数理解析研究所講究録
533巻 1984年 178-187

Sheaf Theoretic L^2-Cohomology

Masayoshi Nagase
(東工大・理 長瀬正義)

If M is a compact manifold, then we have the famous de Rham isomorphism ; $H_i(M) \cong \mathrm{Hom}(H^i_{DR}(M), \mathbb{R})$. Our purpose here is to generalize this isomorphism to the so-called Thom-Mather's stratified space. More explicitly, we aim to show that, exchange the simplicial homology for the Goresky-MacPherson's intersection homology and the de Rham cohomology for the L^2-cohomology of the non-singular part, then the isomorphism is still valid for such a situation.

We discussed this subject at this research institute two years ago ([4]). At that time, I constructed the isomorphism directly. Therefore, at this time, I will select the other plan of proof, in which we will pay little attention to how to construct the isomorphic map. That is, according to Goresky and MacPherson ([3]), I will show the isomorphism axiomatically by sheaf theoretic method.

§1. L^2-cohomology and intersection homology : Main Theorem

From now on, X^n is an n-dimensional compact stratified space without boundary. We will fix a stratification

$$X = X_n \supset X_{n-1} = X_{n-2} (= \Sigma) \supset X_{n-3} \supset \cdots\cdots \supset X_1 \supset X_0 ,$$

and the tubular neighborhood system and, moreover, the

PL-structure compatible with these structures.

Let g be a metric on $X-\Sigma$, and let d_i be the exterior derivative on $X-\Sigma$ with domain

$$\operatorname{dom} d_i = \left\{ \omega \in \Lambda^i(X-\Sigma) \cap L^2\Lambda^i(X-\Sigma) \;\middle|\; d\omega \in L^2\Lambda^{i+1}(X-\Sigma) \right\}.$$

The i-th cohomology group of the cochain complex $\{\operatorname{dom} d_i\}$ is called the i-th <u>L^2-cohomology group</u>, denoted by $H^i_{(2)}(X-\Sigma)$.

Next, taking account of the PL-structure of X, let's define the intersection homology. Let $\bar{p} = (p_2, p_3, \ldots, p_n)$ be a perversity, i.e., a sequence of non-negative integers satisfying $p_2 = 0$ and $p_k \leq p_{k+1} \leq p_k + 1$ for all k. The perversities which are of particular importance are as follows:

$\bar{0} = (0,\ldots,0)$, the zero perversity,

$\bar{m} = (0,0,1,1,2,2,\ldots.)$, $m_k = [\frac{k}{2}]-1$, the (lower) middle perversity,

$\bar{t} = (0,1,2,3,\ldots.)$, the top perversity.

By the way, $\bar{p} \leq \bar{q}$ means that $p_k \leq q_k$ for all k. And we set

$$\bar{p} + \bar{q} = (p_2 + q_2,\ p_3 + q_3, \ldots\ldots).$$

The perversity $\bar{q}$ is said to be the complementary perversity of $\bar{p}$ if $\bar{p} + \bar{q} = \bar{t}$. Then, take an integer i. A subspace Y of X is called $(\bar{p},i)$-allowable if $\dim Y \leq i$ and $\dim(Y \cap X_{n-k}) \leq i - k + p_k$ for all k. For example, that Y is $(\bar{0}, \dim Y)$-allowable means that Y and the strata are in general position. Now, let's set

$$IC_i^{\bar{p}}(X) = \left\{ \xi \in C_i(X) \;\middle|\; \begin{array}{l} |\xi| \text{ is } (\bar{p},i)\text{-allowable and} \\ |\partial\xi| \text{ is } (\bar{p},i-1)\text{-allowable.} \end{array} \right\}.$$

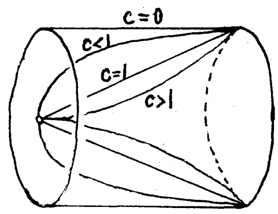
Then the i-th homology group of the chain complex $\{IC_i^{\bar{p}}(X)\}$ is called the i-th <u>intersection homology group</u> with $\bar{p}$ and denoted by $IH_i^{\bar{p}}(X)$.

Now we may remark that the perversities which are interesting here or which we want to treat here are the perversities which are smaller than the middle perversity, i.e., $\bar{p} \leq \bar{m}$. This restriction is not essential; see the remark following Definition 1.1.

Here, once again, we will return to the L^2-cohomology and define the metric associated to a given perversity $\bar{p}$ and then announce the main theorem explicitly.

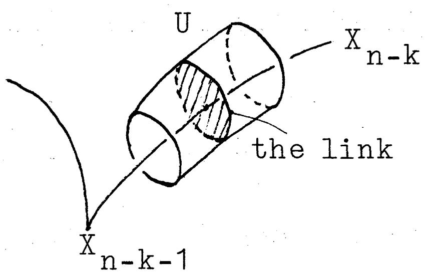
Let Y be a Riemannian manifold with metric g and let's take $c \geq 0$. Then we set

$$C^c(Y) = \text{" the Riemannian manifold } (0,1)\times Y \text{ with metric } dr\otimes dr + r^{2c}g \text{ ".}$$



Now, fix a sequence of non-negative real numbers $\bar{c} = (c_2, \ldots, c_n)$. Then the metic g on $X-\Sigma$ is said to be associated to $\bar{c}$ if, for any point x of any non-empty stratum $X_{n-k} - X_{n-k-1}$, there exists a neighborhood $x \in U \subset X$ such that

$$U \cap (X-\Sigma) \underset{\text{quasi-isometry}}{\sim} C^{c_k}(\,(\text{the link of } X_{n-k}-X_{n-k-1}) \cap (X-\Sigma) \text{ with } g\,) \underset{\text{product}}{\times} (\,U \cap (X_{n-k}-X_{n-k-1}) \text{ with Euclidean metric }).$$



Definition 1.1. The metric g on $X-\Sigma$ is said to be associated to the perversity $\bar{p}\,(\leq \bar{m})$ if g is associated to $\bar{c}=(c_2,\ldots,c_n)$:

$$\begin{cases} (k-1-2p_k)^{-1} \leq c_k < (k-3-2p_k)^{-1} & ;\ 2p_k \leq k-3, \\ 1 \leq c_k < \infty & ;\ 2p_k = k-2. \end{cases}$$

If we want to treat the perversities which are larger than $\bar{m}$ or which cannot be comparable with $\bar{m}$, it will suffice to change (certain) c_k's into suitable negative numbers. By the way, it is noteworthy that, if $1 \leq c_k < \infty$ for all k and the metric g is associated to $\bar{c}$, then g is associated to the important perversity $\bar{m}$: this case was treated by J. Cheeger ([1]).

We will use the notation $(X-\Sigma)_{\bar{p}}$ in order to express clearly that the metric on $X-\Sigma$ under consideration is associated to $\bar{p}$. Then we can now announce

Main Theorem. If $\bar{p} \leq \bar{m}$, then

$$IH_i^{\bar{p}}(X) \cong \mathrm{Hom}(H^i_{(2)}((X-\Sigma)_{\bar{p}}, \mathbb{R})).$$

The following two sections are the preparations for the proof of Main Theorem from the view point of sheaf theoretic method.

§2. Sheaf theoretic L^2-cohomology and intersection homology

As above, the non-singular part $X-\Sigma$ is endowed with metric g. Let $\Omega^{\cdot}$ be the complex of sheaves on X which is defined by

$$\Gamma(U,\Omega^i) = \left\{ \omega \in \Lambda^i(U\cap(X-\Sigma)) \;\middle|\; \begin{array}{l}\text{For any point } x \text{ of } U, \text{ there} \\ \text{exists a neighborhood } x\in V\subset U \\ \text{such that} \\ \int_{V\cap(X-\Sigma)} \omega\wedge *\omega < \infty , \\ \int_{V\cap(X-\Sigma)} d\omega\wedge *d\omega < \infty . \end{array} \right\}$$

with the sheaf maps $d: \Omega^i \longrightarrow \Omega^{i+1}$ induced by the exterior derivative on $X-\Sigma$. In order to indicate that the metric g is associated to the given perversity $\bar{p}$, we will use the notation $\Omega^{\cdot}_{\bar{p}}$.

Next, paying attention to the PL-structure of X, we will define the complex of sheaves $\mathcal{IC}^{\cdot}_{\bar{p}}$. Before it, define the sheaf $\mathcal{C}_i$ by

$$\Gamma(U,\mathcal{C}_i) = \text{" the group of locally finite i-dimensional simplicial chains with respect to the induced PL-structure of U "}.$$

For convenience, we set $\mathcal{C}^{\cdot} = \mathcal{C}_{-\cdot}$ and regard this as a complex of sheaves: the sheaf maps are induced by the simplicial boundary operator. Then we define its subcomplex $\mathcal{IC}^{\cdot}_{\bar{p}}$ by

$$\Gamma(U,\mathcal{IC}^{-i}_{\bar{p}}) = \left\{ \xi\in\Gamma(U,\mathcal{C}^{-i}) \;\middle|\; \begin{array}{l} |\xi| \text{ is } (\bar{p},i)\text{-allowable and } |\partial\xi| \\ \text{is } (\bar{p},i-1)\text{-allowable with} \\ \text{respect to the induced} \\ \text{stratification of } U. \end{array} \right\}.$$

Now $\Omega^{\cdot}_{\bar{p}}$ and $\mathcal{IC}^{\cdot}_{\bar{p}}$ are fine sheaves. Therefore we have

Lemma 2.1.

$$\mathbb{H}^{i}(X, \Omega^{\cdot}_{\bar{p}}) \cong H^{i}_{(2)}((X-\Sigma)_{\bar{p}}),$$

$$\mathbb{H}^{-i}(X, \mathcal{IC}^{\cdot}_{\bar{p}}) \cong IH^{\bar{p}}_{i}(X).$$

Here $\mathbb{H}^{*}(X, \cdot)$ is the hypercohomology.

§3. Key Theorem due to Goresky and MacPherson

Let $\mathcal{S}^{\cdot}$ be a complex of sheaves on X which is constructible with respect to the given stratification $\{X_k\}$ (that is, for any j, $\mathcal{S}^{\cdot}|_{X_j - X_{j-1}}$ is cohomologically locally constant, i.e., its associate cohomology sheaves are locally constant).

Definition 3.1. We shall say $\mathcal{S}^{\cdot}$ satisfies the axiom $[AX1]_{\bar{p}}$ provided:

(a) $\mathcal{S}^{\cdot}|_{X-\Sigma} \cong \mathbb{R}[n]^{\cdot}$ (the isomorphism in the derived category),

(b) $\mathcal{H}^{i}(\mathcal{S}^{\cdot}) = 0$ for all $i < -n$,

(c) $\mathcal{H}^{m}(\mathcal{S}^{\cdot}|_{X-X_{n-k-1}}) = 0$ for all $m > p_k - n$,

(d) the attaching maps (in the derived category)

$$\mathcal{H}^{m}(j_k^{*}\mathcal{S}^{\cdot}|_{X-X_{n-k-1}}) \longrightarrow \mathcal{H}^{m}(j_k^{*}\mathcal{R}i_{k*}\, i_k^{*}\mathcal{S}^{\cdot}|_{X-X_{n-k-1}})$$

are isomorphisms for all $m \leq p_k - n$.

Here $\mathcal{H}^{*}(\cdot)$ is the cohomology sheaf. And $i_k : X-X_{n-k} \longrightarrow X-X_{n-k-1}$ and $j_k : X_{n-k}-X_{n-k-1} \longrightarrow X-X_{n-k-1}$ are the inclusion maps.

Now, according to [3], we have

<u>Key Theorem</u> (Goresky and MacPherson).

(1) The constructible complex of sheaves which satisfies the axiom $[AX1]_{\bar{p}}$ is unique up to isomorphism in the derived category.

(2) $\mathcal{IC}^{\cdot}_{\bar{p}}$ is constructible and satisfies $[AX1]_{\bar{p}}$.

(3) If $\bar{p}+\bar{q}=\bar{t}$, then $\mathcal{IC}^{\cdot}_{\bar{p}} \cong \mathcal{R}\mathcal{H}om(\mathcal{IC}^{\cdot}_{\bar{q}}, \mathcal{D}^{\cdot}_X)[n]^{\cdot}$, where $\mathcal{D}^{\cdot}_X$ is the dualizing complex on X, i.e., $\mathcal{D}^{\cdot}_X = f^{!}\mathbb{R}^{\cdot}_{pt}$ with $f: X \longrightarrow$ (a point).

§4. <u>Proof of Main Theorem</u>

It suffices to prove

<u>Assertion</u>. $\Omega^{\cdot}_{\bar{p}}[n]$ is constructible and satisfies the axiom $[AX1]_{\bar{q}}$, where $\bar{q}$ is the complementary perversity, $\bar{p}+\bar{q}=\bar{t}$.

Actually we have

<u>Proof of Main Theorem</u>. From Key Theorem (1) and Assertion, we have

$$\Omega^{\cdot}_{\bar{p}}[n] \cong \mathcal{IC}^{\cdot}_{\bar{q}} .$$

Therefore, by substituting $\Omega^{\cdot}_{\bar{p}}[n]$ for $\mathcal{IC}^{\cdot}_{\bar{q}}$ at Key Theorem (3), we get

$$\mathcal{IC}^{\cdot}_{\bar{p}} \cong \mathcal{R}\mathcal{H}om(\Omega^{\cdot}_{\bar{p}}, \mathcal{D}^{\cdot}_X)^{\cdot}.$$

Hence, by Verdier duality theorem, we have

$$\mathcal{H}^{-i}(X, \mathcal{IC}^{\cdot}_{\bar{p}}) \cong \mathrm{Hom}(\mathcal{H}^{i}(X, \Omega^{\cdot}_{\bar{p}}), \mathbb{R}).$$

Thus, combined with Lemma 2.1, the proof is complete.

Now we will prove Assertion. It suffices to examine (a)-(d) of $[AX1]_{\bar{q}}$. The constructibility will be referred briefly later on the way.

(a) Since $\Omega^{\cdot}_{\bar{p}}[n]|_{X-\Sigma}$ is a sheaf of C^{∞}-forms on $X-\Sigma$, we have $\Omega^{\cdot}_{\bar{p}}[n]|_{X-\Sigma} \cong \mathbb{R}^{\cdot}_{X-\Sigma}[n]$ because of the usual resolution.

(b) If $i<-n$, then $(\Omega^{\cdot}_{\bar{p}}[n])^{i} = \Omega^{i+n}_{\bar{p}} = 0$. Therefore $\mathcal{H}^{i}(\Omega^{\cdot}_{\bar{p}}[n]) = 0$ for all $i<-n$.

(Preparation for (c) and (d)) For a point x of $X_{n-k}-X_{n-k-1}$, take a suitable neighborhood U and the link L of the stratum at x. Then we have

$$(4.1) \qquad \mathcal{H}^{j}(\Omega^{\cdot}_{\bar{p}})_{x} \cong H^{j}_{(2)}(U \cap (X-\Sigma)).$$

Strictly speaking, the right hand side of (4.1) should be the inductive limit $\varinjlim_{U} H^{j}_{(2)}(U\cap(X-\Sigma))$. But, for sufficiently small U, it is naturally isomorphic to $H^{j}_{(2)}(U\cap(X-\Sigma))$ because the L^{2}-cohomology is invariant under the quasi-isometric transformation. Hence, also, $\Omega^{\cdot}_{\bar{p}}$ can be regarded as constructible. Moreover, (4.1) is isomorphic to

$$H^{j}_{(2)}(C^{c_k}(L\cap(X-\Sigma)) \times (U\cap (X_{n-k}-X_{n-k-1})))$$
$$\cong H^{j}_{(2)}(C^{c_k}(L\cap(X-\Sigma))$$
$$\cong \begin{cases} H^{j}_{(2)}(L\cap(X-\Sigma)) & ; \ j < \frac{1}{2}(k-1+\frac{1}{c_k}), \\ 0 & ; \ j \geq \frac{1}{2}(k-1+\frac{1}{c_k}), \end{cases}$$

through the natural extension maps ([4], [5]). Hence

$$(4.2) \qquad \mathcal{H}^{j}(\Omega^{\cdot}_{\bar{p}})_{x} \cong \begin{cases} H^{j}_{(2)}(L\cap(X-\Sigma)) & ; \ j \leq q_k, \\ 0 & ; \ j > q_k, \end{cases}$$

because $q_k < \frac{1}{2}(k-1+\frac{1}{c_k}) \leq q_k + 1$.

(c) This is equivalent to the assertion that, if $j > q_k$, then $\mathcal{H}^j(\Omega^{\cdot}_{\bar{p}})_x = 0$ for any point x of $X_{n-k}-X_{n-k-1}$. Hence, by (4.2), this is true.

(d) This is equivalent to the assertion that, if $j \leq q_k$, then the attaching maps

$$(4.3) \qquad \mathcal{H}^j(\Omega^{\cdot}_{\bar{p}}|X-X_{n-k-1})_x \longrightarrow \mathcal{H}^j(i_{k*}i_k^*\Omega^{\cdot}_{\bar{p}}|X-X_{n-k-1})_x$$

are isomorphisms for any point x of $X_{n-k}-X_{n-k-1}$.

In order to prove this assertion, first remark that a cross section of $\Omega^{\cdot}_{\bar{p}}|X-X_{n-k-1}$ resp. $i_{k*}i_k^*\Omega^{\cdot}_{\bar{p}}|X-X_{n-k-1}$ is a smooth form which and whose image by the exterior derivative are square integrable near any point of $X-X_{n-k-1}$ resp. $X-X_{n-k}$. (For a cross section ω of $i_{k*}i_k^*\Omega^{\cdot}_{\bar{p}}|X-X_{n-k-1}$, it is not necessary to claim that ω and $d\omega$ are square integrable near any point of $X_{n-k}-X_{n-k-1}$.) Therefore we have the natural sheaf map

$$\Omega^{\cdot}_{\bar{p}}|X-X_{n-k-1} \longrightarrow i_{k*}i_k^*\Omega^{\cdot}_{\bar{p}}|X-X_{n-k-1}.$$

And this just induces the attaching map (4.3). Now, from the property of $i_{k*}i_k^*\Omega^{\cdot}_{\bar{p}}|X-X_{n-k-1}$ mentioned above, we have

$$(4.4) \qquad \mathcal{H}^j(i_{k*}i_k^*\Omega^{\cdot}_{\bar{p}}|X-X_{n-k-1})_x \cong \begin{cases} H^j_{(2)}(L\cap(X-\Sigma)) & ; j<k, \\ 0 & ; j\geq k, \end{cases}$$

for any point x of $X_{n-k}-X_{n-k-1}$. Hence, for $j \leq q_k$, the identity map from the right hand side of (4.2) to the right hand side of (4.2) is just the attaching map (4.3). Thus the proof of (d) is complete.

References

1. J. Cheeger: On the Hodge theory of Riemannian pseudomanifolds, Proc. Sym. Pure Math. 36, Providence: AMS(1980), 91-146.
2. J. Cheeger, M. Goresky and R. MacPherson: L^2-cohomology and intersection homology for singular algebraic varieties, Proceedings of year in differential geometry, I.A.S., S. Yau ed. (1981), Annals of Math. Studies, Princeton.
3. M. Goresky and R. MacPherson: Intersection homology II, Inventiones Math. 72 No.1 (1983), 77-130.
4. M. Nagase: L^2-cohomology on pseudomanifolds, 数理研講究録 450.
5. M. Nagase: L^2-cohomology and intersection homology of stratified spaces, Duke Math. J. 50 No.1 (1983), 329-368.
6. M. Nagase: The fundamental solutions of the heat equations on Riemannian spaces with cone-like singular points, to appear in Kodai Math. J.
7. M. Oka and M. Nagase: 交叉ホモロジー論入門, 総合研究集会報告集2, 於名古屋大学理学部 (1982).
8. 才5回代数セミナー報告集II, 於兵庫県城崎町 (1982).

Department of Mathematics

Tokyo Institute of Technology

Meguro-ku, Tokyo