

ON THE STRUCTURE OF SOLUTIONS  
TO THE SELF-DUAL YANG-MILLS EQUATIONS

Kanehisa TAKASAKI (高崎金之)

RIMS, Kyoto Univ. and Dept. Math., Univ. Tokyo

1. Set-up

Let  $(y, z, \bar{y}, \bar{z})$  denote the coordinates of  $\mathbb{C}^4$ . The (complexified) self-dual Yang-Mills equations we here discuss are written in the following form

$$(1) \quad \begin{aligned} \partial_y A_{\bar{z}} - \partial_{\bar{z}} A_y + [A_y, A_{\bar{z}}] &= 0, & \partial_{\bar{y}} A_{\bar{z}} - \partial_{\bar{z}} A_{\bar{y}} + [A_{\bar{y}}, A_{\bar{z}}] &= 0, \\ \partial_y A_{\bar{y}} - \partial_{\bar{y}} A_y + [A_y, A_{\bar{y}}] + \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z + [A_z, A_{\bar{z}}] &= 0, \end{aligned}$$

where  $\partial_u$ ,  $u = y, z, \bar{y}, \bar{z}$ , denote the derivations  $\partial/\partial u$ ,  $[ , ]$  the commutator  $[A, B] = AB - BA$ , and  $A_u$ ,  $u = y, z, \bar{y}, \bar{z}$ ,  $r \times r$  matrix-valued unknown functions ( $r \geq 2$ ). Our main interest is in their formal power series solutions  $A_u \in \mathfrak{gl}(r, \mathbb{C}[[y, z, \bar{y}, \bar{z}]])$ , where  $\mathfrak{gl}(r, \mathbb{C}[[y, z, \bar{y}, \bar{z}]])$  denotes the set of all  $r \times r$  matrices of

formal power series of  $(y, z, \bar{y}, \bar{z})$  with complex coefficients.

$A_y$  and  $A_z$ , however, may be eliminated from eqs. (1) by performing an appropriate gauge transformation  $A_u \rightarrow g^{-1} A_u g + g^{-1} \partial_u g$  (where  $g$  is an invertible element of  $gl(r, \mathbb{C}[[y, z, \bar{y}, \bar{z}]])$ ). Therefore we may discuss instead of eqs. (1) the following equations and their formal power series solutions.

$$(2) \quad \partial_{\bar{y}} A_{\bar{z}} - \partial_{\bar{z}} A_{\bar{y}} + [A_{\bar{y}}, A_{\bar{z}}] = 0, \quad \partial_y A_{\bar{y}} + \partial_z A_{\bar{z}} = 0.$$

## 2. Construction of solutions — general case

We here discuss how to construct all the formal power series solutions to eqs. (2). Our construction presented below may be thought of as a modification of the method of Sato [1], which originally covered only the case of soliton equations.

Our construction starts with introducing the following equations with  $r \times r$  matrix-valued unknown functions  $\xi_{ij}$ ,  $-\infty < i < \infty$ ,  $j < 0$ .

$$(3) \quad \begin{cases} -\partial_y \xi_{i+1,j} + \partial_{\bar{z}} \xi_{ij} + \xi_{i,-1} \partial_y \xi_{0,j} = 0, & \partial_z \xi_{i+1,j} + \partial_{\bar{y}} \xi_{ij} - \xi_{i,-1} \partial_z \xi_{0,j} = 0, \\ \xi_{i+1,j} = \xi_{i,j-1} + \xi_{i,-1} \xi_{0,j} & \text{for } -\infty < i < \infty, j < 0, \\ \xi_{ij} = \delta_{ij} 1_r & \text{for } i < 0, j < 0. \end{cases}$$

Here  $\delta_{ij}$  denotes the Kronecker delta, and  $1_r$  the  $r \times r$  unit matrix. Eqs. (2) and eqs. (3) are related to each other as follows.

Proposition (4) For any formal power series solution  $\xi_{ij} \in \text{gl}(r, \mathbb{C}[[y, \bar{z}, \bar{y}, \bar{z}]])$  to eqs. (3), the  $r \times r$  matrices

$$A_{\bar{y}} = \partial_z \xi_{0,-1}, \quad A_{\bar{z}} = -\partial_y \xi_{0,-1}$$

give a formal power series solution to eqs. (2), and any formal power series solution to eqs. (2) may be obtained in this way.

Because of this the problem now reduces to the construction of formal power series solutions to eqs. (3).

To construct these solutions, we utilize the framework of the initial value problem of eqs. (3) with initial conditions

$$(5) \quad \xi_{ij} |_{\bar{y}=\bar{z}=0} = \xi_{ij}^{(0)}.$$

Of course the initial values  $\xi_{ij}^{(0)} \in \text{gl}(r, \mathbb{C}[[y, z]])$  are assumed in advance to satisfy the following conditions

$$(6) \quad \begin{cases} \xi_{i+1,j}^{(0)} = \xi_{i,j-1}^{(0)} + \xi_{i,-1}^{(0)} \xi_{0,j}^{(0)} & \text{for } -\infty < i < \infty, j < 0, \\ \xi_{ij}^{(0)} = \delta_{ij} 1_r & \text{for } i < 0, j < 0. \end{cases}$$

As we shall see later,  $\xi_{0,j}^{(0)}$ ,  $j < 0$ , may be arbitrarily given, and they uniquely determine  $\xi_{ij}^{(0)}$ ,  $i > 0, j < 0$ . Now the problem is how to describe the evolution  $\xi_{ij}^{(0)} \rightarrow \xi_{ij}$ .

The evolution  $\xi_{ij}^{(0)} \rightarrow \xi_{ij}$  may be specified in terms of the  $\infty \times \infty$  matrices

$$\xi = (\xi_{ij})_{\substack{-\infty < i < \infty \\ -\infty < j < 0}} = \begin{pmatrix} \dots & \dots & \dots \\ \dots & \xi_{-2-2} & \xi_{-2-1} \\ \dots & \xi_{-1-2} & \xi_{-1-1} \\ \dots & \xi_{0-2} & \xi_{0-1} \\ \dots & \dots & \dots \end{pmatrix}, \quad \xi^{(0)} = (\xi_{ij}^{(0)})_{\substack{-\infty < i < \infty \\ -\infty < j < 0}}.$$

First we define another  $\infty \times \infty$  matrix  $\tilde{\xi} = (\tilde{\xi}_{ij})_{\substack{-\infty < i < \infty \\ -\infty < j < 0}}$ ,  $\tilde{\xi}_{ij} \in \mathfrak{gl}(r, \mathbb{C}[[y, z, \bar{y}, \bar{z}]])$ , by the following formula

$$(7) \quad \tilde{\xi} = \exp(\bar{z}\Lambda\partial_y - \bar{y}\Lambda\partial_z) \xi^{(0)} = \sum_{k=0}^{\infty} (\bar{z}\Lambda\partial_y - \bar{y}\Lambda\partial_z)^k \xi^{(0)} / k!,$$

$$\text{where } \Lambda = (\delta_{i,j-1} 1_r)_{-\infty < i, j < \infty} = \begin{pmatrix} \dots & \dots & \dots \\ \dots & 0_r & 1_r \\ \dots & \dots & 0_r & 1_r \\ \dots & \dots & \dots & \dots \end{pmatrix} \quad (0_r \text{ denotes the}$$

$r \times r$  null matrix). Then the "upper half part"  $\tilde{\xi}_{(-)} \equiv (\tilde{\xi}_{ij})_{-\infty < i, j < 0}$  of  $\tilde{\xi}$  is invertible, the inverse  $\tilde{\xi}_{(-)}^{-1}$  becomes again an  $\infty \times \infty$  matrix of elements of  $\mathfrak{gl}(r, \mathbb{C}[[y, z, \bar{y}, \bar{z}]])$ , and also the product  $\tilde{\xi} \tilde{\xi}_{(-)}^{-1}$  may be defined to be an  $\infty \times \infty$  matrix of elements of  $\mathfrak{gl}(r, \mathbb{C}[[y, z, \bar{y}, \bar{z}]])$ . (Of course these statements require careful justification, though we omit it here.) Finally,

Theorem (8) The components  $\xi_{ij} \in \mathfrak{gl}(r, \mathbb{C}[[y, z, \bar{y}, \bar{z}]])$  of the  $\infty \times \infty$  matrix  $\xi \equiv \tilde{\xi} \tilde{\xi}_{(-)}^{-1}$  satisfy eqs. (4) and initial conditions (5).

### 3. Relation to the inverse scattering formulation

The ordinary inverse scattering formulation of the self-dual Yang-Mills equations may be derived from the above construction of solutions. To see this, we introduce the following equations with  $r \times r$  matrix-valued unknown functions  $W_j, j \geq 0$ .

$$(9) \begin{cases} -\partial_y W_{j+1} + \partial_{\bar{z}} W_j + (\partial_y W_1) W_j = 0, & \partial_{\bar{z}} W_{j+1} + \partial_y W_j - (\partial_{\bar{z}} W_1) W_j = 0 \\ \text{for } j \geq 0, & W_0 = 1_r. \end{cases}$$

These equations are combined with eqs. (3) as follows.

Proposition (10) Eqs. (3) and eqs. (9) are equivalent to each other under the relation

$$W_j = -\xi_{0, -j}, \quad j < 0.$$

In particular this defines a one-to-one correspondence of formal power series solutions between eqs. (3) and eqs. (9).

Now let us introduce another independent variable  $\lambda$ , and try

to rewrite eqs. (9) in terms of the formal power series  $W \equiv \sum_{j=0}^{\infty} W_j \lambda^{-j} \in \mathfrak{gl}(r, \mathbb{C}[[\gamma, z, \bar{\gamma}, \bar{z}, \lambda^{-1}]])$ . In fact we can rewrite eqs. (9) into the following system

$$(11) \quad (-\lambda \partial_{\gamma} + \partial_{\bar{z}} + (\partial_{\gamma} W_1)) W = 0, \quad (\lambda \partial_z + \partial_{\bar{\gamma}} - (\partial_z W_1)) W = 0,$$

which, in view of the relation  $A_{\bar{\gamma}} = \partial_z \xi_{0,-1} = -\partial_z W_1$  and  $A_{\bar{z}} = -\partial_{\gamma} \xi_{0,-1} = \partial_{\gamma} W_1$ , is nothing but the "linear problem" of the self-dual Yang-Mills equations (see [2]).

The algebraic structure of the correspondence between  $W_j$  and  $\xi_{ij}$  as above may be more explicitly described. In fact, the following formulas may be derived.

$$(12) \quad \xi_{ij} = \sum_{k=0}^{-j-1} W_{i-j-k}^* W_k = -\sum_{k=-j}^{i-j} W_{i-j-k}^* W_k, \quad i \geq 0, j < 0,$$

$$(13) \quad (\lambda - \mu) \sum_{i,j=1}^{\infty} \xi_{i-1,-j} \mu^{-i} \lambda^{-j} = \left( \sum_{i=0}^{\infty} W_i^* \mu^{-i} \right) \left( \sum_{j=0}^{\infty} W_j \lambda^{-j} \right) - 1_r,$$

$$(14) \quad \sum_{i=1}^{\infty} \xi_{i-1,-j} \mu^{-i} = \left( \sum_{i=0}^{\infty} W_i^* \mu^{-i} \right) \left( \sum_{k=0}^{j-1} W_k \mu^{j-k-1} \right) - \mu^{j-1} 1_r,$$

$$(15) \quad \sum_{j=1}^{\infty} \xi_{i-1,-j} \lambda^{-j} = \lambda^{i-1} 1_r - \left( \sum_{k=0}^{i-1} W_k^* \lambda^{i-k-1} \right) \left( \sum_{j=0}^{\infty} W_j \lambda^{-j} \right),$$

where  $W_j^*$ ,  $j \geq 0$ , denote the coefficients of  $W^{-1} = \sum_{j=0}^{\infty} W_j^* \lambda^{-j}$  (e.g.  $W_0^* = W_0 = 1_r$ ,  $W_1^* = -W_1$ ,  $W_2^* = W_1^2 - W_2$ , ...), and  $\lambda$  and  $\mu$  are independent variables. These formulas may be derived from

only the last two equations in (3), and therefore they are also valid for the initial values  $\xi_{ij}^{(0)}$  and  $W_j^{(0)} = W_j|_{\bar{y}=\bar{z}=0}$ . This especially means that the initial values of eqs. (3) and eqs. (9) at  $\bar{y}=\bar{z}=0$  uniquely determine each other under the relation  $\xi_{0j}^{(0)} = -W_{-j}^{(0)}$ ,  $j < 0$ , and that  $\xi_{ij}^{(0)}$ ,  $i > 0, j < 0$ , may be calculated from  $W_j^{(0)}$  by means of formula (12).

#### 4. Construction of solutions — special case

We here discuss how the construction presented in Sect. 2 is simplified in the case where

$$(16) \quad W_j = 0 \quad \text{for } j > m$$

for some positive integer  $m$ . Formula (12) shows that condition (16) is equivalent to the following condition

$$(17) \quad \xi_{ij} = 0 \quad \text{for } j > m.$$

These conditions are preserved under the evolution  $\xi_{ij}^{(0)} \rightarrow \xi_{ij}$ ,  $W_j^{(0)} \rightarrow W_j$ . Namely,

Proposition(18) If condition (12) (or (13)) is satisfied when  $\bar{y}=\bar{z}=0$ , then it is satisfied everywhere.

Let us perform the construction of Sect. 3 in the above case.

First, because of the assumption that  $\xi_{ij}^{(0)} = 0$  for  $j > m$ ,  $\xi$  takes the following form

$$(19) \quad \xi = \left( \begin{array}{c|c} \begin{array}{c} \dots \\ 1_r \\ \dots \\ 1_r \end{array} & * \\ \hline 0 & (\tilde{\xi}_{ij})_{\substack{-m \leq i < \infty \\ -m \leq j < 0}} \end{array} \right).$$

Also it is not hard to show that

$$(20) \quad (\tilde{\xi}_{ij})_{\substack{-m \leq i < \infty \\ -m \leq j < 0}} = \exp(\bar{z} \Lambda_{[m]} \partial_y - \bar{y} \Lambda_{[m]} \partial_z) (\xi_{ij}^{(0)})_{\substack{-m \leq i < \infty \\ -m \leq j < 0}}$$

where  $\Lambda_{[m]} = (\delta_{i,j-1} 1_r)_{-m \leq i, j < \infty} = \begin{pmatrix} 0_r & 1_r \\ & 0_r & 1_r \\ & & \dots \end{pmatrix}$ . The inverse of  $\sum_{\Gamma} \xi$

$= (\tilde{\xi}_{ij})_{-\infty < i, j < 0}$  then takes the following form

$$(21) \quad \xi_{(-)}^{-1} = \left( \begin{array}{c|c} \begin{array}{c} \dots \\ 1_r \\ \dots \\ 1_r \end{array} & ** \\ \hline 0 & (\tilde{\xi}_{ij})_{-m \leq i, j < 0}^{-1} \end{array} \right).$$

Therefore

$$(22) \quad \xi = \left( \begin{array}{c|c} \begin{array}{c} \dots \\ 1_r \\ \dots \\ 1_r \end{array} & 0 \\ \hline 0 & (\tilde{\xi}_{ij})_{\substack{-m \leq i < \infty \\ -m \leq j < 0}} (\tilde{\xi}_{ij})_{-m \leq i, j < 0}^{-1} \end{array} \right).$$

In particular this proves Proposition (18) (though Proposition (18) may be proved in a more direct way).

In the above formulas only the blocks at the lower right survive in the final result, and therefore we may reformulate the whole construction in terms of these blocks. In fact, the  $\infty \times r m$  matrices

$$\begin{aligned} \xi_{[m]}^{(0)} &\equiv (\xi_{ij}^{(0)})_{\substack{-m \leq i < \infty \\ -m \leq j < 0}}, & \xi_{[m]} &\equiv (\xi_{ij})_{\substack{-m \leq i < \infty \\ -m \leq j < 0}}, \\ \tilde{\xi}_{[m]} &\equiv (\tilde{\xi}_{ij})_{\substack{-m \leq i < \infty \\ -m \leq j < 0}}, & \tilde{\xi}_{[m](-)} &\equiv (\tilde{\xi}_{ij})_{-m \leq i, j < 0}, \\ & & & \text{(this is an } r m \times r m \text{ matrix)} \end{aligned}$$

satisfy the following formulas

$$\begin{aligned} \tilde{\xi}_{[m]} &= \exp(\bar{z} \Lambda_{[m]} \partial_y - \bar{y} \Lambda_{[m]} \partial_z) \xi_{[m]}^{(0)}, \\ \xi_{[m]} &= \tilde{\xi}_{[m]} \tilde{\xi}_{[m](-)}^{-1}, \end{aligned}$$

and they provide analogues of (7) and (8). Note that these  $\infty \times r m$  matrices play just the same role as those in [3]. In [3], however, the viewpoint of the initial value problem was lacked. Because of this the solutions explicitly constructed there were limited to fairly special ones, and the structure of other solutions remained to be clarified. The above construction shows how to explicitly construct the full family of solutions contained in the framework of [3]. Also note that the above construction is valid not only in the realm of formal power series solutions, but also in other cases such as the case of rational solutions.

### 5. Construction of solutions — more special case

We here discuss a more special case, namely the case where in addition to condition (16) the remaining ones  $W_j$ ,  $1 \leq j \leq m$ , are assumed to be rational with respect to  $(y, z, \bar{y}, \bar{z})$ . Also in this case these solutions are characterized by the initial values.

**Proposition (23)** If  $W_j^{(0)} = 0$  for  $j > m$  and  $W_j^{(0)}$ ,  $1 \leq j \leq m$ , are rational with respect to  $(y, z)$ , then  $W_j = 0$  for  $j > m$  and  $W_j$ ,  $1 \leq j \leq m$ , are rational with respect to  $(y, z, \bar{y}, \bar{z})$ .

This proposition may be proved as follows. By virtue of the calculations in Sect. 4 we have only to show that  $\tilde{\xi}_{ij}$  are rational with respect to  $(y, z, \bar{y}, \bar{z})$ . To show this, let us note the following formula which may be immediately derived from formula (7).

$$\begin{aligned}
 (24) \quad \sum_{i=-\infty}^{\infty} \tilde{\xi}_{ij} \mu^{-i} &= \exp(\bar{z} \mu \partial_y - \bar{y} \mu \partial_z) \sum_{i=-\infty}^{\infty} \xi_{ij}^{(0)} \mu^{-i} \\
 &= \mu^{-j} 1_r + \exp(\bar{z} \mu \partial_y - \bar{y} \mu \partial_z) \sum_{i=0}^{\infty} \xi_{ij}^{(0)} \mu^{-i}.
 \end{aligned}$$

Formula (14) may be applied here to the right hand to conclude that the right hand side of (24) is rational with respect to  $(y, z, \bar{y}, \bar{z}, \mu)$ . Therefore the rationality of  $\tilde{\xi}_{ij}$  follows, and

this completes the proof of the proposition.

The class of solutions discussed above is of special interest because, as implicitly shown in [4], it in principle contains all the instanton solutions, though regretfully I have not been able to find out any effective characterization of the instanton solutions in the framework of the above construction. I expect further developments of our calculation will lead in the future to some good prospects for this problem.

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  5. Full details on the contents of Sect. 1 - Sect. 3 of the present note are presented in:  
K. Takasaki: A new approach to the self-dual Yang-Mills equations. Preprint of RIMS, Kyoto Univ., no. 459. To appear in Commun. Math. Phys.