Observability, Controllability, and Feedback Stabilizability
for Evolution Equations

(発展方程式に対する可観測度、可制御性と
フィードバック安定化可能性)

献・理 山本昌宏 (Masahiro YAMAMOTO)
献・理 鈴木貴 (Takashi SUZUKI)

§1. Introduction.

Recently several authors have been interested in the
feedback stabilization of evolution equations. ([8], [9],
[10], [14], [15], [16], [19], [21] and [22]). Our purpose
here is to give some refinements of [14] by an abstract method.

Let $X$ be a Banach space over $\mathbb{C}$, and let $-A$ be a
generator of a $(C_0)$ semi-group $\{e^{-tA}\}_{t \geq 0}$ in $X$ (Hille and
Phillips [4], Yosida [25]). The mild solution $u = u(t)$
\[\in C([-\infty, 0) \rightarrow X]\]
of
\[
\begin{cases}
(1.1) & \frac{du}{dt} + Au = 0 \ (t \geq 0) \\
(1.2) & u(0) = a
\end{cases}
\]
is given by $u(t) = e^{-tA}a$, and (1.1) is called a "free system".
We suppose that (1.1) is unstable. Then we want to construct
its stable modification in the following sense:

Let $S : X \rightarrow \mathbb{C}^N$ and $T : \mathbb{C}^N \rightarrow X$ be bounded linear
operators, $\mathbb{C}^N$ being the $N$-dimensional complex vector space.
We consider the equation

\[(1.3) \quad \frac{dv}{dt} + Av = Tsv \quad (t \geq 0)\]

with

\[(1.4) \quad v(0) = a .\]

We put \(B = TS\), which is nothing but a bounded linear operator on \(X\) of finite rank, and (1.3) is called a "feedback system" of (1.1). From the practical point of view, \(S\) and \(T\) may be called a "sensor" and a "controller", respectively, and the pair \(<T, S>\) is called a "feedback".

It is known that \(-(A - B)\) generates a \((C_0)\) semi-group, which is denoted by \(\{e^{-t(A-B)}\}_{t \geq 0}\). The mild solution \(v = v(t) \in C([0, \infty) \to X)\) of (1.3) with (1.4) is given by \(v(t) = e^{-t(A-B)}a\) (Kato [5]).

**Definition 1.**

For \(\omega \in \mathbb{R}, \quad <T, S>\) is said to be "\(\omega\)-(feedback) stabilizable" (in \(X\)) with respect to \(e^{-tA}\) if

\[(1.5) \quad \|e^{-t(A-B)}\|_{X \to X} \equiv Me^{-t\omega} \quad (t \geq 0)\]

holds for some constant \(M > 0\).

We are interested in constructing \(\omega\)-stabilizable feedback
\[ \langle T, S \rangle \] for \( \omega \geq 0 \), in the case that the free system (1.1) is unstable, that is,

\[ \inf \left\{ \Re \lambda ; \lambda \in \sigma(A) \right\} < 0. \]

Here \( \sigma(A) \) denotes the spectrum of \( A \).

We introduce the following notions for the sensor \( S \) and for the controller \( T \), according to Pattorini [2], Sakawa [12, 13], Triggiani [20], etc.

Let \( Y \subset X \) be a closed linear subspace:

**Definition 2.**

A sensor \( S : X \to \mathbb{C}^N \) is said to be "\( Y \)-observable" (in \( X \)) with respect to \( e^{-tA} \) if the conditions \( a \in Y \) and \( Se^{-tA}a = 0 \) \((0 \leq t < \infty)\) imply \( a = 0 \).

Henceforth \( \overline{Z} \) denotes the closure of \( Z \subset X \) in \( X \).

**Definition 3.**

A controller \( T : \mathbb{C}^N \to X \) is said to be "\( Y \)-controllable" (in \( X \)) with respect to \( e^{-tA} \) if \( \overline{Z} \supset Y \), where \( Z = \bigcup_{t>0} Z_t \) with \( Z_t = \left\{ \int_0^t e^{-(t-s)A}Tf(s)ds ; f \in L^2(0, t)^N \right\} \).

Here we note that \( v(t) = \int_0^t e^{-(t-s)A}Tf(s)ds \) is a mild solution of
\begin{align*}
(1.6) \quad \frac{dv}{dt} + Av &= Tf(t) \quad (t \geq 0) \\
\text{with} \\
(1.7) \quad v(0) = 0 .
\end{align*}

As for the free system (1.1), we assume the following hypotheses:

**Assumption 1.**
The spectrum of $A$, $\sigma(A)$, is divided into two subsets $\Sigma_0$ and $\Sigma_1$. Furthermore, $\Sigma_0$ is bounded, and the relation

\begin{equation}
(1.8) \quad \kappa_0 = \sup \left\{ \Re \lambda ; \lambda \in \Sigma_0 \right\} < \inf \left\{ \Re \lambda ; \lambda \in \Sigma_1 \right\}
\end{equation}

holds true.

There exists a Jordan curve $\Gamma$ which surrounds $\Sigma_0$ and, at the same time, separates $\Sigma_0$ and $\Sigma_1$. We put

\begin{equation}
(1.9) \quad P = \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma} (\lambda - A)^{-1} \, d\lambda .
\end{equation}

**Assumption 2.**
The estimate

\begin{equation}
(1.10) \quad \| (1 - P)e^{-tA} \|_{X \rightarrow X} \leq M_1 e^{-t\kappa_1} \quad (t \geq 0)
\end{equation}

holds for a $\kappa_1 > \kappa_0$.
Assumption 3.
We have the finite dimensionality of $X_0$, that is,

$$m = \dim X_0 < \infty,$$

where

$$X_0 = PX.$$

Now our theorems are stated as follows:

**Theorem 1.**
Let a sensor $S : X \to \mathcal{CN}$ and a real number $\omega$ satisfying $\kappa_0 < \omega \leq \kappa_1$ be given. Then, there exists a controller $T : \mathcal{CN} \to X$ such that the feedback $< T, S >$ is $\omega$-stabilizable with respect to $e^{-tA}$ if and only if $S$ is $X_0$-observable with respect to $e^{-tA}$.  \[ \square \]

**Theorem 2.**
Let a controller $T : \mathcal{CN} \to X$ and a real number $\omega$ satisfying $\kappa_0 < \omega \leq \kappa_1$ be given. Then, there exists a sensor $S : X \to \mathcal{CN}$ such that the feedback $< T, S >$ is $\omega$-stabilizable with respect to $e^{-tA}$ if and only if $T$ is $X_0$-controllable with respect to $e^{-tA}$.  \[ \square \]

These theorems are abstract versions of the results in Sakawa and Matsushita (14) for parabolic equations in bounded domains.
In fact, in later sections, $X_0$-observability and $X_0$-controllability are shown to be equivalent to similar rank conditions given in [14]. However, the following points should be noted: First we shall propose to use a simple argument in the proof, applicable to other problems on the feedback stabilization. For this point, see our forthcoming papers. Secondly our theorems actually give refinements of [14].

In fact,

(a) The operator need not be self-adjoint nor a generator of a holomorphic semi-group.

(b) The best possible exponent $\kappa_1$ can be taken as $\omega$ in constructing the feedback $\langle T, S \rangle$.

For example, we can apply our theorems to the parabolic equation

$$\frac{\partial u}{\partial t} = \Delta u - V(x)u \quad \text{in} \quad L^2(\mathbb{R}^3)$$

with $V(x) = O(|x|^{-2-\varepsilon})$ ($\varepsilon > 0$) as $|x| \to \infty$, to construct zero-stabilizable feedback $\langle T, S \rangle$, since this operator $-\Delta + V$ is self-adjoint and $\sigma(-\Delta + V) = (0, \infty) \cup \{ \lambda_i; 1 \leq i \leq l \}$ with $\lambda_i < 0$. See Kuroda [7], for instance.

Theorems 1 and 2 are constructively proved. In a forthcoming paper, we shall discuss certain concrete problems, where, applying Theorems 1 and 2, we shall refine some theorems due to Nambu [9, 10] and Triggiani [21] on "boundary sensors", Yamamoto [24] on "pointwise sensors" and Triggiani [22] on
"boundary controllers".

This paper is composed of six sections. In §2, we show that the observability is equivalent to certain rank conditions even if A is not self-adjoint. In §3, we show a key estimate on \( \| e^{-t(A-B)} \|_{X \rightarrow X} \). Theorem 1 is proved in §4. In §5, we state the duality between observability and controllability, and show the equivalence of controllability and certain rank conditions. In virtue of this duality, we reduce Theorem 2 to Theorem 1 in §6.

§2. Observability and rank conditions.

In this section, we show that the observability is equivalent to what is called, the rank conditions even if A is not self-adjoint. We recall that

\begin{equation}
X_0 = PX, \quad m = \dim X_0 < \infty,
\end{equation}

and we put

\begin{equation}
A_0 = A \big|_{X_0} = \frac{1}{2\pi \sqrt{-1}} \int_{\Gamma} \lambda (\lambda - A)^{-1} d\lambda \big|_{X_0}.
\end{equation}
The operator $A_0 : X_0 \rightarrow X_0$ is linear on the finite dimensional space $X_0$, so that $e^{-tA_0} : X_0 \rightarrow X_0$ is well-defined. We first note the followings.

**Lemma 1.**
A sensor $S : X \rightarrow \mathbb{C}^N$ is $X_0$-observable in $X$ with respect to $e^{-tA}$ if and only if

$$(2.3) \quad S_0 = S|_{X_0} : X_0 \rightarrow \mathbb{C}^N$$

is $X_0$-observable in $X_0$ with respect to $e^{-tA_0}$.

**Proposition 1** (the first rank condition).
A sensor $S : X \rightarrow \mathbb{C}^N$ is $X_0$-observable in $X$ with respect to $e^{-tA}$ if and only if

$$(2.4) \quad \text{rank} \left( S_0^*, A_0^*S_0^*, \ldots, (A_0^*)^{m-1} S_0^* \right) = m,$$

where $^*$ denotes the dual operator in consideration.

Let $\sigma_p(A)$ be the set of the eigenvalues of $A$.

The assumption of $\dim X_0 = m < \infty$ implies that $\Sigma_0$ is contained in $\sigma_p(A)$, and also that the number of elements in $\Sigma_0$ is finite. We set $\Sigma_0 = \{ \lambda_i \}_{1 \leq i \leq l}$. For a sufficiently small circle $\Gamma_i$ with center at $\lambda_i$, the projection

$$(2.5) \quad P_i = \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_i} (\lambda - A)^{-1} d\lambda$$
can be defined, and we get \( \sum_{i=1}^{\ell} p_i = p \). Let

\[
(2.6) \quad m_i = \dim p_i X \quad \text{and} \quad n_i = \dim \ker(\mathcal{L}_i - A).
\]

Then we have \( m = \sum_{i=1}^{\ell} m_i \) and \( n_i \leq m_i \) (Kato [5]).

Let \( \{ \varphi_{ij} \}_{1 \leq j \leq n_i} \) be a basis of \( \ker(\mathcal{L}_i - A) \) \( (1 \leq i \leq \ell) \).

The following proposition was shown by Pattorini [2], Sakawa [12, 13] and Triggiani [20], in the case that \( A \) is self-adjoint. We can prove it along the line of Suzuki [17, 18]. Its proof is given in [26]. See also Hautus [3]. Henceforth \( X^* \) means the dual space of \( X \):

**Proposition 2** (the second rank condition).

Let \( S : X \to \mathbb{F}^N \) be a sensor:

\[
(2.7) \quad S = T(S_1, S_2, \ldots, S_N) : X \to \mathbb{F}^N,
\]

\( S_k \in X^* \quad (1 \leq k \leq N) \).

Then \( S \) is \( X_0 \)-observable in \( X \) with respect to \( e^{-tA} \) if and only if

\[
(2.8) \quad \text{rank } M_i = n_i \quad (1 \leq i \leq \ell),
\]

where \( M_i \) is an \( N \times n_i \) matrix defined by

\[
(2.9) \quad M_i = ( x^* \langle S_k, \varphi_{ij} \rangle X )_{1 \leq k \leq N, 1 \leq j \leq n_i}.
\]
§3. A stabilization estimate.

In this section, we drive an estimate on \( \| e^{-t(A-B)} \|_{X \to X} \) which is useful later. We recall that \(-A\) is a generator of a \((C_0)\) semi-group in \(X\), satisfying Assumptions 1-3. The operator \(B = TS\) is a bounded linear operator on \(X\) of finite rank, and the equation

\[
(3.1) \quad \frac{du}{dt} + Au = Bu \quad (t \geq 0)
\]

represents the feedback system of (1.1). We set

\[
(3.2) \quad X_0 = PX, \quad X_1 = (1 - P)X,
\]

\[
(3.3) \quad A_0 = A|X_0, \quad A_1 = A|X_1
\]

and

\[
(3.4) \quad B_0 = PB|X_0.
\]

The operators \(A_0, B_0 : X_0 \to X_0\) are linear on the finite dimensional space \(X_0\), hence \(e^{-t(A_0 - B_0)}\) is well-defined. On the other hand, \(-A_1 : X_1 \to X_1\) generates a \((C_0)\) semi-group \(e^{-tA_1}\) on \(X_1\), which can be written as

\[
(3.5) \quad e^{-tA_1} = (1 - P)e^{-tA}|X_1.
\]
Therefore the estimate

\[(3.6) \quad \|e^{-tA_1}\|_{X_1 \to X_1} \leq M_1 e^{-t\kappa_1} \quad (t \geq 0)\]

holds true by (1.10). Here we further assume the following. There exists some constant \(\omega_0\) with \(\omega_0 > \kappa_1\) such that the estimate

\[(3.7) \quad \|e^{-t(A_0-B_0)}\|_{X_0 \to X_0} \leq M_0 e^{-t\omega_0} \quad (t \geq 0)\]

holds true. Then we can show the following (\(\subseteq 26\)).

**Theorem 3.**

Under the assumption stated above, the estimate

\[(3.8) \quad \|e^{-t(A-B)}\|_{X \to X} \leq M \exp(-t \left\{ \kappa_1 - M_1 \left( B_1 + \frac{\beta_1 P_0 M_0}{\omega_0 - \kappa_1} \right) \right\}) \]

holds for some constant \(M > 0\), where

\[(3.9) \quad \beta_0 = \|PB(1-P)\|_{X \to X} , \]
\[(3.10) \quad B_1 = \|(1-P)B(1-P)\|_{X \to X} \]
and

\[(3.11) \quad \beta_1 = \|(1-P)BP\|_{X \to X} . \]

\[\square\]
§4. Proof of Theorem 1.

Now we prove Theorem 1 stated in §1:

Firstly we suppose that there exists a controller $T$:

$$
\phi^N \rightarrow X \ 	ext{such that}
$$

(4.1) \[ \|e^{-t(A-TS)}\|_{X \rightarrow X} \leq Me^{-t\omega} \]

for some $\omega$ satisfying $\kappa_0 < \omega \leq \kappa_1$. We show that $S : X \rightarrow \mathbb{C}^N$ is $X_0$-observable with respect to $e^{-tA}$.

To this end, we assume

(4.2) \[ a \in X_0 \ 	ext{and} \ Se^{-tA}a = 0 \quad (0 \leq t < \infty), \]

and put $u(t) = e^{-tA}a$. Since $a \in X_0 \subseteq \mathcal{D}(A)$, $u = u(t)$ $C^1(0, \infty) \rightarrow X$ is the strong solution of

$$
\begin{cases}
\frac{du}{dt} + Au = 0 = T Su \quad (t \geq 0) \\
u(0) = a .
\end{cases}
$$

Therefore we may get

(4.3) \[ u(t) = e^{-t(A-TS)}a , \]

whence we have
\[(4.4) \quad \|u(t)\| \leq M e^{-t\omega} \|a\| \quad (t \geq 0)\]

by (4.1). On the other hand, in view of \( a \in X_0 \)
we have for some positive integers \( q_i \),

\[(4.5) \quad u(t) = e^{-tA_0} a = \sum_{i=1}^{\ell} e^{-t\lambda_i} \sum_{j=0}^{q_i-1} \frac{(-t)^j}{j!} D_i^j a \quad (t \geq 0). \]

We now substitute (4.5) into (4.4). Noting \( \Re \lambda_i \leq \kappa_0 < \omega \)
\((1 \leq i \leq \ell)\), we obtain \( D_i^j a = 0 \quad (1 \leq i \leq \ell, 0 \leq j \leq q_i-1) \),
and, in particular, \( a = Pa = 0 \).

Conversely we suppose that \( S \) is \( X_0 \)-observable. Then,

\[ \text{rank} \left( S_0^*, A_0 S_0^*, \ldots, (A_0^*)^{m-1} S_0^* \right) = m \]

by Proposition 1 of \( \S 2 \). By Wonham's pole assignment theorem
\(( \mathcal{23} \)\), there exists a linear operator \( T_0 : \mathfrak{N} \rightarrow X_0 \)
such that

\[ \inf \left\{ \Re \lambda ; \lambda \in \sigma(A_0 - T_0 S_0) \right\} > \kappa_1. \]

We then have

\[(4.6) \quad \|e^{-t(A_0-T_0 S_0)}\|_{X_0 \rightarrow X_0} \leq M_0 e^{-t\omega} \quad (t \geq 0)\]
for some constants $M_0 > 0$ and $\omega_0 > \kappa_1$. We put

\[ (4.7) \quad B = T_0 S : X \longrightarrow X. \]

Since $(1 - P)B = 0$, we can take $\alpha_1 = \beta_1 = 0$ in Theorem 3 of §3. Therefore we get

\[ (4.8) \quad \| e^{-t(A-B)} \| \leq Me^{-t\kappa_1} \quad (t \geq 0), \]

so that $< T_0, S >$ is $\kappa_1$-stabilizable with respect to $e^{-tA}$. 

\[ \square \]

§5. Duality between observability and controllability.

The duality between observability and controllability is well-known (Dolecki and Russell). We here show it as a version concerning "$Y$-observability" and "$Y$-controllability".

For a closed linear subspace $Y \subseteq X$, $Y^*$ and $Y^\perp$ denotes the dual space of $Y$ and the orthogonal complement of $Y$, respectively;

\[ Y^* = \{ g ; \text{ g is a bounded linear functional on } Y \}. \]

\[ Y^\perp = \{ g \in X^* ; g|_Y = 0 \}. \]

We recall
\[
P = \frac{1}{2\pi\sqrt{-1}} \int_{\mathcal{P}} (\lambda - A)^{-1} \, d\lambda,
\]

\[X_0 = PX\]

and

\[X_1 = (1 - P)X.\]

From the direct sum decomposition

(5.1) \[X = X_0 \oplus X_1,\]

we have the isomorphisms

(5.2) \[x_1^\perp \cong x_0^*, \quad \text{and} \quad x_0^\perp \cong x_1^*.\]

In particular, we have \(\dim x_1^\perp = \dim x_0^* = m < \infty.\)

Also, we have

(5.3) \[x_1^\perp = P^*x^*, \quad x_0^\perp = (1 - P^*)x^*,\]

(5.4) \[P^* = \frac{1}{2\pi\sqrt{-1}} \int_{\mathcal{P}} (\lambda - A^*)^{-1} \, d\lambda\]

and

(5.5) \[x^* = x_1^\perp \oplus x_0^\perp.\]
For the operator $A_0 = A|_{X_0}$, we denote its dual operator by $A_0^\star$. Then $e^{-tA_0^\star} : X_0^\star \rightarrow X_0^\star$ is defined and we have

\[(5.6) \quad (e^{-tA})^\star \mid_{X_1} = e^{-tA_0^\star} \quad \text{(by } X_1 \cong X_0^\star \text{)}.
\]

Also, for a controller $T$, we define the operator $T_0^\star : X_0^\star \rightarrow \mathcal{C}^N$ such that

\[(5.7) \quad T_0^\star \cong T^\star \mid_{X_1} \quad \text{(by } X_0^\star \cong X_1 \text{)}.
\]

Then we have the followings (§ 26 §).

**Lemma 2.**

$T : \mathcal{C}^N \rightarrow X$ is $X_0$-controllable in $X$ with respect to $e^{-tA}$ if and only if $T_0^\star : X_0^\star \rightarrow \mathcal{C}^N$ is $X_0^\star$-observable in $X_0^\star$ with respect to $e^{-tA_0^\star}$. □

It is known that $\dim \text{Ker}(\lambda_i - A^\star) = \dim \text{Ker}(\lambda_i - A) = n_i$ holds. (Kato § 5 §, for example.) Let $\{\varphi_{ij}^\star\}_{1 \leq i \leq n_1}$ be a basis of $\text{Ker}(\lambda_i - A^\star)$.

**Proposition 3** (the second rank condition for controllability).

Let $T : \mathcal{C}^N \rightarrow X$ be a controller:

\[(5.8) \quad T : \mathcal{C}^N \rightarrow X
\]

\[\omega \downarrow
\]

\[(\alpha_k)_{1 \leq k \leq N} \rightarrow \sum_{k=1}^{N} \alpha_k \psi_k , \quad \psi_k \in X_0^\star.
\]
Then, \( T \) is \( X_0 \)-controllable in \( X \) with respect to \( e^{-tA} \) if and only if

\[
\text{rank } L_i = n_i \quad (1 \leq i \leq l),
\]

where \( L_i \) is an \( N \times n_i \) matrix defined by

\[
L_i = \begin{pmatrix} x^* & \phi^*_{ij} & \psi_k \end{pmatrix} \quad \begin{pmatrix} 1 \leq k \leq N \\ 1 \leq j \leq n_i \end{pmatrix}.
\]

\[\square\]

§6. **Proof of Theorem 2.**

We now prove Theorem 2 stated in §1.

**Proof of the "if" part of Theorem 2:**

We assume that \( T : c^N \rightarrow X \) is \( X_0 \)-controllable in \( X \) with respect to \( e^{-tA} \). By Lemma 2 in §5, \( T_0^* : X_0^* \rightarrow c^N \) is \( X_0^* \)-observable in \( X_0^* \) with respect to \( e^{-tA_0^*} \). By Proposition 1 in §2 and Wonham's pole assignment theorem, there exists a linear operator \( F_0 : c^N \rightarrow X_0^* \) such that

\[
e^{-t(A_0^* - F_0^*t^*)} \quad x_0^* \rightarrow x_0^* = M_0 e^{-t\omega_0}
\]

for some constants \( M_0 > 0 \) and \( \omega_0 > \kappa'_1 \).

Let \( S_0 : X_0 \rightarrow c^N \) be the dual operator of \( F_0 \). Then, since \( \dim X_0 = m < \infty \), (6.1) implies
\begin{equation}
\| e^{-t(A_0-PTS_0)} \|_{X_0 \rightarrow X_0} \leq M_0 e^{-t\omega_0}.
\end{equation}

We set
\begin{equation}
S = \begin{cases} 
S_0 & \text{(on } X_0 \text{)} \\
0 & \text{(on } X_1 \text{)},
\end{cases}
\end{equation}
and apply Theorem 3 in §3 for $B = TS$. Since $F_0 = F_1 = 0$ by (6.3), we then have
\begin{equation}
\| e^{-t(A-TS)} \|_{X \rightarrow X} \leq e^{-t\mathcal{H}_1}
\end{equation}
for some constant $M > 0$, which shows that $< T, S >$ is $\mathcal{H}_1$-stabilizable in $X$ with respect to $e^{-tA}$.

Proof of the "only if" part of Theorem 2:
Here we show the "only if" part in the case that $X$ is reflexive. For the general Banach space $X$, the proof is given in [26].
We assume that $< T, S >$ is $\mathcal{C}_0$-stabilizable with respect to $e^{-tA}$ for $\mathcal{H}_0 < \mathcal{C}_0 \leq \mathcal{H}_1$. Then we have
\begin{equation}
\| e^{-t(A-TS)} \|_{X \rightarrow X} \leq Me^{-t\omega}
\end{equation}
for some constant $M > 0$.

The reflexivity of $X$ implies
\begin{align*}
(e^{-tA})^* &= e^{-tA^*} \\
(e^{-t(A-TS)})^* &= e^{-t(A-TS^*)} & (R.S. Phillips [11]).
\end{align*}
Therefore we get

\[(6.6) \quad \| e^{-t(A^* - S^* T^*)} \|_{X^* \to X^*} \leq M e^{-t \omega} \quad (t \geq 0). \]

We show that \( T \) is \( X_0 \)-controllable with respect to \( e^{-tA} \).
To this end, we have only to show (6.7);

\[(6.7) \quad T e^{-tA} a = 0 \quad (0 \leq t < \infty) \quad \text{and} \quad a \in X_1^\perp \quad \text{imply} \quad a = 0. \]

We put \( u(t) = e^{-tA} a \). Since \( a \in X_1^\perp \subseteq \mathcal{D}(A^*) \), \( u = u(t) \in C^1([0, \infty) \to X) \) is the strong solution of

\[
\begin{aligned}
\frac{du}{dt} + A^* u &= 0 = S^* T^* u \quad (t \geq 0) \\
u(0) &= a
\end{aligned}
\]

Therefore we get \( u(t) = e^{-t(A^* - S^* T^*)} a \) \( a \) and

\[(6.8) \quad \| e^{-tA^*} a \|_{X^*} \leq M e^{-t \omega} \| a \|_{X^*} \quad (t \geq 0). \quad \text{(by (6.6)).} \]

Now by (6.8), \( a \in X_1^\perp \) and \( \Re \lambda_i \leq \kappa_0 < \omega \), we see \( a = 0 \).
Appendix. Example:

Here we apply Theorem 1 to the following stabilization problem.

\[
\begin{aligned}
X &= L^2(0, \pi) \\
Au &= -\frac{d^2u}{dx^2} - 5u \\
\Theta(A) &= H^2(0, \pi) \cap H^1_0(0, \pi)
\end{aligned}
\]

In Assumption 1, we set

\[
\sum_0 = \{-4, -1\} \quad \text{and} \quad \sum_1 = \{n^2 - 5\} \quad n=3.
\]

Then we have

\[
X_0 = \text{Span}\{\sin x, \sin 2x\}.
\]

1) We define \( S : X \longrightarrow \mathbb{C} \) by

\[
Su = \int_0^\pi u(x)dx.
\]

Then by Proposition 2, this sensor \( S \) is \( X_0 \)-observable with respect to \( e^{-tA} \). By Theorem 1, we define \( T : \mathbb{C} \longrightarrow X \) by

\[
T(\alpha) = \alpha(35\sin x - 17\sin 2x),
\]

for example. Then this feedback \( \langle T, S \rangle \) is 3-feedback stabilizable in \( X \) with respect to \( e^{-tA} \).
2) We define $S : X \rightarrow \Phi^2$ by

$$Su = \left( \int_0^\pi u(x)dx, \frac{2}{\pi} \int_0^\pi u(x) xdx \right).$$

Then by Proposition 2, this sensor $S$ is $X_0$-observable. By Theorem 1, we define $T : \Phi^2 \rightarrow X$ by

$$T(\alpha_1, \alpha_2) = \alpha_1\left( \frac{11}{2} \sin x + 13 \sin 2x \right) + \alpha_2\left( \frac{1}{2} \sin x - 3 \sin 2x \right),$$

for example. Then this feedback $\langle T, S \rangle$ is 3-feedback stabilizable. \(\square\)
References


[10] T.Nambu, Feedback stabilization for distributed parameter systems of parabolic type, II,


Addendum, Ibid., 56 (1976), 492.


Department of Mathematics
Faculty of Science
University of Tokyo
Hongo Tokyo
113 Japan.