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Observability, Controllability, and Feedback Stabilizability for Evolution Equations

(発展方程式に対する可観測性, 可制御性とフィードバック安定化可能性)

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§1. Introduction.

Recently several authors have been interested in the feedback stabilization of evolution equations. ([8], [9], [10], [14], [15], [16], [19], [21] and [22]). Our purpose here is to give some refinements of [14] by an abstract method.

Let X be a Banach space over $\mathbb{C}$, and let $-A$ be a generator of a (C_0) semi-group $\{e^{-tA}\}_{t\geqq 0}$ in X (Hille and Phillips [4], Yosida [25]). The mild solution $u = u(t) \in C([0,\infty) \longrightarrow X)$ of

$$\begin{cases} (1.1) & \dfrac{du}{dt} + Au = 0 \quad (t \geqq 0) \\ (1.2) & u(0) = a \end{cases}$$

is given by $u(t) = e^{-tA}a$, and (1.1) is called a "free system". We suppose that (1.1) is unstable. Then we want to construct its stable modification in the following sense :

Let $S : X \longrightarrow \mathbb{C}^N$ and $T : \mathbb{C}^N \longrightarrow X$ be bounded linear operators, $\mathbb{C}^N$ being the N-dimensional complex vector space.

We consider the equation

$$(1.3) \qquad \frac{dv}{dt} + Av = TSv \qquad (t \geqq 0)$$

with

$$(1.4) \qquad v(0) = a \quad .$$

We put $B = TS$, which is nothing but a bounded linear operator on X of finite rank, and (1.3) is called a "feedback system" of (1.1). From the practical point of view, S and T may be called a "sensor" and a "controller" , respectively, and the pair $\langle T, S \rangle$ is called a "feedback".

It is known that $-(A - B)$ generates a (C_0) semi-group, which is denoted by $\{e^{-t(A-B)}\}_{t \geqq 0}$. The mild solution $v = v(t) \in C([0, \infty) \longrightarrow X)$ of (1.3) with (1.4) is given by $v(t) = e^{-t(A-B)}a$ (Kato [5]).

<u>Definition 1.</u>

For $\omega \in \mathbb{R}$, $\langle T, S \rangle$ is said to be " ω-(feedback) stabilizable" (in X) with respect to e^{-tA} if

$$(1.5) \qquad \|e^{-t(A-B)}\|_{X \to X} \leqq Me^{-t\omega} \qquad (t \geqq 0)$$

holds for some constant $M > 0$. $\square$

We are interested in constructing ω-stabilizable feedback

$\langle T, S \rangle$ for $\omega \geqq 0$, in the case that the free system (1.1) is unstable, that is,

$$\inf \{ \mathrm{Re}\lambda ; \lambda \in \sigma(A) \} < 0.$$

Here $\sigma(A)$ denotes the spectrum of A.

We introduce the following notions for the sensor S and for the controller T, according to Fattorini [2], Sakawa [12, 13], Triggiani [20], etc.

Let $Y \subset X$ be a closed linear subspace:

Definition 2.

A sensor $S : X \longrightarrow \mathbb{C}^N$ is said to be "Y-observable" (in X) with respect to e^{-tA} if the conditions $a \in Y$ and $Se^{-tA}a = 0$ $(0 \leqq t < \infty)$ imply $a = 0$. $\square$

Henceforth $\overline{Z}$ denotes the closure of $Z \subset X$ in X.

Definition 3.

A controller $T : \mathbb{C}^N \longrightarrow X$ is said to be "Y-controllable" (in X) with respect to e^{-tA} if $\overline{Z} \supset Y$, where $Z = \bigcup_{t>0} Z_t$

with $Z_t = \left\{ \int_0^t e^{-(t-s)A} Tf(s)ds \ ; \ f \in L^2(0, t)^N \right\}$. $\square$

Here we note that $v(t) = \int_0^t e^{-(t-s)A} Tf(s)ds$ is a mild solution of

$$(1.6) \qquad \frac{dv}{dt} + Av = Tf(t) \qquad (t \geqq 0)$$

with

$$(1.7) \qquad v(0) = 0 \ .$$

As for the free system (1.1), we assume the following hypotheses:

Assumption 1.

The spectrum of A, $\sigma(A)$, is divided into two subsets Σ_0 and Σ_1. Furthermore, Σ_0 is bounded, and the relation

$$(1.8) \qquad \kappa_0 = \sup \{ \operatorname{Re} \lambda \ ; \ \lambda \in \Sigma_0 \} < \inf \{ \operatorname{Re} \lambda \ ; \ \lambda \in \Sigma_1 \}$$

holds true.

There exists a Jordan curve Γ which surrounds Σ_0 and, at the same time, separates Σ_0 and Σ_1. We put

$$(1.9) \qquad P = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma} (\lambda - A)^{-1} \, d\lambda \ . \qquad \square$$

Assumption 2.

The estimate

$$(1.10) \qquad \| (1 - P) e^{-tA} \|_{X \to X} \leqq M_1 e^{-t\kappa_1} \qquad (t \geqq 0)$$

holds for a $\kappa_1 > \kappa_0$. $\square$

Assumption 3.

We have the finite dimensionality of X_0 , that is,

$$(1.11) \qquad m = \dim X_0 < \infty ,$$

where

$$(1.12) \qquad X_0 = PX . \qquad \square$$

Now our theorems are stated as follows:

Theorem 1.

Let a sensor $S : X \longrightarrow \mathbb{C}^N$ and a real number ω satisfying $\kappa_0 < \omega \leqq \kappa_1$ be given. Then, there exists a controller $T : \mathbb{C}^N \longrightarrow X$ such that the feedback $\langle T, S \rangle$ is ω-stabilizable with respect to e^{-tA} if and only if S is X_0-observable with respect to e^{-tA} . $\square$

Theorem 2.

Let a controller $T : \mathbb{C}^N \longrightarrow X$ and a real number ω satisfying $\kappa_0 < \omega \leqq \kappa_1$ be given. Then, there exists a sensor $S : X \longrightarrow \mathbb{C}^N$ such that the feedback $\langle T, S \rangle$ is ω-stabilizable with respect to e^{-tA} if and only if T is X_0-controllable with respect to e^{-tA} . $\square$

These theorems are abstract versions of the results in Sakawa and Matsushita [14] for parabolic equations in bounded domains.

In fact, in later sections, X_0-observability and X_0-controllability are shown to be equivalent to similar rank conditions given in [14]. However, the following points should be noted: First we shall propose to use a simple argument in the proof, applicable to other problems on the feedback stabilization. For this point, see our forthcoming papers. Secondly our theorems actually give refinements of [14]. In fact,

(a) The operator need not be self-adjoint nor a generator of a holomorphic semi-group.

(b) The best possible exponent κ_1 can be taken as ω in constructing the feedback $\langle T, S \rangle$.

For example, we can apply our theorems to the parabolic equation

$$\frac{\partial u}{\partial t} = \Delta u - V(x)u \qquad \text{in } L^2(\mathbb{R}^3)$$

with $V(x) = O(|x|^{-2-\varepsilon})$ $(\varepsilon > 0)$ as $|x| \longrightarrow \infty$, to construct zero-stabilizable feedback $\langle T, S \rangle$, since this operator $-\Delta + V$ is self-adjoint and $\sigma(-\Delta + V) = [0, \infty) \cup \{\lambda_i ; 1 \leqq i \leqq \ell\}$ with $\lambda_i < 0$. See Kuroda [7], for instance.

Theorems 1 and 2 are constructively proved. In a forthcoming paper, we shall discuss certain concrete problems, where, applying Theorems 1 and 2, we shall refine some theorems due to Nambu [9, 10] and Triggiani [21] on "boundary sensors", Yamamoto [24] on "pointwise sensors" and Triggiani [22] on

"boundary controllers".

This paper is composed of six sections. In §2, we show that the observability is equivalent to certain rank conditions even if A is not self-adjoint. In §3, we show a key estimate on $\| e^{-t(A-B)} \|_{X \to X}$. Theorem 1 is proved in §4. In §5, we state the duality between observability and controllability, and show the equivalence of controllability and certain rank conditions. In virtue of this duality, we reduce Theorem 2 to Theorem 1 in §6.

§2. Observability and rank conditions.

In this section, we show that the observability is equivalent to what is called, the rank conditions even if A is not self-adjoint. We recall that

$$(2.1) \qquad X_0 = PX, \qquad m = \dim X_0 < \infty ,$$

and we put

$$(2.2) \qquad A_0 = A|_{X_0}$$

$$= \frac{1}{2\pi\sqrt{-1}} \int_\Gamma \lambda(\lambda - A)^{-1} d\lambda |_{X_0} \quad .$$

The operator $A_0 : X_0 \longrightarrow X_0$ is linear on the finite dimensional space X_0, so that $e^{-tA_0} : X_0 \longrightarrow X_0$ is well-defined. We first note the followings.

Lemma 1.

A sensor $S : X \longrightarrow \mathbb{C}^N$ is X_0-observable in X with respect to e^{-tA} if and only if

$$(2.3) \qquad S_0 = S|_{X_0} \quad : X_0 \longrightarrow \mathbb{C}^N$$

is X_0-observable in X_0 with respect to e^{-tA_0}. $\square$

Proposition 1 (the first rank condition).

A sensor $S : X \longrightarrow \mathbb{C}^N$ is X_0-observable in X with respect to e^{-tA} if and only if

$$(2.4) \qquad \operatorname{rank}\,(S_0^*,\ A_0^* S_0^*,\ \cdots,\ (A_0^*)^{m-1} S_0^*) = m\ ,$$

where $\cdot^*$ denotes the dual operator in consideration. $\square$

Let $\sigma_p(A)$ be the set of the eigenvalues of A. The assumption of $\dim X_0 = m < \infty$ implies that Σ_0 is contained in $\sigma_p(A)$, and also that the number of elements in Σ_0 is finite. We set $\Sigma_0 = \{\lambda_i\}_{1 \le i \le \ell}$. For a sufficiently small circle Γ_i with center at λ_i, the projection

$$(2.5) \qquad P_i = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_i} (\lambda - A)^{-1} d\lambda$$

can be defined, and we get $\sum_{i=1}^{\ell} P_i = P$. Let

(2.6) $\quad m_i = \dim P_i X$ and $n_i = \dim \mathrm{Ker}(\lambda_i - A)$.

Then we have $m = \sum_{i=1}^{\ell} m_i$ and $n_i \leqq m_i$ (Kato [5]) .

Let $\{\varphi_{ij}\}_{1 \leqq j \leqq n_i}$ be a basis of $\mathrm{Ker}(\lambda_i - A)$ $(1 \leqq i \leqq \ell)$. The following proposition was shown by Fattorini [2] , Sakawa [12, 13] and Triggiani [20] , in the case that A is self-adjoint. We can prove it along the line of Suzuki [17, 18] . Its proof is given in [26] . See also Hautus [3] . Henceforth X^* means the dual space of X :

<u>Proposition 2</u> (the second rank condition).
Let $S : X \longrightarrow \mathbb{C}^N$ be a sensor :

(2.7) $\quad S = {}^t(S_1, S_2, \cdots, S_N) : X \longrightarrow \mathbb{C}^N$,
$\quad S_k \in X^*$ $(1 \leqq k \leqq N)$.

Then S is X_0-observable in X with respect to e^{-tA} if and only if

(2.8) $\quad \mathrm{rank}\, M_i = n_i \qquad (1 \leqq i \leqq \ell)$,

where M_i is an $N \times n_i$ matrix defined by

(2.9) $\quad M_i = ({}_{X^*}\langle S_k , \varphi_{ij} \rangle_X)_{1 \leqq k \leqq N, 1 \leqq j \leqq n_i}$. $\square$

§3. A stabilization estimate.

In this section, we drive an estimate on $\| e^{-t(A-B)}\|_{X \to X}$, which is useful later. We recall that $-A$ is a generator of a (C_0) semi-group in X, satisfying Assumptions 1-3. The operator $B = TS$ is a bounded linear operator on X of finite rank, and the equation

$$(3.1) \qquad \frac{du}{dt} + Au = Bu \quad (t \geqq 0)$$

represents the feedback system of (1.1). We set

$$(3.2) \qquad X_0 = PX \ , \quad X_1 = (1 - P)X,$$

$$(3.3) \qquad A_0 = A_{|X_0} \ , \quad A_1 = A_{|X_1}$$

and

$$(3.4) \qquad B_0 = PB_{|X_0} \quad .$$

The operators A_0, $B_0 : X_0 \longrightarrow X_0$ are linear on the finite dimensional space X_0, hence $e^{-t(A_0 - B_0)}$ is well-defined. On the other hand, $-A_1 : X_1 \longrightarrow X_1$ generates a (C_0) semi-group e^{-tA_1} on X_1, which can be written as

$$(3.5) \qquad e^{-tA_1} = (1 - P)e^{-tA}{}_{|X_1} \quad .$$

Therefore the estimate

$$(3.6) \qquad \| e^{-tA_1} \|_{X_1 \to X_1} \leqq M_1 e^{-t\kappa_1} \quad (t \geqq 0)$$

holds true by (1.10). Here we further assume the following. There exists some constant ω_0 with $\omega_0 > \kappa_1$ such that the estimate

$$(3.7) \qquad \| e^{-t(A_0 - B_0)} \|_{X_0 \to X_0} \leqq M_0 e^{-t\omega_0} \quad (t \geqq 0)$$

holds true. Then we can show the following ([26]) .

<u>Theorem 3.</u>

Under the assumption stated above, the estimate

$$(3.8) \qquad \| e^{-t(A-B)} \|_{X \to X} \leqq$$

$$M \exp\left(-t \left\{ \kappa_1 - M_1 \left(\beta_1 + \frac{\alpha_1 \beta_0 M_0}{\omega_0 - \kappa_1} \right) \right\} \right)$$

holds for some constant $M > 0$, where

$$(3.9) \qquad \beta_0 = \| PB(1 - P) \|_{X \to X} ,$$

$$(3.10) \qquad \beta_1 = \| (1 - P)B(1 - P) \|_{X \to X}$$

and

$$(3.11) \qquad \alpha_1 = \| (1 - P)BP \|_{X \to X} . \qquad \square$$

§4. Proof of Theorem 1.

Now we prove Theorem 1 stated in §1 :

Firstly we suppose that there exists a controller $T : \mathbb{C}^N \longrightarrow X$ such that

$$(4.1) \qquad \|e^{-t(A-TS)}\|_{X \to X} \leqq Me^{-t\omega}$$

for some ω satisfying $\kappa_0 < \omega \leqq \kappa_1$. We show that $S : X \longrightarrow \mathbb{C}^N$ is X_0-observable with respect to e^{-tA} . To this end, we assume

$$(4.2) \qquad a \in X_0 \text{ and } Se^{-tA} a = 0 \qquad (0 \leqq t < \infty) ,$$

and put $u(t) = e^{-tA}a$. Since $a \in X_0 \subset \mathcal{D}(A)$, $u = u(t)$ $C^1([0, \infty) \longrightarrow X)$ is the strong solution of

$$\begin{cases} \dfrac{du}{dt} + Au = 0 = TSu & (t \geqq 0) \\ u(0) = a \ . \end{cases}$$

Therefore we may get

$$(4.3) \qquad u(t) = e^{-t(A-TS)} a ,$$

whence we have

$$(4.4) \qquad \|u(t)\| \leqq Me^{-t\omega}\|a\| \qquad (t \geqq 0)$$

by (4.1). On the other hand, in view of $a \in X_0$ we have for some positive integers q_i,

$$(4.5) \qquad u(t) = e^{-tA_0} a$$

$$= \sum_{i=1}^{\ell} e^{-t\lambda_i} \sum_{j=0}^{q_i-1} \frac{(-t)^j}{j!} D_i^j a \quad (t \geqq 0) .$$

We now substitute (4.5) into (4.4). Noting $\mathrm{Re}\,\lambda_i \leqq \kappa_0 < \omega$ $(1 \leqq i \leqq \ell)$, we obtain $D_i^j a = 0$ $(1 \leqq i \leqq \ell,\ 0 \leqq j \leqq q_i - 1)$, and, in particular, $a = Pa = 0$.

Conversely we suppose that S is X_0-observable. Then,

$$\mathrm{rank}\,(S_0^*, A_0^* S_0^*, \cdots, (A_0^*)^{m-1} S_0^*) = m$$

by Proposition 1 of §2. By Wonham's pole assignment theorem ([23]), there exists a linear operator $T_0 : \mathbb{C}^N \longrightarrow X_0$ such that

$$\inf \{ \mathrm{Re}\,\lambda ;\ \lambda \in \sigma(A_0 - T_0 S_0) \} > \kappa_1 .$$

We then have

$$(4.6) \qquad \|e^{-t(A_0 - T_0 S_0)}\|_{X_0 \to X_0} \leqq M_0 e^{-t\omega_0} \qquad (t \geqq 0)$$

for some constants $M_0 > 0$ and $\omega_0 > \kappa_1$. We put

(4.7) $$B = T_0 S : X \longrightarrow X .$$

Since $(1 - P)B = 0$, we can take $\alpha_1 = \beta_1 = 0$ in Theorem 3 of §3. Therefore we get

(4.8) $$\| e^{-t(A-B)} \| \leqq M e^{-t\kappa_1} \qquad (t \geqq 0) ,$$

so that $\langle T_0, S \rangle$ is κ_1-stabilizable with respect to e^{-tA}. $\square$

§5. Duality between observability and controllability.

The duality between observability and controllability is well-known (Dolecki and Russell [1]). We here show it as a version concerning "Y-observability" and "Y-controllability".

For a closed linear subspace $Y \subset X$, Y^* and $Y^\perp$ denotes the dual space of Y and the orthogonal complement of Y, respectively ;

$$Y^* = \{ g \; ; \; g \text{ is a bounded linear functional on } Y . \}$$

$$Y^\perp = \{ g \in X^* \; ; \; g_{|Y} = 0 \} .$$

We recall

$$P = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma} (\lambda - A)^{-1} \, d\lambda \ ,$$

$$X_0 = PX$$

and

$$X_1 = (1 - P)X \ .$$

From the direct sum decomposition

$$(5.1) \qquad X = X_0 \oplus X_1 \ ,$$

we have the isomorphisms

$$(5.2) \qquad X_1^{\perp} \cong X_0^{*} \quad \text{and} \quad X_0^{\perp} \cong X_1^{*} \ .$$

In particular, we have $\dim X_1^{\perp} = \dim X_0^{*} = m < \infty$. Also, we have

$$(5.3) \qquad X_1^{\perp} = P^{*}X^{*} \ , \quad X_0^{\perp} = (1 - P^{*})X^{*} \ ,$$

$$(5.4) \qquad P^{*} = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma} (\lambda - A^{*})^{-1} \, d\lambda$$

and

$$(5.5) \qquad X^{*} = X_1^{\perp} \oplus X_0^{\perp} \ .$$

For the operator $A_0 \equiv A_{|X_0}$, we denote its dual operator by A_0^*. Then $e^{-tA_0^*} : X_0^* \longrightarrow X_0^*$ is defined and we have

$$(5.6) \qquad (e^{-tA})^*_{|X_1^\perp} \cong e^{-tA_0^*} \qquad (\text{by } X_1^\perp \cong X_0^*) .$$

Also, for a controller T, we define the operator $T_0^* : X_0^* \longrightarrow \mathbb{C}^N$ such that

$$(5.7) \qquad T_0^* \cong T^*_{|X_1^\perp} \qquad (\text{by } X_0^* \cong X_1^\perp) .$$

Then we have the followings ([26]) .

<u>Lemma 2.</u>
$T : \mathbb{C}^N \longrightarrow X$ is X_0-controllable in X with respect to e^{-tA} if and only if $T_0^* : X_0^* \longrightarrow \mathbb{C}^N$ is X_0^*-observable in X_0^* with respect to $e^{-tA_0^*}$. $\square$

It is known that $\dim \mathrm{Ker}(\lambda_i - A^*) = \dim \mathrm{Ker}(\lambda_i - A) = n_i$ holds. (Kato [5], for example.) Let $\{\varphi^*_{ij}\}_{1 \leq j \leq n_i}$ be a basis of $\mathrm{Ker}(\lambda_i - A^*)$.

<u>Proposition 3</u> (the second rank condition for controllability).
Let $T : \mathbb{C}^N \longrightarrow X$ be a controller :

$$(5.8) \qquad \begin{array}{rccc} T : & \mathbb{C}^N & \longrightarrow & X \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & (\alpha_k)_{1 \leq k \leq N} & \longmapsto & \sum_{k=1}^N \alpha_k \psi_k \quad , \quad \psi_k \in X . \end{array}$$

Then, T is X_0-controllable in X with respect to e^{-tA} if and only if

$$(5.9) \qquad \operatorname{rank} L_i = n_i \qquad (1 \leqq i \leqq \ell),$$

where L_i is an $N \times n_i$ matrix defined by

$$(5.10) \qquad L_i = \left({}_{X^*}\langle \varphi^*_{ij}, \psi_k \rangle_X \right)_{\substack{1 \leqq k \leqq N \\ 1 \leqq j \leqq n_i}} . \qquad \square$$

§6. Proof of Theorem 2.

We now prove Theorem 2 stated in §1.

Proof of the "if" part of Theorem 2 :

We assume that $T : \mathbb{C}^N \longrightarrow X$ is X_0-controllable in X with respect to e^{-tA}. By Lemma 2 in §5, $T_0^* : X_0^* \longrightarrow \mathbb{C}^N$ is X_0^*-observable in X_0^* with respect to $e^{-tA_0^*}$. By Proposition 1 in §2 and Wonham's pole assignment theorem, there exists a linear operator $F_0 ; \mathbb{C}^N \longrightarrow X_0^*$ such that

$$(6.1) \qquad \| e^{-t(A_0^* - F_0 T_0^*)} \|_{X_0^* \to X_0^*} \leqq M_0 e^{-t\omega_0}$$

for some constants $M_0 > 0$ and $\omega_0 > \kappa_1$.

Let $S_0 : X_0 \longrightarrow \mathbb{C}^N$ be the dual operator of F_0. Then, since $\dim X_0 = m < \infty$, (6.1) implies

$$(6.2) \qquad \| e^{-t(A_0-PTS_0)} \|_{X_0 \to X_0} \leqq M_0 e^{-t\omega_0} .$$

We set

$$(6.3) \qquad S = \begin{cases} S_0 & (\text{on } X_0) \\ 0 & (\text{on } X_1) , \end{cases}$$

and apply Theorem 3 in §3 for $B = TS$. Since $\beta_0 = \beta_1 = 0$ by (6.3), we then have

$$(6.4) \qquad \| e^{-t(A-TS)} \|_{X \to X} \leqq M e^{-t\kappa_1}$$

for some constant $M > 0$, which shows that $\langle T, S \rangle$ is κ_1-stabilizable in X with respect to e^{-tA}. $\square$

<u>Proof of the "only if" part of Theorem 2</u> :

Here we show the "only if" part in the case that X is reflexive. For the general Banach space X, the proof is given in [26].

We assume that $\langle T, S \rangle$ is ω-stabilizable with respect to e^{-tA} for $\kappa_0 < \omega \leqq \kappa_1$. Then we have

$$(6.5) \qquad \| e^{-t(A-TS)} \|_{X \to X} \leqq M e^{-t\omega}$$

for some constant $M > 0$.

The reflexivity of X implies

$$(e^{-tA})^* = e^{-tA^*}$$

$$(e^{-t(A-TS)})^* = e^{-t(A-TS)^*} \qquad \text{(R.S.Phillips [11])}.$$

Therefore we get

$$(6.6) \qquad \| e^{-t(A^*-S^*T^*)} \|_{X^* \to X^*} \leqq Me^{-t\omega} \qquad (t \geqq 0) .$$

We show that T is X_0-controllable with respect to e^{-tA}. To this end, we have only to show (6.7) ;

$$(6.7) \qquad T^* e^{-tA^*} a = 0 \quad (0 \leqq t < \infty) \text{ and } a \in X_1^{\perp} \text{ imply } a = 0 .$$

We put $u(t) = e^{-tA^*} a$. Since $a \in X_1^{\perp} \subset \mathcal{D}(A^*)$, $u = u(t) \in C^1([0, \infty) \longrightarrow X)$ is the strong solution of

$$\begin{cases} \dfrac{du}{dt} + A^* u = 0 = S^* T^* u & (t \geqq 0) \\ u(0) = a & . \end{cases}$$

Therefore we get $u(t) = e^{-t(A^*-S^*T^*)} a$ and

$$(6.8) \qquad \| e^{-tA^*} a \|_{X^*} \leqq Me^{-t\omega} \|a\| \quad (t \geqq 0) \text{ (by (6.6))} .$$

Now by (6.8), $a \in X_1^{\perp}$ and $\operatorname{Re} \lambda_i \leqq \kappa_0 < \omega$, we see $a = 0$. $\square$

Appendix. <u>Example</u> :

Here we apply Theorem 1 to the following stabilization problem.

$$\begin{cases} X = L^2(0, \pi) \\ Au = -\dfrac{d^2u}{dx^2} - 5u \\ \mathcal{D}(A) = H^2(0, \pi) \cap H_0^1(0, \pi) \end{cases} .$$

In Assumption 1, we set

$$\Sigma_0 = \{ -4, -1 \} \quad \text{and} \quad \Sigma_1 = \{ n^2-5 \}_{n=3}^{\infty} .$$

Then we have

$$X_0 = \text{Span} \{ \sin x, \sin 2x \} .$$

1) We define $S : X \longrightarrow \mathbb{C}$ by $Su = \int_0^{\frac{\pi}{2}} u(x)dx$.
Then by Proposition 2, this sensor S is X_0-observable with respect to e^{-tA} . By Theorem 1, we define $T : \mathbb{C} \longrightarrow X$ by

$$T(\alpha) = \alpha(35\sin x - 17\sin 2x) ,$$

for example. Then this feedback $< T, S >$ is 3-feedback stabilizable in X with respect to e^{-tA} . $\square$

2) We define $S : X \longrightarrow \mathbb{C}^2$ by

$$Su = \left(\int_0^{\pi} u(x)dx, \quad \frac{2}{\pi} \int_0^{\pi} u(x)\, xdx \right) .$$

Then by Proposition 2, this sensor S is X_0-observable. By Theorem 1, we define $T : \mathbb{C}^2 \longrightarrow X$ by

$$T(\alpha_1, \alpha_2) = \alpha_1 \left(\frac{11}{2} \sin x + 13 \sin 2x \right) +$$

$$\alpha_2 \left(\frac{1}{2} \sin x - 3 \sin 2x \right) ,$$

for example. Then this feedback $\langle T, S \rangle$ is 3-feedback stabilizable.

$\square$

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