthogonal polynomials [15]. Let

$$\phi_k(z) = \frac{1}{\det T_{k-1}} \det \begin{bmatrix} c_0 & c_1 & \cdots & c_k \\ c_1 & c_0 & \cdots & c_{k-1} \\ & \cdots\cdots & & \\ c_{k-1} & c_{k-2} & \cdots & c_1 \\ 1 & z & \cdots & z^k \end{bmatrix}$$

$$\phi_0(z) = 1.$$

These polynomials are obtained by Gram-Schmidt orthogonalization procedure of the basis $\{1, z, z^2, \cdots\}$ with respect to the inner product

$$\langle f(z), g(z) \rangle = [f(z)g(z^{-1})(\Gamma(z) + \Gamma(z^{-1}))]_0 \tag{17}$$

$$\Gamma(z^{-1}) = \frac{1}{2} + c_1 z^{-1} + \cdots + c_n z^{-n}. \tag{18}$$

Thus, $\phi_n(z)$ satisfies

$$<\phi_n(z),\ z^i> = 0, \quad i = 0, 1, \cdots, n-1 \tag{19}$$

$$<\phi_n(z),\ z^n> = \sigma_n^{\,2}, \tag{20}$$

for some number σ_n. The polynomials $\{\phi_k(z)\}$ can alternatively be defined by the recursion

$$\phi_{j+1}(z) = z\phi_j(z) - k_{j+1}\phi_j^{\,*}(z) \tag{21}$$

$$\phi_{j+1}^{\,*}(z) = -k_{j+1}z\phi_j(z) + \phi_j^{\,*}(z) \tag{22}$$

$$\phi_0(z) = \phi_0^{\,*}(z) = 1,$$

where $k_j = -\phi_j(0)$ and $\phi_j^{\,*}(z) = z^j\phi_j(z^{-1})$ is the reciprocal polynomial of $\phi_j(z)$. Note that $<\phi_j^{\,*}(z),\ z^j> = <z^j\phi_j(z^{-1}),\ z^j> = <1,\ \phi_j(z)> = <\phi_j(z),\ 1> = 0$. From (21)(22), $\sigma_j^{\,2} = <\phi_j(z),\ z^j> = -k_j<\phi_j^{\,*}(z),\ z^j> + (1 - k_j^{\,2})<z\phi_{j-1}(z),\ z^j> = (1 - k_j^{\,2})<\phi_{j-1}(z),\ z^{j-1}> = (1 - k_j^{\,2})\sigma_{j-1}^{\,2}$. Therefore, noting that $\sigma_0^{\,2} = 1$, we have

$$\sigma_j^{\,2} = \prod_{k=1}^{i} (1 - k_i^{\,2}). \tag{23}$$

The parameters k_j in (21)(22) play an essential role in stochastic realization, and are referred to as the reflection coefficients associated with $C^{(n)}$. It is known that (16) holds if and only if

$$|k_j| < 1. \quad j = 1, \cdots, n. \tag{24}$$

Also, it is well-known that $\{\phi_j(z)\}$ are stable polynomials (their zeros are all inside the unit disc) if and only if (24) holds (see [16] for details).

Define the polynomials $\{\psi_i(z)\}$ by

$$\psi_i(z) = 2[\phi_i(z)\Gamma(z^{-1})]_+. \tag{25}$$

From the definition, we can write $\psi_i(z) = 2\phi_i(z)\Gamma(z^{-1}) + f_1 z^{-1} + f_2 z^{-2} + \cdots$ for some numbers $f_1, f_2, \cdots$. From this, we conclude that

$$\ell(z) = \psi_n(z)/2\phi_n(z) \tag{26}$$

is a rational interpolation of the sequence $C^{(n)} = \{1, c_1, \cdots, c_n\}$. We shall examine the relation between $\{\psi_i(z)\}$ and $\{\phi_i(z)\}$. Let $\tilde{C}^{(n)} = \{1, \tilde{c}_1, \cdots, \tilde{c}_n\}$ be the sequence of numbers which is defined by

$$\frac{1}{4\Gamma(z^{-1})} = \frac{1}{2} + \tilde{c}_1 z^{-1} + \tilde{c}_2 z^{-2} + \cdots + \tilde{c}_n z^{-n} + 0(z^{-(n+1)}) \tag{27}$$

Note that $\tilde{c}_i$ is a linear function of $(c_1, \cdots, c_i)$.

Lemma 3 The polynomials $\{\psi_i(z)\}$ are the Szegö orthogonal polynomials associated with $\tilde{C}^{(n)}$. The reflection coefficients associated with $\tilde{C}^{(n)}$ are given by $\psi_j(0) = -k_j$, where k_j are the reflection coefficients associated with $C^{(n)}$.

(Proof is found in Appendix.)

Due to Lemma 3, $\{\psi_i(z)\}$ satisfy the recursion

$$\psi_{j+1}(z) = z\psi_j(z) + k_{j+1}\psi_j^*(z) \tag{27}$$

$$\psi_{j+1}^*(z) = k_{j+1} z\psi_j(z) + \psi_j^*(z) \tag{28}$$

$$\psi_0(z) = \psi_0^*(z) = 1.$$

Combining (27)(28) with (21)(22), we have $\psi_{j+1}^*(z)\phi_{j+1}(z) + \phi_{j+1}^*(z)\psi_{j+1}(z) = z(1 - k_{j+1}^2)(\psi_j^*(z)\phi_j(z) + \phi_j^*(z)\psi_j(z))$. It follows that $\psi_n^*(z)\phi_n(z) + \phi_n^*(z)\psi_n(z) = 2\sigma_n^2 z^n$. Therefore, we conclude that

$$\frac{1}{2}\left(\frac{\psi_n(z)}{\phi_n(z)} + \frac{\psi_n(z^{-1})}{\phi_n(z^{-1})}\right) = \frac{\sigma_n^2}{\phi_n(z)\phi_n(z^{-1})} \tag{29}$$

The right-hand side is the well-known spectrum estimator of AR type. This relation shows that the rational interpolation (25) leads to the spectrum factors

$$r_k(z) = \frac{\sigma_n z^{n-k}}{\phi_n(z)}, \quad k = 0, 1, \cdots, n. \tag{30}$$

In the next section, we discuss a class of state-space realizations of (30).

4. SCALED SCHWARZ MATRIX

In this section, we discuss a class of system matrices which leads to a specific class of state space realizations of AR spectrum factors (30). This is a generalization of the *state space generator* introduced by Kailath and Porat [11].

Let F_c be a companion matrix

$$F_c = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\phi_n \\ 1 & 0 & & 0 & -\phi_{n-1} \\ & & \cdots\cdots & & \\ 0 & 0 & \cdots & 1 & -\phi_1 \end{bmatrix}$$

corresponding to the orthogonal polynomial $\phi_n(z) = z^n + \phi_1 z^{n-1} + \cdots + \phi_n$. It is not difficult to see that F_c satisfies

$$T_{n-1} - F_c^T T_{n-1} F_c = \sigma_n^2 e_n e_n^T, \tag{31}$$

where T_{n-1} is defined in (16) and $e_n = [0 \cdots 0\ 1]^T$ (see [17] for the proof). Let Φ be the matrix defined by

$$[\phi_0(z)\ \phi_1(z)\ \cdots\ \phi_{n-1}(z)] = [1\ z\ \cdots\ z^{n-1}]\Phi. \tag{32}$$

It is obvious that Φ is upper triangular with all the diagonal elements equal to 1. The matrix Φ diagonalizes the Toeplitz matrix T_{n-1}, as is shown by the relation

$$\Phi^T T_{n-1} \Phi = \Sigma \tag{33}$$

$$\Sigma = \mathrm{diag}[\sigma_0^2\ \sigma_1^2\ \cdots\ \sigma_{n-1}^2] \tag{34}$$

The relation (33) is well-known (e.g. [18]). An alternative proof based on the orthogonality (19)(20) of $\Phi_i(z)$ is found in Appendix.

Define

$$F^* = \Sigma^{1/2}\Phi^{-1}F_c\Phi\Sigma^{-1/2}. \tag{35}$$

The premultiplication by $\Sigma^{-1/2}\Phi^T$ and the postmultiplication by $\Phi\Sigma^{1/2}$ of (31) yield

$$I - (F^*)^T F^* = (k_n^c)^2 e_n e_n^T, \tag{36}$$

where we have used the relation (33) and the notation $k_n^c = \sqrt{1 - k_n^2}$. The matrix F^* is the state space generator defined by Kailath and Porat [11] in a different way. They derived its explicit form as

$$
F^* = \begin{bmatrix}
k_1 & k_1^c k_2 & k_1^c k_2^c k_3 & \cdots & k_1^c k_2^c & \cdots & k_{n-1}^c k_n \\
k_1^c & -k_1 k_2 & -k_1 k_2^c k_3 & \cdots & -k_1 k_2^c & \cdots & k_{n-1}^c k_n \\
0 & k_2^c & -k_2 k_3 & \cdots & -k_2 k_3^c & \cdots & k_{n-1}^c k_n \\
\cdots & & \cdots & & \cdots\cdots\cdots & & \\
0 & 0 & 0 & & & -k_{n-1} k_n &
\end{bmatrix}.
$$

The elements of F^* have the definite meaning. Let $\hat{f}_{ji}$ ($j=1, \cdots, i$) be the coefficients of representing $\phi^*_{i-1}(z)$ in terms of $\phi_0(z)$, $\phi_1(z)$, $\cdots$, $\phi_{n-1}(z)$ i.e.,

$$\phi^*_{i-1}(z) = \hat{f}_{1i}\phi_0(z) + \hat{f}_{2i}\phi_1(z) + \cdots + \hat{f}_{ni}\phi_{n-1}(z).$$

Obviously, $\hat{f}_{ji} = 0$ for $j \geq i + 1$.

Lemma 4 The elements f_{ji} of F^* are given by

$$f_{ji} = \frac{\sigma_{i-1}}{\sigma_{i-1}} k_i \hat{f}_{ji}, \qquad j = 1, \cdots, i, \; i = 1, \cdots, n$$

$$f_{i+1\, i} = k_i^c. \qquad i = 1, \cdots, n - 1.$$

(Proof will be found in Appendix.)

We slightly generalize F^* in order to allow the state scaling of the realization. Let

$$L = \mathrm{diag}[\rho_1 \; \rho_2 \; \cdots \; \rho_n], \quad \rho_i = 0. \tag{38}$$

The scaled version $F = L^{-1} F^* L$ is explicitly written as

$$F = \begin{bmatrix} k_1 & \dfrac{k_1^c k_2 \rho_2}{\rho_1} & \dfrac{k_1^c k_2^c k_3 \rho_3}{\rho_1} & \cdots & \dfrac{k_1^c k_2^c \cdots k_{n-1}^c k_n \rho_n}{\rho_1} \\ \dfrac{k_1^c \rho_1}{\rho_2} & -k_1 k_2 & -\dfrac{k_1 k_2^c k_3 \rho_3}{\rho_2} & \cdots & -\dfrac{k_1 k_2^c \cdots k_{n-1}^c k_n \rho_n}{\rho_2} \\ 0 & \dfrac{k_2^c \rho_2}{\rho_3} & -k_2 k_3 & \cdots & -\dfrac{k_2 k_3^c \cdots k_{n-1}^c k_n \rho_n}{\rho_3} \\ & \cdots\cdots\cdots\cdots & & & \\ 0 & 0 & 0 & \cdots & -k_{n-1} k_n \end{bmatrix} \tag{39}$$

If we choose $\rho_i = \prod_{j=i}^{n} k_i^c$, then (39) becomes

$$F = \begin{bmatrix} k_1 & k_2 & k_3 & \cdots & k_n \\ (k_1^c)^2 & -k_1 k_2 & -k_1 k_3 & \cdots & -k_1 k_n \\ 0 & (k_2^c)^2 & -k_2 k_3 & \cdots & -k_2 k_n \\ & \cdots\cdots & & & \\ 0 & 0 & 0 & \cdots & -k_{n-1} k_n \end{bmatrix} \tag{40}$$

This is the discrete-time Schwarz matrix which was discussed extensively by Mansour [19] and Anderson, Jury and Mansour [20]. This form was also derived by Morf [10] connected with the unnormalized lattice filter. We call the form (39) a *scaled Schwarz matrix*. The most important characteristic feature of the scaled Schwarz form is its nesting property. If we denote (39) by F_n for emphasizing its size n, F_n contains F_{n-1} as its principal submatrix. More explicitly, F_n can be written as

$$F_n = \begin{bmatrix} F_{n-1} & \dfrac{k_{n-1}^c k_n \rho_n}{k_{n-1}\rho_{n-1}} F_{n-1} e_{n-1} \\ \dfrac{k_{n-1}^c \rho_{n-1}}{\rho_n} e_{n-1}^T & -k_{n-1}k_n \end{bmatrix} \tag{41}$$

This nesting property was considered to be a defining property of the state space generator (37) in [10]. We shall discuss its implication in Section 6.

The scaled Schwarz form (39) has a number of interesting algebraic properties, some of which are stated below.

Lemma 5 The following relations hold (for F in (39)):

(i) $\phi_n(F) = 0$

(ii) $\det F = (-1)^{n-1} k_n$

(iii) $L^2 - F^T L^2 F = (k_n^c)^2 \rho_n^2 e_n e_n^T$ (42)

(iv) $e_n^T F^i e_1 = 0, \quad i = 0, 1, \cdots, n-2.$

$e_n^T F^{n-1} e_1 = \rho_1 \sigma_{n-1}/\rho$

(v) $e_1^T F^i e_1 = c_i, \quad i = 0, 1, \cdots, n.$

(vi) If $k_n \neq 0$, then

$$L^{-2} - FL^{-2}F^T = (k_n^c/k_n\rho_n)^2 F e_n e_n^T F^T \tag{43}$$

(vii) $e_1^T \phi_i(F)^T \phi_j(F) e_1 = \begin{cases} 0 & i = j \\ (\sigma_i \rho_1/\rho_{i+1})^2 & i = j. \end{cases}$

(The proof will be found in Appendix.)

5. LATTICE REALIZATION

According to Lemma 5 (V),

$$\ell(z) = \frac{1}{2} + e_1^T F(zI - F)^{-1} e_1 \tag{44}$$

is a rational interpolation of $C^{(n)}$. A direct manipulation yields

$$\ell(z) = \det(zI - F + 2e_1e_1^TF)/2\cdot\det(zI - F).$$

From the construction, $\det(zI - F) = \phi_n(z)$. It is easily seen from (39) that the matrix $F - 2e_1e_1^TF$ is obtained from F by replacing k_i with $-k_i$ for each i. Therefore, from Lemma 3, $F - 2e_1e_1^TF$ is the scaled Schwarz matrix associated with the sequence $\tilde{C}^{(n)}$. Hence, $\det(zI - F + 2e_1e_1^TF) = \psi_n(z)$. Thus the rational interpolation (44) is identical to (26). In this section, we assume $k_n \neq 0$ for the minimality of (44).

Now we shall derive the spectrum factor of $\ell(z)$ using LMI. In this case, M(P) in (12) is written as

$$M(P) = \begin{bmatrix} P - FPF^T & (I - FPF^T)e_1 \\ e_1^T(I - FPF^T) & 1 - e_1^TFPF^Te_1 \end{bmatrix} \geq 0 \tag{45}$$

We write the scaling factor ρ_i in (38) as

$$\rho_i = \lambda_i\lambda_{i+1} \cdots \lambda_n \tag{46}$$

and define

$$\Lambda = \mathrm{diag}[1 \;\; \lambda_1^2 \;\; \lambda_1^2\lambda_2^2 \;\cdots\; \lambda_1^2\lambda_2^2 \cdots \lambda_{n-1}^2]. \tag{47}$$

Since $\Lambda = \lambda_1^2 \cdots \lambda_n^2L^{-2}$, we have, from (43),

$$\Lambda - F\Lambda F^T = (\lambda k_n^c/k_n)^2Fe_ne_n^TF^T, \tag{48}$$

where $\lambda = \lambda_1\lambda_2 \cdots \lambda_{n-1}$.

Lemma 6 Under the assumption $k_n \neq 0$,

$$(I - F^{-i}\Lambda(F^{-i})^T)e_1 = 0, \quad i = 0, 1, \cdots, n-1.$$

(The proof will be found in Appendix.)

Now we derive the state space realization of the spectrum factors corresponding to the rational interpolation (44).

Theorem 1 The LMI (45) has n+1 covariance solutions which are given by

$$P_i = F^{-i}\Lambda(F^{-i})^T, \quad i = 0, 1, \cdots, n. \tag{49}$$

Each solution gives the state space realization

$$\mathcal{L}^0 = (e_1{}^T F,\ F,\ \frac{\lambda k_n{}^c}{k_n} Fe_n,\ \frac{\lambda k_n{}^c}{k_n} e_1{}^T Fe_n) \tag{50}$$

$$\mathcal{L}^i = (e_1{}^T F^{-(i-1)},\ F,\ \frac{\lambda k_n{}^c}{k_n} F^{-(i-1)} e_n,\ 0), \quad i = 1, \cdots, n, \tag{51}$$

which corresponds respectively to the spectrum factor $r_i(z)$.

(Proof) Note that $(I - F\Lambda F^T)e_1 = (I - \Lambda + \Lambda - F\Lambda F^T)e_1 = (\Lambda - F\Lambda F^T)e_1$. Therefore, due to (48), we have

$$M(\Lambda) = (\frac{\lambda k_n{}^c}{k_n})^2 \begin{bmatrix} Fe_n \\ e_1{}^T Fe_n \end{bmatrix} [e_n{}^T F^T \quad e_n{}^T F^T e_1].$$

This implies that $\mathcal{L}^0$ given by (50) is a spectrum factor according to Lemma 5.

Due to (48), $P_i - FP_iF^T = F^{-i}(\Lambda - F\Lambda F^T)(F^{-i})^T = (\lambda k_n{}^c/k_n)^2 F^{-(i-1)} e_n e_n{}^T (F^{-(i-1)})^T$. Also, due to Lemma 6, $(I - FP_iF^T)e_1 = (I - \Lambda + \Lambda - FP_iF^T)e_1 = (\Lambda - F^{-(i-1)}\Lambda(F^{-(i-1)})^T)e_1 = 0$, for $i = 1, \cdots, n$. Therefore,

$$M(P_i) = \left(\frac{\lambda k_n^c}{k_n}\right)^2 \begin{bmatrix} F^{-(i-1)}e_n \\ 0 \end{bmatrix} [e_n^T (F^{-(i-1)})^T \quad 0].$$

This implies that $\mathcal{L}^i$ in (51) gives a spectrum factor.

It remains to prove that each spectrum factor corresponds to $r_i(z)$, respectively. Since $P_i - P_{i-1} = F^{-i}(\Lambda - F\Lambda F^T)(F^{-i})^T = (k_n^c/k_n)^2 F^{-(i-1)} e_n e_n^T (F^{-(i-1)})^T \geq 0$, we have $P_0 \leq P_1 \leq \cdots \leq P_n$. Therefore, P_0 is the minimal solution. According to Lemma 1, it is obvious that $\mathcal{L}^0$ corresponds to the minimum phase factor $r_0(z)$. Assume that $\mathcal{L}^i$ is a realization of $r_i(z)$, i.e., $(\lambda k_n^c/k_n) e_1^T F(zI - F)^{-1} F^{-(i-1)} e_n = r_i(z)$. Then, due to Lemma 5 (vi) and (vi), $(\lambda k_n^c/k_n) e_1^T F(zI - F)^{-1} F^{-(i-1)} e_n = (\lambda k_n^c/k_n) e_1^T F(zI - F)^{-1}(F - zI + zI)F^{-i} e_n$ $= -(\lambda k_n^c/k_n) e_1^T F^{-(i-1)} e_n + z(\lambda k_n^c/k_n) e_1^T F(zI - F)^{-1} F^{-i} e_n = z(\lambda k_n^c/k_n) e_1^T F (zI - F)^{-1} F^{-i} e_n$. This implies that $\mathcal{L}_{i+1}$ is a realization of $z^{-1} r_i(z) = r_{i+1}(z)$. Therefore, the assertion has been established. □

The block diagram of the realization $\mathcal{L}_0$ with the scaling factors (46) is shown in Fig. 1. This is the celebrated lattice filter with the scaling factors λ_i. The parameter λ is used for the input scaling. If $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = 1$, it becomes the normalized lattice filter [11]. If we choose $\lambda_i = k_i^c$, $i = 1, \cdots, n - 1$, then it becomes the unnormalized lattice filter whose block diagram is shown in Fig. 2.

The rational interpolation (44) can be written in the dual form

$$\ell(z) = \frac{1}{2} + e_1^T (zI - F^T)^{-1} F^T e_1. \tag{52}$$

This leads to the dual LMI

$$\hat{M}(P) = \begin{bmatrix} P - F^T P F & F^T (I - P) e_1 \\ e_1^T (I - P) F & 1 - e_1^T P e_1 \end{bmatrix} \geq 0. \tag{53}$$

Under the same parametrization (46) of scaling factors, we can derive the dual lattice filter in the analogous way to Theorem 1. We only state the result without the proof.

Theorem 2. The LMI(53) has n+1 covariance solutions which are given by $\hat{P}_i = P_i^{-1}$, i = 0, 1, $\cdots$, n. Each solution gives the realization

$$\hat{\mathcal{L}}^i = (k_n^c e_1^T,\ F^T,\ (F^T)^i e_n,\ 0), \qquad i = 0, 1, \cdots, n$$

$$\hat{\mathcal{L}}^n = (k_n^c e_1^T,\ F^T,\ (F^T)^n e_n,\ k_n^c e_1^T (F^T)^{n-1} e_1),$$

which corresponds respectly to the spectrum factor $\hat{w}_i(z) = \sigma_n z^i/\phi_n(z)$.

The realizations $\mathcal{L}^i$ and $\hat{\mathcal{L}}^{n-i}$ represent the same spectrum factor $\sigma_n z^{n-i}/\phi_n(z)$ with different state covariance matrices. It is interesting to note that a version of $\mathcal{L}^i$ given by

$$\mathcal{L}^{i'} = ((k_n^c/k_n) e_1^T F^{-(i-1)},\ F,\ Fe_n,\ 0) \tag{54}$$

is a realization of the same spectrum factor $\sigma_n z^{n-i}/\phi_n(z)$ but has the state covariance Λ. The realization (54) is obtained from the rational interpolation

$$\ell(z) = \frac{1}{2} + e_1^T F^{-(i-1)} (zI - F)^{-1} F^i e_1.$$

The dual realization $\hat{\mathcal{L}}^n$ has an analogous block diagram representation as in Figs. 1 and 2, which is relatively complicated compared with $\mathcal{L}^o$. The realization $\hat{\mathcal{L}}^o$ representing the maximum phase factor has the simplest block diagram which is shown in Fig. 3.

6. NESTING PROPERTIES OF LATTICE REALIZATION

In the previous section, we derived a particular realization (50) of spectrum factor $\sigma_n z^n/\phi_n(z)$ using the scaled Schwarz form. In this section, we consider the state-space implication of this realization. Denote by F_n^* the scaled Schwarz form (37) with the suffix n representing its size. This corresponds to the scaling factors $\rho_1 = \rho_2 = \cdots = \rho_n = 1$ in (39) and hence, due to (42) and (43),

$$I - (F_n^*)^T F_n^* = (k_n^c)^2 e_n e_n^T \tag{55}$$

$$I - F_n^*(F_n^*)^T = (k_n^c/k_n)^2 F_n^* e_n e_n^T (F_n^*)^T \tag{56}$$

From the form of F_n^*, we can write

$$F_n^* = V_n X_n \tag{57}$$

$$X_n = \begin{bmatrix} I & 0 \\ 0 & k_n \end{bmatrix}. \tag{58}$$

It follows, from (55) and the definition of k_n^c, that $V_n V_n^T = I$.

Consider the sequence of matrices $\{F_i^*\}$, $i = 1, \cdots, n$ and the sequence of unitary matrices $\{U_i\}$. From (41), $\{F_i^*\}$ is nested, i.e.,

$$F_{i+1}^* = \begin{bmatrix} F_i^* & \dfrac{k_i^c k_{i+1}}{k_i} F_i^* e_i \\ k_i^c e_i^T & -k_i k_{i+1} \end{bmatrix}, \quad i \geq 2, \cdots, n \tag{59}$$

where e_i denotes the i-vector $(0 \cdots 0\ 1)^T$. We shall show the condition on the sequence $\{U_i\}$ for which the unitary transformation

$$F_i = U_i F_i^* U_i^T \tag{60}$$

preserves the nesting property, i.e. F_i is represented as

$$F_i = \begin{bmatrix} F_{i-1} & * \\ * & * \end{bmatrix}. \tag{61}$$

Lemma 7 The sequence $\{U_i\}$ of unitary matrices preserves the nesting property of F_i^* in the transformation (60), if and only if, U_{i+1} is selected in either of the following two ways

$$\text{(i)} \quad U_{i+1} = \begin{bmatrix} U_i & 0 \\ 0 & 1 \end{bmatrix} \tag{62}$$

$$\text{(ii)} \quad U_{i+1} = \begin{bmatrix} U_i & 0 \\ 0 & 1 \end{bmatrix} V_{i+1}^T, \tag{63}$$

for each $i = 1, \cdots, n - 1$.

(The proof will be found in Appendix.)

The above lemma shows that we have 2^{n-1} selections of $\{U_i\}$, $i = 2 \cdots n$, depending on the two selections (i) and (ii) at each step i. This implies that there are 2^{n-1} sequences of $\{F_i\}$ of the unitary transformations (60) which are nested. If we always choose the option (i) at each i, we have the original sequence $\{F_i^*\}$. Now we shall show that choosing the option (ii) at each i leads to the sequence $\{(F_i^*)^T\}$. Indeed, direct calculation verifies $V_2 F_2^* V_2^T = (F_2^*)^T$. Assume that

$$U_i F_i^* U_i^T = (F_i^*)^T. \tag{64}$$

Due to (57), $V_{i+1}^T F_{i+1}^* V_{i+1} = X_{i+1} V_{i+1} = X_{i+1} F_{i+1}^* X_{i+1}^{-1}$. Therefore, the selection of the option (ii) at step i+1 leads to

$$F_{i+1} = \begin{bmatrix} U_i & 0 \\ 0 & 1 \end{bmatrix} X_{i+1} F_{i+1}^{*} X_{i+1}^{-1} \begin{bmatrix} U_i^T & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} U_i & 0 \\ 0 & k_{i+1} \end{bmatrix} F_{i+1}^{*} \begin{bmatrix} U_i^T & 0 \\ 0 & k_{i+1}^{-1} \end{bmatrix}. \tag{65}$$

Substituting (59) in (65) and using the induction hypothesis (64) yield

$$F_{i+1} = \begin{bmatrix} (F_i^{*})^T & \frac{k_i^c}{k_i} U_i F_i^{*} e_i \\ k_{i+1} k_i^c e_i^T U_i^T & -k_i k_{i+1} \end{bmatrix}. \tag{66}$$

Due to (55), (56) and (64), $(k_i^c)^2 e_i e_i^T = I - U_i F_i^{*} (F_i^{*})^T U_i^T = I - U_i(I - (k_i^c/k_i)^2 F_i^{*} e_i e_i^T (F_i^{*})^T) U_i^T = (k_i^c/k_i)^2 U_i F_i^{*} e_i e_i^T (F_i^{*})^T U_i^T$. Therefore, we conclude that

$$(k_i^c/k_i) U_i F_i^{*} e_i = k_i^c e_i \tag{67}$$

Analogously, we can show that

$$k_i^c e_i^T U_i = (k_i^c/k_i) e_i^T (F_i^{*})^T \tag{68}$$

Substituting (67) and (68) in (66) and comparing the result with (59) establish that $F_{i+1} = (F_{i+1}^{*})^T$. Therefore, we have proved the following:

Lemma 8 The sequence $\{F_i^{*}\}$ and its dual $\{(F_i^{*})^T\}$ are the two extremal cases of the 2^{n-1} selections of the sequence $\{U_i\}$. The former is the one which obtained by the selection of option (i) and the latter obtained by the selection of option (ii) at each i.

The nesting property of $\{F_i\}$ in (61) leads to the notion of nested realizations. Let $\mathcal{L}_i = \{f_i, F_i, g_i, \nu_i\}$ be a realization of $\sigma_i z^i/\phi_i(z)$. The sequence $\{R_i\}$ of realizations is called a *nested realizations,* if

$$h_{i+1} = [h_i \quad *] \qquad F_{i+1} = \begin{bmatrix} F_i & * \\ * & * \end{bmatrix} \tag{69}$$

$$g_{i+1} = \begin{bmatrix} g_i \\ * \end{bmatrix} \rho_{i+1} \qquad \nu_{i+1} = \nu_i \rho_{i+1},$$

where ρ_{i+1} is some number. The implication of nested realizations is shown in Fig.4. The nested realizations allow us to build up the realizations sequentially preserving the structure of the previous realizations. An example of such realization was given by Kalman [21], which is closely related to the state space algorithm of recursive realization proposed by Rissanen [22]. There are a number of ways of constructing nested realizations of $\sigma_i z^i/\phi_i(z)$. However, there is one and only one nested realizations which has the diagonal state convariance. This is the lattice realizations $\{\mathcal{L}_i\}$ given by

$$\mathcal{L}_i = (e_1^T F_i \; F_i, \; \frac{k_i^c}{k_i} F_i e_i, \; \frac{k_i^c}{k_i} e_1^T F_i e_i), \tag{70}$$

which corresponds to $\mathcal{L}^0$ in (50) for the order i. Here, we denote the scaled Schwarz form (39) of order i by F_i.

Theorem 3 The lattice realization (50) is the one and the only one nested realizations of $\sigma_i z^i/\phi_i(z)$, i = 1, $\cdots$, n, having the diagonal state covariances $\{P_i\}$. The state covariance matrices $\{P_i\}$ are nested, i.e., P_{i+1} is written in the form

$$P_{i+1} = \begin{bmatrix} P_i & 0 \\ 0 & * \end{bmatrix}.$$

(Proof) It is easily seen by direct manipulations that $\{\mathcal{L}_i\}$ is nested with the diagonal state covariance $P_i = \Lambda_i = \mathrm{diag}(1\ \lambda_1^2 \cdots\cdots \lambda_1^2 \cdots \lambda_{i-1}^2)$. To prove the converse we note that every minimal realization of $\sigma_i z^i/\phi_i(z)$ is obtained by a similar transformation of the realization $(e_1^T F_i^*,\ F_i^*,\ \frac{k_i^c}{k_i} F_i^* e_i,\ \frac{k_i^c}{k_i} e_1^T F_i^* e_i)$. Let Y_i be the transformation matrix, i.e.,

$$h_i = e_1^T F_i^* Y_i^{-1}, \quad F_i = Y_i F_i^* Y_i^{-1}, \tag{71}$$

$$g_i = \frac{k_i^c}{k_i} Y_i F_i^* e_i, \quad \nu_i = \frac{k_i^c}{k_i} e_1^T F_i^* e_i. \tag{72}$$

Since $I - F_i^*(F_i^*)^T = (k_i^c/k_i)^2 F_i^* e_i e_i^T (F_i^*)^T$, we have $Y_i Y_i^T - F_i(Y_i Y_i^T)F_i^T = g_i g_i^T$. Therefore, $P_i = Y_i Y_i^T$. From the assumption that $Y_i Y_i^T$ is diagonal, Y_i can be written $Y_i = \Lambda_i U_i$ with Λ_i diagonal and U_i unitary. Then $F_i = \Lambda_i U_i F_i^* U_i^T \Lambda_i^{-1}$. In order that F_i is nested, $U_i F_i^* U_i^T$ must be nested. Therefore, according to Lemma 7, U_i should be chosen in either of the two ways (62) and (63). Keeping the selection of (62) at each i leads to $F_i = \Lambda_i F_i^* \Lambda_i^{-1}$ which is exactly the scaled Schwarz form. Therefore, this selection generates the lattice realizations.

To prove the assertion, it is sufficient to show that a selection of U_{i+1} according to (63) destructs the nesting property. Let $U_{i+1} = V_{i+1}^T$. Then, due to (72), $g_{i+1} = (k_{i+1}^c/k_{i+1})\Lambda_{i+1} V_{i+1}^T F_{i+1}^* e_{i+1} = (k_{i+1}^c/k_{i+1})\Lambda_{i+1} Y_{i+1} e_{i+1} = k_{i+1}^c \Lambda_{i+1} e_{i+1} = \text{const. } e_{i+1}$. This obviously contradicts to the nesting property. □

Theorem 3 reveals the essential characteristic feature of lattice realizations in terms of state space representations. It should be remarked that the requirement of both of the nesting property and the orthogonality (which is reflected in the requirement that the state covariance is diagonal) is quite strong.

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(Appendix is omitted due to space limitation.)

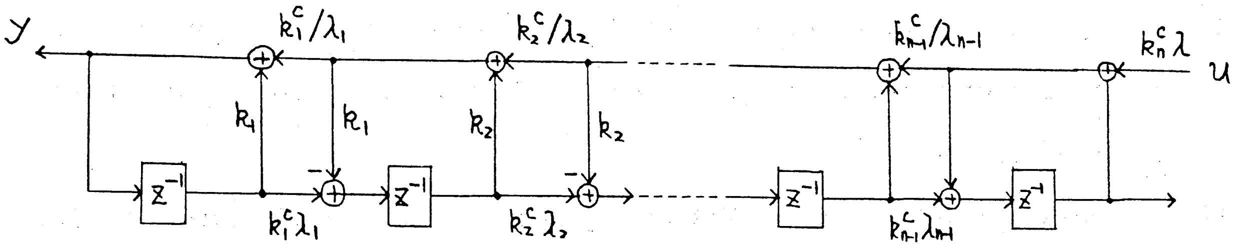

Fig. 1 Normalized Lattice Filter

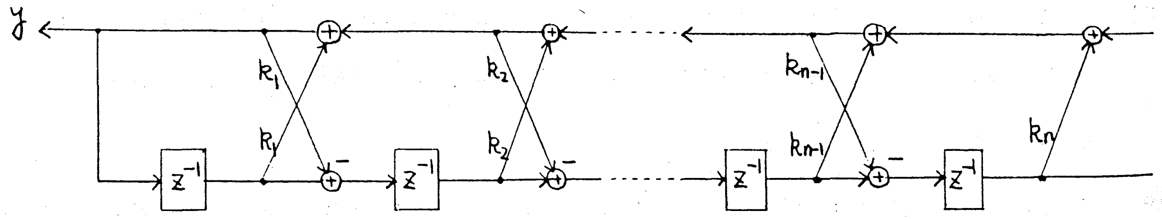

Fig. 2 Unnormalized Lattice Filter

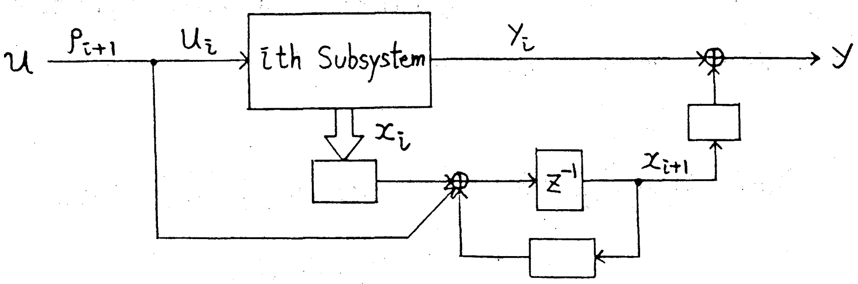

Fig. 3 Block Diagram of Nested Realization

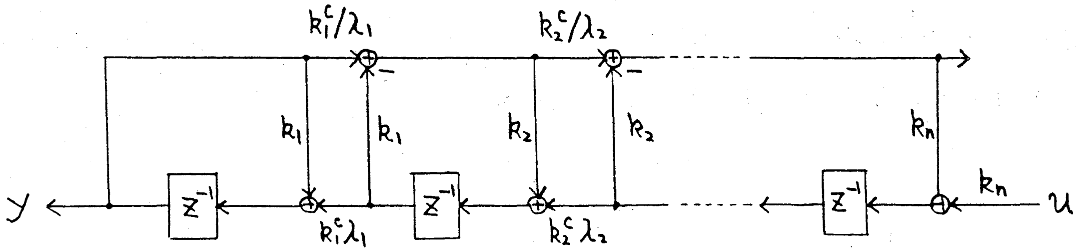

Fig. 4 Maximum Phase Dual Lattice Filter