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A Cech cohomological method of construction of holomorphic vector bundles

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1. This is a continuation of our previous papers [6], [7], and we propose an (elementary) Cech theoretical construction of holomorphic vector bundles over a complex variety. In this note we take the universal quotient bundle over the Grassmann variety as our guiding model for the construction (cf. §1.3) and the arguments will be done by extending vector bundles to closed subvarieties of codimension $\geqq 2$ (cf. §1.1). The content of this note is very provisional, but provides a general method of construction of bundles up to codimension $\leqq 5$ (for Stein manifolds and for projective manifolds up to tensor product of line bundles). Among explicit computations in this note, the following may be worthwhile pointing out: (1) An analogue of Bertini's theorem on moving singularity (for divisors of a linear system) for what we call 'Grassmann system of divisors' (Lemma 1.4.2), (2) Some conditions for the locally freeness of the sheaves in question (Lemma 1.5.1 $\sim$ 1.5.3). The above two types of results concern singularities of certain varieties which appear in our construction, and

would clarify that the existence of singularities provide the hard part of the construction.*) (3) A type of residue theorem(Theorem 2.1), which represent the characteristic class by the residue of certain meromorphic differential forms. This part is based on the Cech theoretical treatment of the characteristic class in Atiyah [1], and, in our context, the validity of the residue theorem(residue condition) is a basic condition for frames of bundles in question.

2. Very many important results on constructions of vector bundles are known(cf.,for example, Hartshorne [3] and Schneider [9]. See also Maruyama [5] and Grauert-Müllich [2] for constructions in general situations.) However, a systematic constructions of bundles by Cech methods seems to have been not taken up for varieties of dimension $\geqq 2$. (Note that classical treatments of bundles as in Weil [11] and Tjurin [10] may be regarded as Cech theoretical. One of our motivation of this note is to try generalizations of the classical approaches in [11], [10] in terms of stratification theory.)

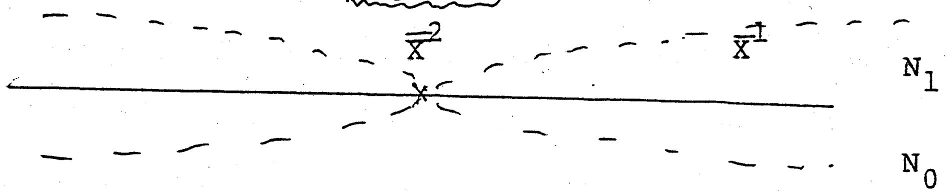
*) By this this manuscript may be very suited to be sent to the proceeding of the singularity seminar.

§ 1.1. Frame condition. Let $\overline{X}$ be a complex variety. Then our construction of holomorphic vector bundles will start with data as follows:

(1.1.1-1) $D_1 = (\overline{X}^2, E_X)$, where $\overline{X}^2$ is a codimension two subvariety of $\overline{X}$ and E_X is a holomorphic bundle over $X := \overline{X} - \overline{X}^2$.

(1.1.1-2) $D_2 = (\overline{X}^1, N_1; \underline{e}^0, \underline{e}^1)$, where $\overline{X}^1$ is a codimension one subvariety of $\overline{X}$ containing $\overline{X}^2$, N_1 is an open neighborhood of $X^1 := \overline{X}^1 - \overline{X}^2$ and $\underline{e}^i = (e^i_1, .., e^i_r)$, r=rank of E_X, is a frame of $E_X|_{N_i}$ (i=0,1). (Here we set $N_0 = X - X^1$.)

Figure I



Let i be the injection: $X \hookrightarrow \overline{X}$ and $E_{\overline{X}} := i_* E_X$ the zero-th direct image sheaf of (i, E_X). Then our proposed method of construction of holomorphic vector bundles is as follows:

(*-1) To settle data $D = (D_1, D_2)$ just above.

(*-2) To investigate structures of $E_{\overline{X}}$.

Because $E_{\overline{X}}$ is determined uniquely by D, (*-1) (or, more precisely, settling conditions on D and examining validities of conditions) will be a basic factor. In this provisional note, we do not enter into seriously (*-1). (Some arguments will be given in § 1.1 soon below and in § 1.3.) Concerning (*-2) our hope is to investigate $E_{\overline{X}}$ in the following devices:

(**-1) To give conditions of locally freeness of $E_{\overline{X}}$,

(**-2) To stratify $\overline{X}^2$, to attach an open neighborhood of each stratum and to consruct a frame of $E_{\overline{X}}$ in each neighborhood.

(**-3) To use frames as in (**-2) for investigations of structures of $E_{\overline{X}}$.

Our results in this note give a provisional result for (**-1) $\sim$ (**-3). Very roughly, one can handle (**-1) $\sim$ (**-3) outside of X^1_{sing} (cf. §1.5 and §2), and we hope that the content of the present note provides a basis for further considerations.

2. The following proposition would show that our start with the data $D=(D_1,D_2)$ as in (1.1.1),(1.1.2) will not lose generarilties:

Proposition 1.1.1. Assume that $\overline{X}$ is a Stein variety or a projective variety. Then, for a holomorphic vector bundle $E_{\overline{X}}$ over $\overline{X}$, there is a coimension two subvariety $\overline{X}^2$ of $\overline{X}$ and a datum $D_2=(\overline{X}^1,N_1;\underline{e}^0,\underline{e}^1)$ for $D_1:=(\overline{X}^2,E_X)$.

Proof. The proof is given similarly in the both cases. Here we give a proof only for the case where $\overline{X}$ is a projective variety. Then we know that there is a codimension one variety $\overline{X}^1$ such that $E_{\overline{X}|\overline{X}-\overline{X}^1}$ is a product bundle (and we fix a frame

$\underline{e}^0$ of $E_{\overline{X}|\overline{X}-\overline{X}^1}$). Next taking a codimension two subvariety $\overline{X}^2$ of $\overline{X}$, which is contained in $\overline{X}^1$, one can assume the following:

(a) the restriction of $E_{\overline{X}}$ to $X^1:=\overline{X}^1-\overline{X}^2$ is trivial.

Assuming that X^1 is an affine variety, if necessary, one can find an open neighborhood N_1 of X^1 and a frame $\underline{e}^1$ of $E_{X|N_1}$. From the above arguments we see that ($\overline{X}^2$ and $\underline{D}_2:=$ $(\overline{X}^1, N_1; \underline{e}^0, \underline{e}^1)$ satisfy the condition in this proposition.

q.e.d.

Next assume that dim $\overline{X}$=1. Then $\overline{X}^2 = \phi$. Then remarking that $\overline{X}^1$ is a compact Riemann surface or a Stein variety, we may say that the role of the datum D_2 is as follows:

§ 1.2. Representations of E_X. In § 1.2 we assume that there is an element $y \in \Gamma(X, \mathcal{O}_X)$ such that

(1.2.1) $\quad X^1 = (y)_{0,red}$.

Proposition 1.2.1. Assume that h_{01} admits an expression:

(1.2.2) $\quad h_{01} = y^{-a} \cdot h'_{01}$, with elements $a \in \mathbb{Z}_{+0}$ and $h'_{01} \in M_r(N_1, \mathcal{O}_X)$.

Then E_X is imbedded into $\mathcal{O}_X^r$.

(Note that (1.2.2) implies that the coefficients of h_{01} are meromorphic with respect to X^1.)

Proof. Define an $\mathcal{O}_X$-homomorphism $\tau : E_X \longrightarrow \mathcal{O}_X^r$ by

$$(1.2.3)\begin{cases} \tau_{|N_1} : E_{X|N_1} \ni \underline{e}^1 \cdot \zeta^1 \longrightarrow \mathcal{O}_{N_1}^r \ni h'_{01} \cdot \zeta^1 \\ \tau_{|N_0} : E_{X|N_0} \ni \underline{e}^0 \cdot \zeta^0 \longrightarrow \mathcal{O}_{N_0}^r \ni y^a \cdot \zeta^0 \end{cases}$$

where $\zeta^i = (\zeta^i_1, .., \zeta^i_r)$ is an element of $\mathcal{O}_{N_i}^r$ and $\underline{e}^i \cdot \zeta^i = \sum_{j=1}^r \zeta^i_j e^i_j$.
Note that (1.2.2) implies $\tau_{|N_1} = \tau_{|N_0}$ in N_{10}. It is also easy to see that τ is injective. q.e.d.

Denote by L_X and L'_X the determinant bundles of E_X and $E'_X := \tau(E_X)$. Then we have the following commutative diagram:

$$(1.2.4)\quad \begin{array}{ccc} E_X^{\otimes r} & \xrightarrow{\tau^{\otimes r}} & E_X'^{\otimes r} \\ \Lambda^r \downarrow & (\Lambda^r \tau) & \downarrow \Lambda^r \\ L_X & \longrightarrow & L'_X \end{array}$$

Λ^r denotes the r-th exterior produc-

Note that, by taking $\underline{f}^i := \Lambda^r \underline{e}^i$ to be a frame of $\check{L}_{X|N_i}$ (i=0,1), the isomorphism $(\Lambda^r \tau)$ is explicitly as follows:

$$(1.2.5)\begin{cases}(\wedge^r\tau)|_{N_1} : L_{X|N_1} \ni \underline{f}^1\cdot\zeta_1 \longrightarrow \underset{\sim}{O}_{N_1} \ni (\det h'_{01})\cdot\zeta_1 \\ (\wedge^r\tau)|_{N_0} : L_{X|N_0} \ni \underline{f}^0\zeta_0 \longrightarrow \underset{\sim}{O}_{N_0} \ni y^{ar}\cdot\zeta_0\end{cases}$$

where ζ_i is an element of $\underset{\sim}{O}_{N_i}$ $(i=0,1)$.

Take an element $\varphi\in\Gamma^r(X,E_X)$. We mean by <u>the divisor of</u> φ the one of $\wedge^r\varphi\in\Gamma(X,L_X)$. Letting D_φ and D'_φ be the divisors of φ and $\tau(\varphi)\in\Gamma^r(X,E'_X)$, we have the following from (1.2.5):

(1.2.6) $\qquad D'_\varphi = D_\varphi + D_0$,

where the divisor D_0 of X is defined as follows:

(1.2.7) $\qquad D_{0|N_1}$ = locus of $\det h'_{01}$ and $D_{0|N_0}$ = that of 1 .

(Note that D'_φ is the divisor of $\wedge^r\varphi\in\Gamma(X,\underset{\sim}{O}_X)$, and treatements of it are much easier than those of D_φ .) Divisors like D_φ play very basic role in our arguments henceforth;some properties of such divisors will be investigated in later arguments(cf. §1.4).

Next note that (1.2.2) concerns the growth property of the matrix h_{01} with respect to the divisor X^1. We discuss here growth properties of h'_{01} with respect to the codimension two subvariety $\overline{X}^2$. For this letting the element $h'_{01}\in M_r(N_1,\underset{\sim}{O}_X)$ be the one in (1.2.2), we assume that the inverse h'_{10} of h'_{01} admits the following expression:

(1.2.8) $\qquad h'_{10} = y^{-b}(x^{-c}\cdot h'_{10,0} + y^b h'_{10,1})$, where $h'_{10,0}$ and $h'_{10,1}$ are elements of $M_r(X,\underset{\sim}{O}_X)$ and $M_r(N_1,\underset{\sim}{O}_X)$. Moreover, c is an element of $\mathbb{Z}_{+0}$ and x is an element of $\Gamma(X,\underset{\sim}{O}_X)$ which does not vanish on N_1.

(Thus we may say that the main part of h'_{10} is meromorphic with respect to x and y.) Now letting $\mathcal{X}$ and μ denote the $\underset{\sim}{O}_{\overline{X}}$-morphism: $\underset{\sim}{O}^r_{\overline{X}} \ni \zeta \longrightarrow \underset{\sim}{O}^r_{\overline{X}} \ni h'_{10,0} \cdot \zeta$ and the quotient morphism: $\underset{\sim}{O}^r_{\overline{X}} \longrightarrow \underset{\sim}{O}^r_{\overline{X}}/y^b \underset{\sim}{O}^r_{\overline{X}}$, we have:

Proposition 1.2.2. $E'_{\overline{X}}$ is the kernel of $\mu\mathcal{X}: \underset{\sim}{O}^r_{\overline{X}} \longrightarrow \underset{\sim}{O}^r_{\overline{X}}/y^b \underset{\sim}{O}^r_{\overline{X}}$.

Proof. Take a point $p \in \overline{X}^1$ and an element $\zeta_p \in \underset{\sim}{O}^r_{\overline{X},p}$. Then we easily have the following;

(1.2.9) $\zeta_p \in E'_{\overline{X},p} \Leftrightarrow (y^b \cdot h'_{10}) \cdot \zeta_p \equiv 0(y^b) \Leftrightarrow h'_{10,0} \cdot \zeta_p \equiv 0(y^b)$.

This implies that $E'_{\overline{X},p} =$ (kernel of $\mathcal{X}\mu)_p$. On the otherhand, for a point $p \in N_0 (=\overline{X}-\overline{X}^1)$, we obviously have: $E'_{X,p} =$ (kernel of $\mathcal{X}\mu)_p$ $= \underset{\sim}{O}^r_{\overline{X},p}$. Thus we have this proposition. q.e.d.

Remark. Assume that $\overline{X}$ is smooth. Then, by a simple observation, we have:

(*) $E_{\overline{X}}$ is coherent $\Longleftrightarrow$ $D=(D_1,D_2)$ satisfies (1.2.8)

(i.e., the main part of the matrix h'_{10} is meromorphic with respect to X^1 and $\overline{X}^2$).

§1.3. Frames of type (G) $\cdots$1. In §1.3 we assume that $\overline{X}$ is normal. Then we introduce the following

Definition 1.3.1. We say that $\underline{D}_2=(\overline{X}^1,N_1;\underline{e}^0,\underline{e}^1)$ is of type (G), if there are elements $\underline{e}=(e_1,\ldots,e_{r+1})\subset\Gamma(X,E_X)$, with which the following hold:

(1.3.1-1) The frames $\underline{e}^0$ and $\underline{e}^1$ are of the form: $\underline{e}^0=(e_1,\ldots,e_{r-1},e_r)$ and $\underline{e}^1=(e_1,\ldots,e_{r-1},e_{r+1})$, and $\bigwedge^r\underline{e}^i\,(i=0,1)$ does not vanish identically on X.

(1.3.1-2) The closure $\overline{D}_0(\subset\overline{X})$ of the divisor D_0 of $\underline{e}^0$ is reduced and irreducible, and coincides with X^1. (Thus X^1 is irreducible).

The bundle E_X is of type (G), if it admits a datum D_2 of type (G). Note that Definition 1.3.1 concerns essentially the elements $\underline{e}=(e_1,\ldots,e_{r+1})\subsetneq\Gamma(X,E_X)$. We say that $\underline{e}$ is of type (G), if it satisfies (1.3.1-1), (1.3.1-2) and ;

(1.3.1-3) $\bigwedge^r\underline{e}^1$ does not vanish at all on X^1.

Note that, in this case, the closure $\overline{X}'^1$ of the divisor D_1 of $\bigwedge^r\underline{e}^1$ satisfies:

(1.3.1-4) $\quad(\overline{X}^1\cap\overline{X}'^1_{red})\subset\overline{X}^2$.

Also note that, by taking a suitable open neighborhood N_1 of X^1 in X, the datum $\underline{D}_2=(\overline{X}^1,N_1;\underline{e}^0,\underline{e}^1)$ is of type (G). Here we make some very simple remarks for $\underline{D}_2=(\overline{X}^1,N_1;\underline{e}^0,\underline{e}^1)$ just above. Some more delicate computations will be given in §1.3 and §1.5.

The datum $\underline{D}=(\overline{X}^1;N_i;\underline{e}^i)$ $(i=0,1)$ and the divisors $\overline{X}^1$ and $\overline{X}'^1$ have the similar meaning to the ones just above, and denote by $\underset{\sim}{O}_X[X^1]$ the sheaf of meromorphic functions over X with the pole X^1.

Proposition 1.3.1. The transition matrix h_{10} for $(\underline{e}^0,\underline{e}^1)$: $\underline{e}^0=\underline{e}^1 h_{10}$: is explicitly as follows:

(1.3.2-1) $h_{10} = \begin{bmatrix} I_{r-1} & \underline{c} \\ 0 & y \end{bmatrix}$, with $c\in\Gamma^{r-1}(X,\underset{\sim}{O}_X[X^1])$ and $y\in\Gamma(X,\underset{\sim}{O}_X[X^1])$

(Remark that this implies that y defines the divisor X^1 in $X-X'^1$.)

Proof. Remark that the first (r-1)-terms of $\underline{e}^0$ and $\underline{e}^1$ are $(e_1,\ldots,e_{r-1})$, and, for the proof of (1.3.2-1), it suffices to see that

(1.3.2-2) $e_r = \underline{e}^1\begin{bmatrix}\underline{c}\\ y\end{bmatrix}$.

But, because $\wedge^r e^1$ does not vanish on $X-X'^1$, we have such an expression in $X-X'^1$, by understanding that $c\in\Gamma^{r-1}(X-X'^1,O_X)$ and $y\in\Gamma(X-X'^1,O_X)$. On the otherhand, taking a suitable open neighborhood N_1' of X'^1 in X, one can write:

(1.3.2-3) $e_{r+1} = (c_1' e_1+\cdots+c_{r-1}' e_{r-1})+x\cdot e_r$, with elements $c_1',\ldots,c_{r-1}'$ and $x\in\Gamma(N_1',\underset{\sim}{O}_X)$,

and one can also write:

(1.3.2-4) $(-1)\cdot e_r = x^{-1}(c_1'\cdot e_1+\cdots+c_{r-1}'\cdot e_{r-1}-e_{r+1})$.

Because x defines X'^1, we have this proposition. q.e.d.

Let the elements $\underline{c}$ and y be as in (1.3.2-1) and denote by N_0' the open variety $X-X'^1$. Then, letting μ' the $\mathcal{O}_0'(:=\mathcal{O}_{N_0'})$-morphism: $\mathcal{O}_0'^r \ni \zeta \longrightarrow \mathcal{O}_0'^{r-1} \ni [I_{r-1},\underline{c}]\cdot\zeta$ and χ' the quotient morphism: $\mathcal{O}_0'^{r-1} \longrightarrow \mathcal{O}_0'^{r-1}/y\mathcal{O}_0'^{r-1}$, one can rewrite Proposition 1.2.2 as follows:

Proposition 1.3.2. $E_{X|N_0'}$ ~~is isomorphic to the kernel~~ of $\chi'\cdot\mu' : \mathcal{O}_0'^r \longrightarrow \mathcal{O}_0'^{r-1}/y\mathcal{O}_0'^{r-1}$.

Assume that there are elements y and x $\in \Gamma(\overline{X},\mathcal{O}_{\overline{X}})$ such that

(1.3.2-1) y and x generate the ideal of $\overline{X}^1$ and $\overline{X}'^1$.

Then one can write:

(1.3.2-2) $\underline{c} = x^{-1}\cdot \underline{c}'$, with $\underline{c}' \in \Gamma^{r-1}(\overline{X},\mathcal{O}_{\overline{X}})$.

Then, letting μ be the $\mathcal{O}_X$-morphism: $\mathcal{O}_X^r \ni \zeta \longrightarrow \mathcal{O}_X^{r-1} \ni [xI_{r-1},\underline{c}']\cdot\zeta$ and χ the quotient morphism: $\mathcal{O}_X^{r-1} \longrightarrow \mathcal{O}_X^{r-1}/y\mathcal{O}_X^{r-1}$, one can rewrite Proposition 1.2.3 as follows:

Proposition 1.3.3. The direct image sheaf $E_{\overline{X}}$ is isomorphic to the kernel of $\chi\cdot\mu : \mathcal{O}_{\overline{X}}^r \longrightarrow \mathcal{O}_X^{r-1}/y\mathcal{O}_{\overline{X}}^{r-1}$.

(Note that, writing ζ as $(\zeta_j)_{j=1}^r$, we set $\zeta' = (\zeta_j)_{j=1}^{r-1}$. Then ζ is in the kernel of $\chi\cdot\mu$ if and only if:

(1.3.2-3) $\quad x\cdot\zeta' + \zeta_r\underline{c}' \equiv 0 \pmod{y}$.)

The above simple proposition will be a starting point for our explicit computations done for $E_{\overline{X}}$ from now on.

Next by a simple computation, we see that the injection $\tau : E_X \longrightarrow \mathcal{O}_X^r$ (as in (1.2.3)) is given explicitly as follows:

$$(1.3.3\text{-}1) \begin{cases} \tau_{|X-X'1} : E_{X|X-X'1} \ni e^1 \cdot \zeta^1 \rightarrow \mathcal{O}^r_{X-X'1} \ni h'_{01} \cdot \zeta^1 \\ \tau_{|X-X1} : E_{X|X-X1} \ni e^0 \cdot \zeta^0 \longrightarrow \mathcal{O}^r_{X-X1} \ni y \cdot \zeta^0 , \end{cases}$$

where $h'_{01} = \begin{bmatrix} yI_{r-1}, & -\underline{c} \\ 0 & x \end{bmatrix}$.

Moreover, let $\underline{f}_j$ be the element of $\Gamma(X, E_X)$, whose i-th component = 1 or 0, according as i=j or $\neq$ j. Then we easily have:

Proposition 1.3.4. $\tau(e_j) = y\underline{f}_j \, (1 \leqq j \leqq r)$ and $\tau(e_{r+1}) = \left(\begin{smallmatrix} \underline{c} \\ x \end{smallmatrix} \right)$

Thirdly let $\underline{e}^j$ denote the element $(e_1, \ldots, \hat{e}_{r+1-j}, \ldots, e_{r+1}) \in \Gamma^r(X, E_X)$ $(0 \leqq j \leqq r)$. Then we easily have:

Proposition 1.3.5. *The closure $\overline{X}^1_j$ (in $\overline{X}$) of the divisor $(\wedge^r \underline{e}^j)_0$ is as follows:*

(1.3.4) $\overline{X}_0 = (y)_0$, $\overline{X}_1 = (x)_0$ and $\overline{X}_{r+1-j} = (c_j)_0 \, (1 \leqq j \leqq r-1)$.

Now we define a subvariety $\overline{Y}$ of $\overline{X}$ by

(1.3.5) $\overline{Y} = \bigcap_{j=0}^{r} \overline{X}^1_j \quad (\subset \overline{X}^2)$.

Proposition 1.3.6. *The direct image sheaf $E_{\overline{X}}$ is locally free over $\overline{X}-\overline{Y}$.*

Proof. For a point $p \in \overline{X}$ it is clear that $E_{\overline{X},p}$ is $\mathcal{O}_{\overline{X},p}$-free if and only if:

(1.3.6) there is an element $\zeta = (\zeta_j)_{j=1}^{r} \in E^r_{\overline{X},p}$ such that the divisor D_ζ of ζ does not contain the point p.

Thus, in $\overline{X}-\overline{X}^1_k$, it is clear that $\underline{e}^{r+1-k}$ is a frame, and we have this proposition. q.e.d.

Remark. In arguments henceforth, we take $\underset{\sim}{e}^{r+1-k}$ to be a standard frame of $\bar{X}-\bar{X}_k^1$.

The locally freeness condition of $E_{\bar{X},p}$ $(p \in \bar{Y})$ will be discussed in §1.5 (in a more detail).

Remark. Assume that $\bar{X}$ is the Grassmann variety of subspaces of dimension d in $\mathbb{C}^n$. Then letting $T_{\bar{X}} = \bar{X}\times\mathbb{C}^n$ be the trivial bundle over $\bar{X}$, we have the following exact sequence of the universal bundles:

(1.3.7)
$$\begin{tikzcd} 0 \arrow[r] & R_{\bar{X}} \arrow[r] \arrow[dr] & T_{\bar{X}}(=\bar{X}\times\mathbb{C}^n) \arrow[r, "\omega"] \arrow[d] & Q_{\bar{X}} \arrow[r] \arrow[dl] & 0 \\ & & \bar{X} & & \end{tikzcd}$$

where $R_{\bar{X}}$ and $Q_{\bar{X}}$ are the universal sub and quotient bundles. Now take a basis $\underset{\sim}{e}' = (e_1', \ldots, e_n')$ of $\mathbb{C}^n$, and let $\tilde{e}' = (\tilde{e}_1', \ldots, \tilde{e}_n')$ be the corresponding elements of $\Gamma(\bar{X}, T_{\bar{X}})$. Moreover, let $\hat{e} = (\tilde{e}_1, \ldots, \tilde{e}_n)$ denote $\omega(\tilde{e}) \in \Gamma^n(\bar{X}, Q_{\bar{X}})$, and we set $\tilde{e} = (\tilde{e}_1, \ldots, \tilde{e}_{r+1})$. Then, from the Scubert calculus (cf. Musili [ ]), we easily have the following:

(1.3.8-1) $\quad \bar{X}_j^1 (= \bigwedge^r e_1 \wedge \cdots \wedge \hat{e}_{r+1-j} \wedge \cdots \wedge e_{r+1})_0$ represnts the first Chern class $c_1(Q_X)$ $(0 \leq j \leq r)$,

and

(1.3.8-2) $\quad \bar{Y} := \bigcap_{j=0}^{r} \bar{X}_j^1$ is the Scubert cycle of codimension 2 (in $\bar{X}$) such that

$$\bar{X}_0^1 \cap \bar{X}_1^1 = \bar{Y} \uplus \bar{X}_0^2, \text{ with } \bar{X}_0^2 = (\bigwedge^{r-1} e_1 \wedge \cdots \wedge e_{r-1})_0$$

(= represnting cycle of the secind Chern class $c_2(Q_X)$).

Also, for each $j=1,\ldots,r$, we have:

(1.3.8-3) $\bar{X}_0^1 \cap \bar{X}_j^1 = \bar{Y} \cup \bar{X}_j^2$, with $\bar{X}_j^2 = (\bigwedge^{r-1} e_1\wedge\cdots\wedge\hat{e}_{r-1-j}\wedge\cdots\wedge\hat{e}_{r-1})_C$

Remark. The above example of the universal quotient bundle would show that it is not, in general, legitimate to take the following as a generic condition:

(1.3.8-4) $\operatorname{codim}_{\bar{X}} \bar{Y} = r+1$.

Some other explicit computaions for the universal quotient bundles will be given in § 1.5 and in Appendix I.*)

*) This appendix is not included here...

§ 1.4. Frames of type (G)⋯2. Here we confirm some situations, where the 'bundles of type G' appear. First we remark that the most typical example of 'bundles of type G' (in our sense) is provided by the universal quotient bundle over the Grassmann variety. Letting $\bar{X}$ denote the Grassmann variety and $Q_{\bar{X}}$ the quotient universal bundle, we have:

Lemma 1.4.1. There is a codimension two subvariety $\bar{X}^2$ of $\bar{X}$ such that

(1.4.1) $\quad Q_{\bar{X}|\bar{X}-\bar{X}^2}$ is of type (G).

Using the notation in Remark in § 1.3, one can take $\bar{X}^2=\bar{X}^1_0 \cap \bar{X}^1_1$. Some detailed arguments on the univesal bundle $Q_{\bar{X}}$ is given elsewhere*).

Assume that $\bar{X}$ is a projective variety. Then Lemma 1.4.1 and a generic position argument in Kleiman[ ] implies that following:

Lemma 1.4.2. For a bundle $E_{\bar{X}}$ over $\bar{X}$ there are a codimension two subvariety $\bar{X}^2$ of $\bar{X}$ and an element $m \in \mathbb{Z}_{+0}$ such that

(1.4.2) $\quad (E_{\bar{X}} \otimes L^m_{\bar{X}})_{|\bar{X}-\bar{X}^2}$ is of type (G),

where $L_{\bar{X}}$ denotes the line bundle corresponding to the hyper plane cut.

A similar fact holds for a holomorphic bundle over a Stein variety, without taking the tensor product with line bundles.

Lemma 1.4.2 and the remark soon below would insure generalities to work with bundles of type (G) in our construction of holomorphic bundles.

2. Here we fix elements $s,t \in \mathbb{Z}_+$ satisfying $s > t$, and, for each index $I=(i_1,\ldots,i_t): 1 \leqq i_1 < \cdots < i_t \leqq s$, we fix an element $f_I \in \Gamma(\bar{X}, \mathcal{O}_{\bar{X}})$. Let $M_{st}(\mathbb{C})$ denote the $\mathbb{C}$-vector space of $s \times t$-matrices with coefficients in $\mathbb{C}$ and we identify $M_{st}(\mathbb{C})$ with $\mathbb{C}^{st}$. For an element $A \in M_{st}(\mathbb{C})$ we write A^I for the submatrix of A consisting of $I(=(i_1,\ldots,i_t))$-rows of A. We then define an element $F \in \Gamma(\bar{X} \times \mathbb{C}^{st}, \mathcal{O}_t)$, $\mathcal{O}_t$=structure sheaf of $\bar{X} \times \mathbb{C}^{st}$, by

$$(1.4.3) \qquad \tilde{F}(p,A) = \sum_I (\det A^I) \cdot F_I(p) .$$

Let i_A be the injection: $\bar{X} (\simeq \bar{X} \times \{A\}) \hookrightarrow \bar{X} \times \mathbb{C}^{st}$, and let D_A be the divisor $i_A^* \tilde{F}$ (of $\bar{X}$). Denote by $\mathcal{F}$ the collection $\{f_I\}_I$, and we set:

$$(1.4.4) \qquad B_{\mathcal{F}} (=\text{base locus of } \mathcal{F}) = \bigcap_I (f_I, 0)_{red} .$$

Now take a point $p \in \bar{X}$ and an element $A \in \mathbb{C}^{st}$. Then, taking a suitable open neighborhood U of A and a proper subvariety V of A such that the following hold:

Lemma 1.4.3. *For each $A' \in U-V$ we have:*

$$(1.4.5) \qquad \operatorname{codim}_{\bar{X}_p} ((D_{A'}, p)_{sing} - (\bar{X}_{p,sing} \cup B_{\mathcal{F},p})) \geqq 4,$$

where $\bar{X}_p, \ldots$ denote the germs of $\bar{X}, \ldots$ at p.

(We may say that the ststem $\{D_A\}_{A \in \mathbb{C}^{st}}$ is a Grassmannian system of divisors, because D_A depends only the Grassmannian coordinate determined by A.)

Here we give the key points of the proof of Lemma 1.4.2 (and the content soon below may suffice to claim a regourous proof.) Our basic idea is to reduce Lemma 1.4.2 to the original Bertini's theorem on the moving singularity of the divisors (in a linear system). For this letting $A=[a^i_j]$ $(1\leq i\leq s, 1\leq j\leq t)$, we expand A^I as:

(1.4.6-1) $\det A^I = \sum_{v=1}^{s} (-1)^{t+v}\cdot a_t^{i_v}\cdot \Delta_{i_v,t}(A^I)$, where $\Delta_{i_v,t}(A^I) = (i_v,t)$-cofactor of A^I,

and substituting this to (1.4.3), we write $\tilde{F}(p,A)$ as follows[*)]:

(1.4.6-2) $\tilde{F}(p,A) = \sum_{v=1}^{s} a_t^v \cdot F_v(p,A_{t-1})$, where A_{t-1} is the submatrix of A consisting of the first (t-1)-rows

Note that (1.4.6-2) is linear with respect to $(a_t^v)_{v=1}^{s}$, and, in order to apply the Bertini's theorem to this linear system, we define the following 'base locus' for $A_{t-1}\in \mathbb{C}^{s(t-1)}$:

(1.4.6-3) $B(A_{t-1}) = \bigcap_{v=1}^{s}(F_v,0)_{red}$.

Also, for our proof, we set:

(1.4.6-4) $R'_k(A_{t-1}) = \{p\in \overline{X}_{reg};\ \text{rank of}\left[\frac{\partial F_v(p,A_{t-1})}{\partial x_u}\right]\leq k\}$,

where $(x_1,\ldots,x_{\dim \overline{X}})$ are coordinates of $\overline{X}$ at p,

and we also set:

(1.4.6-5) $R_k(A_{t-1})$ = closure of $R'_k(A_{t-1})$ in $\overline{X}$.

Letting the point p be as in Lemma 1.4,2, the key points of the proof are as follows[**)]:

*),**) See [*] for an analogue of Lemma 1.4.2 in an algebraic situation. The proof is given similarly to [*].

(1.4.7-1) Each irreducible component $B(A_{t-1})_{p,j}$ of $B(A_{t-1})_p$ satisfying $B(A_{t-1})_{p,j} \not\subset B_p$, we have:

$$B(A_{t-1})_{p,j} \not\subset R_1(A_{t-1})_j \quad .$$

(1.4.7-2) For each irreducible germ V_p at p satisfying $V_p \not\subset B_p$ we have:

$$V_p \not\subset R_0(A_{t-1})_p \quad .$$

(In the above $B_p, \ldots$ denote the germs of B,.. at p, and the element $A_{t-1} \in \underset{\sim}{C}^{s(t-1)}$ is understood to be chosen generally.)
By a simple observation we have the following from the above:

(1.4.7-3) $(D_A)_{sing} \cap (B(A_{t-1}) - (X_{sing} \cup R_1(A_{t-1}))_p$ is of codiemnsion $\geqq 2$ in $B(A_{t-1})$.

On the otherhand a simple computation also leads to:

(1.4.7-4) $\operatorname{codim}_{X_p} B(A_{t-1})_p \geqq 2$.

Thus from (1.4.7-3) and (1.4.7-4) we have Lemma 1.4.2.

Next recall that a basic fact on reflexive sheavs (cf.Siu-Trautmann [ ]) implies:

(1.4.7-5) $\operatorname{codim}_{\overline{X}} S(E_{\overline{X}}) \geqq 3$, where $S(E_{\overline{X}}) := \{ p \in \overline{X}; E_{\overline{X},p}$ is not $\underset{\sim}{O}_{\overline{X},p}$-free $\}$.

This and Lemma 1.4.3 will insure that there are generalities to start with bundles of type (G) in local situations.

§ 1.5. Locally freeness conditions. Here we assume that $\bar{X}$ is smooth and that the datum $D=(\bar{X}^1, N_1; \underline{e}^0, \underline{e}^1)$ is of type (G) (cf. Definition 1.3.1). Let $\bar{X}^1$ and $\bar{X}'^1$ be the extensions of $(\Lambda^r \underline{e}^0)_0$ and $(\Lambda^r \underline{e}^1)_0$ to $\bar{X}$. Without loss of generality*) we assume that

$$(1.5.1) \qquad \bar{X}^2 = \bar{X}^1 \cap \bar{X}'^1_{red} \quad .$$

Letting $\mathcal{O}_{\bar{X}}[\bar{X}'^1]$ be the sheaf over $\bar{X}$ of meromorphic functions with the pole $\bar{X}'^1$, recall that the transition matrix h_{10} for $(\underline{e}^0, \underline{e}^1)$ is explicitly as follows:

$$(1.5.2) \quad \begin{cases} h_{10} = \begin{bmatrix} I_{r-1}, & \underline{c}' \\ 0 & , y' \end{bmatrix}, \text{ with } y' \in \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}[\bar{X}'^1]) \text{ and} \\ \underline{c}' = (c_j')_{j=1}^{r-1} \in \Gamma^{r-1}(\bar{X}, \mathcal{O}_{\bar{X}}[\bar{X}'^1]) \quad . \end{cases}$$

We assume that there is an element $x \in \Gamma(\bar{X}, \mathcal{O}_{\bar{X}})$ such that

$$(1.5.3) \quad \begin{cases} y' = y/x \text{ and } \underline{c}' = \underline{c}/x \text{ , with } y \in \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}) \text{ and} \\ \underline{c} = (c_j)_{j=1}^{r-1} \in \Gamma^{r-1}(\bar{X}, \mathcal{O}_{\bar{X}}) \quad . \end{cases}$$

(In view of Lemma 1.4.2 this assumption has generalities in the local situation.) Let τ be the injection: $E_{\bar{X}} \hookrightarrow \mathcal{O}_{\bar{X}}^r$, which is defined by (1.3.6.1), and we write $E'_{\bar{X}}$ for $\tau(E_{\bar{X}})$. Moreover, let ω be the quotient morphism: $\mathcal{O}_{\bar{X}} \longrightarrow \mathcal{O}_{\bar{X}^1}$. Taking a point $p \in \bar{X}^1$, recall that an element $\zeta = (\zeta_j)_{j=1}^r \in \mathcal{O}_{\bar{X},p}$ is in $E'_{\bar{X},p}$ if and only if the following holds (cf. (1.3.6.2)):

(1.5.4) $\tilde{x} \cdot (\tilde{\zeta}') + \tilde{\zeta}_r \cdot \tilde{c} = 0$, where $\tilde{x} = \omega(x)$ and $\tilde{c} = \omega(c)$. Moreover, $\tilde{\zeta}' (:= (\tilde{\zeta}_j)_{j=1}^{r-1}) = \omega((\zeta_j)_{j=1}^{r-1})$ and $\tilde{\zeta}_r = \omega(\zeta_r)$.

Now we will give some conditions for locally freeness of $E_{\bar{X}}$, by analyzing the very simple equation (1.5.4). Our basic idea in the analysis is:

(*) to discuss structures of $E_{\overline{X}',p}$ for each $p \in \overline{X}^2 \cap \overline{X}^1_{reg}$ and to use results of the discussions for the structure of $E_{X',\overline{p}}(p \in \overline{X}^2 \cap \overline{X}^1_{sing})$.

As will be seen soon below, the structure of $E_{\overline{X}',p}$ is given explicitly for $p \in \overline{X}^2 \cap \overline{X}^1_{reg}$. We hope to discuss the structure of $E_{\overline{X}',p}$ for each $p \in \overline{X}^2 \cap \overline{X}^1_{sing}$ (cf. the end of this section).

Now let $F_{\overline{X}^1}$ denote the $\mathcal{O}_{\overline{X}^1}$(=structure sheaf of $\overline{X}^1$)-submodule of $\mathcal{O}^r_{\overline{X}^1}$, which is defined by (1.5.4). We determine the structure as follows: First we write the irreducible decomposition of $\tilde{x}$ in the form:

(1.5.5-1) $\tilde{x}=x_1^{a_1}\cdot\,\cdot\,\cdot\,\tilde{x}_u^{a_u}$, with $a_i \in \mathbb{Z}_+$ and $\tilde{x}_i \in \mathcal{O}_{\overline{X}^1,p}$ vanishes at p and is irreducible $(1 \leqq i \leqq u)$,

and write $\tilde{c}_j$ $(1 \leqq j \leqq r-1)$ as follows:

(1.5.5-2) $\tilde{c}_j = \tilde{x}_1^{b_1(j)}\cdot\,\cdot\,\cdot\,\tilde{x}_u^{b_u(j)}\cdot\tilde{c}_j'$, with $b_1(j),\dots,b_u(j) \in \mathbb{Z}_{+0}$

and $\tilde{c}_j' \in \mathcal{O}_{\overline{X}^1,p}$ is not divided by $\tilde{x}_t$ $(1 \leqq t \leqq u)$.

Define a subset $I(j)$ of $\{1,\dots,u\}$ by

(1.5.6) $\quad t \in I(j) \Leftrightarrow a_t > b_t(j)$,

and we set:

(1.5.7)' $\quad I = \bigcup_{j=1}^{r-1} I_j$, and $b_i = \min_{j\in I} b_i(j)\ (i \in I)$.

Now define an element $\eta \in F_{\overline{X}^1,p}\ (\subset \mathcal{O}^r_{\overline{X}^1,p})$ by

(1.5.7) $\eta_r = (\prod_{i\in I} \tilde{x}_i^{a_i-b_i})$, and $\eta_j = (\prod_{i\in I}\tilde{x}_i^{b_i(j)-b_i})(\prod_{i\notin I}\tilde{x}_i^{b_i(j)-a_i})$

$\cdot\tilde{c}_j'$ $(1\leqq j \leqq r-1)$. (Here η_j=j-th component of η .)

<u>Proposition 1.5.1</u>. $F_{\overline{X}^1,p}$ <u>is spanned by the single element</u> η.

Proof. First remark that , for each element

$j \in \{1,..,r-1\}$, we have: $\tilde{x}\cdot\tilde{\zeta}_j + \tilde{c}_j\cdot\tilde{\zeta}_r = 0$. This implies that if $i \in I_j$ then we have $\tilde{\zeta}_r \equiv 0 \pmod{\tilde{x}_i^{a_i - b_i}}$, and by (1.5.5.3) we have:

(1.5.8.1) $\tilde{\zeta}_r = \tilde{\delta}' \cdot (\prod_{i \in I} \tilde{x}_i^{a_i - b_i})$, with an element $\tilde{\delta} \in \mathcal{O}_{\bar{X}^1,p}$.

By substituting this to the equation: $\tilde{x}.\tilde{\zeta}_j + \tilde{c}_j\cdot\tilde{\zeta}_r = 0$ we have:

(1.5.8.2) $(-)\cdot\tilde{\zeta}_j = (\prod_{i \in I} \tilde{x}_i^{b_i(j)-b_i}) \cdot (\prod_{i \notin I} \tilde{x}_i^{b_i(j)-a_i}) \cdot \tilde{c}_j'$.

Thus we have the 'only if' part of this proposition. But it is easy to check that the element $\tilde{\eta}$ satisfies (1.5.3), and we have the present proposition. q.e.d.

As before, let f_j be the element of $\Gamma(\bar{X}, \mathcal{O}_{\bar{X}}^r)$ whose i-th component $=1$ or 0 according as $i=j$ or $\neq j$ $(1 \leq j \leq r)$. Moreover, take an element $\eta \in \mathcal{O}_{\bar{X},p}^r$ such that $\omega(\eta) = \tilde{\eta}$. Then from Proposition 1.3.4 and 1.5.1 we have:

<u>Lemma 1.5.1.</u> <u>The stalk $E'_{\bar{X},p}$ is generated over $\mathcal{O}_{\bar{X},p}$ by $y\cdot f_j$ $(=\tau(e_j))$, $(\frac{-c}{x})$ $(=\tau(e_{r+1}))$ and η.</u>

We derive a condition for the $\mathcal{O}_{\bar{X},p}$-freeness of $E'_{\bar{X},p}$ from Lemma 1.5.1. For notational concordance we write x and y as c_r and c_{r+1}, and we use the symbol $\bar{X}_j^1$ for the divisor $(c_{r+1-j})_0$ $(0 \leq j \leq r)$. Thus we have:

(1.5.9) $\bar{X}^1 = \bar{X}_0^1$ and $\bar{X}^2 = \bar{X}_0^1 \cap \bar{X}_{1,red}^1$.

Then setting

(1.5.10) $\bar{Y} = \bigcap_{j=0}^{r} \bar{X}_{j,red}^1 (\subset \bar{X}^2)$

recall that Proposition 1.3.7 implies:

(1.5.11) $E_{\overline{X}|\overline{X}-\overline{Y}}$ is locally free .

Next letting $\tilde{c}_j$ denote $\omega(c_j)$ we have:

Lemma 1.5.2. Take a point $p \in \overline{Y} \cap \overline{X}^1_{reg}$. Then $E_{\overline{X}',p}$ is $\mathcal{O}_{\overline{X}',p}$-free if and only if, for an element $j \in \{1,..,r\}$, we have:

(1.5.12) $\tilde{c}_k \equiv 0 \,(\text{mod.}\, \tilde{c}_j)$ in $\mathcal{O}_{\overline{X}',p}$ $(1 \leq k \leq r)$.

Proof. Let $\eta \in E'_{\overline{X}',p}$ be as in Lemma 1.5.1, and let $D_{j,k;\eta}$ denote the divisor of $(e_1,..,\hat{e}_j,.\hat{e}_k,..,e_{r+1},\eta)$ $(1 \leq j < k \leq r+1)$.

By Lemma 1.5.1 we have:

(1.5.13) $E'_{\overline{X}',p}$ is $\mathcal{O}_{\overline{X}',p}$-free if and only if $(\bigcap_{1 \leq j < k \leq r+1} D_{j,k;\eta})_p = \phi$ ..

To analyze the above condition, we first remark:

(1.5.14) $\omega\tau(e_{r+1}) = (\prod_{i \in I} \tilde{x}_i^{b_i})(\prod_{i \notin I} \tilde{x}_i^{a_i})$

and $e_1 \wedge \cdots \wedge \tilde{e}_j \wedge \cdots \wedge \tilde{e}_k \wedge \cdots \wedge \tilde{e}_{r+1} \equiv 0$ on $\mathfrak{X}$, unless $k=r+1$. Thus the condition in (1.5.13) is rewritten in the form:

(1.5.15) $\bigcap_{1 \leq j \leq r} D_{j,r+1;\eta} \not\ni p$.

Writing η as $(\eta_j)_{j=1}^r$ we get the following from Proposition 1.3. and (1.2.):

(1.5.16) $D_{j,r+1;\eta} = (\eta_j)_0 \; (1 \leq j \leq r)$.

Thus we rewrite (1.5.13) as follows:

(1.5.17) $E_{\overline{X}',p}$ is $\mathcal{O}_{\overline{X}',p}$-free if and only if one of η_j $(1 \leq j \leq r)$ does not vanish at p.

But from the explicit form of η (cf.(1.5.4.3)), we have:

(1.5.18-1) $\eta_r = 0$ at $p (\Leftrightarrow b_i(j) \geqq a_i ; 1 \leqq j \leqq r-1, 1 \leqq i \leqq s)$.

$\Leftrightarrow \tilde{c}_j \equiv 0 (\text{mod.} \tilde{c}_r) (1 \leqq j \leqq r)$.

Also, for an element $j \in \{1,..,r-1\}$, we easily have:

(1.15.18-2) $\eta_j \neq 0$ at $p \Leftrightarrow (I = \phi$, and $\tilde{c}_j = \tilde{c}_r \cdot \varepsilon$, with a unit ε),

or $(I \neq \phi$, $b_i(j) = b_i (i \in I)$ and $b_i(j) = a_i (i \notin I)) \Leftrightarrow \tilde{c}_k \equiv 0 (\text{mod.} \tilde{c}_j)$.

Thus we have:

(1.15-8-3) $D_{j, r+1; \eta} \ni p \Leftrightarrow \tilde{c}_k \equiv 0 (\text{mod.} \tilde{c}_j) (1 \leqq k \leqq r)$.

From the above we have this lemma. q.e.d.

The following is easily derived from the above lemma and shows that the variety $\overline{Y}$ has a very restricted property.

Corollary 1.5.2. Assume that $\overline{Y} \cap \overline{X}^1_{reg} \neq \phi$ and $E_{\overline{X} | \overline{X} - \overline{X}^1_{sing}}$ is locally free. Then we have:

(1.15.19) $(\overline{Y} \cap \overline{X}^1_{reg})$ is of codimension one in $\overline{X}^1_{reg}$.

The arguments hitherto concerning the locally freeness conditions are purely local in the sense that they are given for each point on $\overline{X}^1_{reg}$. Here we give conditions of more global nature: For this letting $\overline{Y}^2_j$ denote $(\tilde{c}_j)_{0, red} \cap \overline{X}^1_{reg}$, we assume that $\overline{Y}^2_j$ admits the (finite) irreducible decomposition, and we write it as follows:

(1.5.20-1) $\overline{Y}^2_j = \overline{Y}^2_{j1} \cup \cdots \cup \overline{Y}^2_{j, s(j)}$

Then the following is easily checked:

Proposition 1.5.4. For each $u \in \{1,..,s(j)\}$ there is an element $m_u(j) \in \mathbb{Z}_+$ such that

(1.5.20-2) $\tilde{c}_j \equiv 0 (I_{j,u,p}^{m_u(j)})$ but $\not\equiv 0 (I_{j,u,p}^{m_u(j)+1})$ for each $p \in \tilde{Y}^2_{j,u}$

where $I_{j',u}$ denotes the ideal of $\widetilde{Y}^2_{j',s(j)}$.

We write (1.5.20-1) and (1.5.20-2) symbolically as follows:

(1.5.20-3) $\quad (\widetilde{c}_j)_0|\widetilde{X}^l_{reg} = m_1(j)\widetilde{Y}^2_{j',1} + \cdots + m_{s(j)}(j)\cdot\widetilde{Y}^2_{j',s(j)}$.

Let $\widetilde{Y}_1,\ldots,\widetilde{Y}_s$ be the set of all irreducible subvarieties of $\widetilde{X}^1_{reg}$ that appear as the irreducible component of $\widetilde{Y}^2_j$ $(1 \leq j \leq r)$. Assume that the indices in (1.5.20-3) are so chosen that

(1.5.20-4) $\quad (\widetilde{c}_j)_0|\widetilde{X}^l_{reg} = (m_1(j)\widetilde{Y}^2_1 + \cdots + m_s(j)\widetilde{Y}^2_s) + (m_{s+1}(j)\widetilde{Y}^2_{j',s+1} +$

$$+ \cdots + m_{s(j)}(j)\widetilde{Y}^2_{j',s(j)}) .$$

We will give a locally freeness condition, by using the above expression: For this take a subset $I=(i_1,\ldots,i_u)$ of $(1,\ldots,s)$ satis-fying

(1.5.20-5) $\quad \widetilde{Y}^2_I (:= Y^2_{i_1} \cap \cdots \cap Y^2_{i_u}) \neq \phi$.

For each $k \in \{1,\ldots,r\}$ we set:

(1.5.20-6) $\quad m_I(k) = (m_{i_1}(k),\ldots m_{i_u}(k))$,

and we write $m_I(k) \leqq m_I(k')$ if $m_{i_1}(k) \leqq m_{i_1}(k'),\ldots,m_{i_u}(k) \leqq m_{i_u}(k')$.

We say that $j \in \{1,\ldots,r\}$ is <u>of type</u> I, if $m_I(j) \leqq m_I(k)$ for all $k \in \{1,\ldots,r\}$. Define a Zariski open set $\widetilde{Y}'_I$ of $\widetilde{Y}_I$ by

(1.5.20-7) $\quad \widetilde{Y}'_I = \widetilde{Y}_I - \bigcup_{I' \supsetneqq I} \widetilde{Y}_{I'}$.

<u>Proposition 1.5.5. We have the disjoince union of $\widetilde{Y}$ as foll-ows:</u>

(1.5.20-8) $\quad \widetilde{Y} = \coprod_I \widetilde{Y}'_I$, <u>where a subset</u> I <u>of</u> $\{1,\ldots,s\}$ <u>satisfies</u>

(1.5.20-5) and $\widetilde{Y}'_I \neq \phi$.

Proof. For a point $p \in \widetilde{Y}$ we define a subset $I(p)$ of $\{1,..,s\}$ By

(1.5.20-9-1) $\quad I(p) = \{ j \in \{1,..,s\} ; \widetilde{Y}_j \ni p \}$.

The subset $I(p)$ is also characterized by

(1.5.20-9-2) $\quad \widetilde{Y}_{I(p)} \ni p$, and $I(p) \geqq I$ if $\widetilde{Y}_I \ni p$,

and we see easily that, for a subset I of $\{1,..,s\}$ satisfying (1.5.20-5) and $\widetilde{Y}_I \neq \phi$, we have:

(1.5.20-10) $\quad \widetilde{Y}'_I = \{ p \in \widetilde{Y} ; I(p) = I \}$.

From this we easily have (1.5.20-8). q.e.d.

Now using the stratification (1.5.20-8) of $\widetilde{Y}$, we will give a locally freeness condition of $E_{\overline{X}|\overline{X}-\overline{X}^1_{sing}}$. For this taking a subset I of $\{1,..,s\}$, we set:

(1.5.20-11) $\quad T_I = \{ j \in \{1,...,r\} ;\ j$ is of type $I \}$.

Lemma 1.5.3. $E_{\overline{X}|\overline{X}-\overline{X}^1_{sing}}$ <u>is locally free if and only if the following holds for each subset</u> I <u>of</u> $\{1,..,s\}$ <u>satisfying</u> (1.5.20-8) <u>and</u> $\widetilde{Y}'_I \neq \phi$.

(1.5.21-1) $\quad T_I \neq \phi$, <u>and</u> $\widetilde{Y}'_I \cap (\bigcap_{j \in T_I} \widetilde{Y}^{2}_j) = \phi$, <u>where the subvariety</u> $\widetilde{Y}^{\prime 2}_j$ <u>of</u> $\widetilde{Y}_j$ <u>is defined to be the second component of the expression</u> (1.5.20-4).

Proof. Take a point $p \in \widetilde{Y}$ and we set $I = I(p)$ $(\subset \{1,..,s\})$. By Lemma 1.5.1, $E_{\overline{X}',p}$ is $\mathcal{O}_{\overline{X}',p}$-free if and only if there is an index $j \in \{1,..,r\}$ such that

(a) $\quad \widetilde{c}_k \equiv 0 \pmod{\widetilde{c}_j}$ for $k = 1,..,r$.

From a simple observation we have:

(b) $\quad j \in T_I$ and $\widetilde{Y}^{\prime 2}_j \not\ni p$.

This implies the following implication:

(c) "$E_{\bar{X},p}$ is $\mathcal{O}_{\bar{X},p}$-free for each $p \in \tilde{Y}_I'$" $\Leftrightarrow$ "(1.5.21-1) for I".

Conversely, assume the right hand side of (c). Then, for a point $p \in Y_I'$, take an element $j \in T_I$ such that $p \in \tilde{Y}_I' - \tilde{Y}_j'^2$. But from (1.5.20-4) we have: $\tilde{c}_k \equiv 0 \,(\mathrm{mod}.\, \tilde{c}_j)\ (1 \leqq k \leqq r)$ at p. Thus we have the converse of (c). This, toghether with the disjoint union (1.5.20-8), implies the lemma. q.e.d.

Examples. The simplest case may be that there is an irreducible divisor $\tilde{Y}'$ such that

(1.5.22-1) $(\tilde{c}_j)_{0,red} = \tilde{Y}'$ for all $1 \leqq j \leqq r$.

In this case we see easily that $E_{\bar{X}|\bar{X}-\bar{X}^1_{sing}}$ is locally free. Next assume that each $(\tilde{c}_j)_0$ can be written as follows:

(1.5.22-2) $(\tilde{c}_j)_0 = \tilde{Y}' + \tilde{Y}_j''$, where $\tilde{Y}'$ is irreducible and $\tilde{Y}_j''$ does not contain $\tilde{Y}'$.

Then Lemma 1.5.3 can be rewritten as follows:

(1.5.22-3) "$E_{\bar{X}|\bar{X}-\bar{X}^1_{sing}}$ is locally free" $\Leftrightarrow$ "$\bigcap_j \tilde{Y}_j'' = \phi$" .

We summarize the arguments hitherto in the following diagram:

$$(1.5.23) \qquad \begin{array}{ccccc} \bar{X} & \hookleftarrow & \bar{Y} & \hookleftarrow & \bar{Z} = \bar{Y} \cap \bar{X}^1_{sing} \\ & \text{$E_{\bar{X}}$ is locally free} & & \text{The locally freeness condition is given explicitly} & \end{array}$$

We hope to discuss the locally freeness of $E_{\bar{X}}$ for $\bar{Z}$, by our arguments (Lemma 1.5.1 $\sim$ 1.5.3) in an another place.

Remark. Using the same notation to Remark in § 1.3 for Scubert calculus, we write $(\tilde{c}_j)_0 = (e_1 \wedge \cdots \wedge \tilde{e}_{r+1-j} \wedge \cdots \wedge \tilde{e}_{r+1})_0$, and we rewrite (1.3.8-3) as follows:

(1.5.24) $(\tilde{c}_j)_0 = \tilde{Y} \cup \tilde{X}_j^2$, with $\tilde{X}_j^2 = (e_1 \wedge \cdots \wedge \hat{e}_{r-1-j} \wedge \cdots \wedge e_{r-1})_0$.

Also from Scubert calculus we have:

(1.5.25) $\tilde{X}^1_{sing} = \bigcap_{j=1}^{r} (e_1 \wedge \cdots \wedge \hat{e}_j \wedge \cdots \wedge e_r)_0 \,(= \bigcap_{j=1}^{r} \overline{X}_j^2)$,

and $\operatorname{codim}_{\overline{X}} \overline{X}^1_{sing} = 4$.

Note that, in this case, one can change the role of $(\tilde{c}_1, \ldots, \tilde{c}_{r+1})$ to $(\tilde{c}_{\sigma(1)}, \ldots, \tilde{c}_{\sigma(r+1)})$ for any permutaion σ of (r+1)-letters, and using the notation in (1.3.8-1), we set:

(1.5.26-1) $\overline{W} := \bigcap_{j=0}^{r} \overline{X}^1_{j,sing} \,(= \bigcap_{1 \leq j < k \leq r+1} (\tilde{e}_1 \wedge \cdots \wedge \hat{\tilde{e}}_j \wedge \cdots \wedge \hat{\tilde{e}}_k \wedge \cdots \wedge \tilde{e}_{r+1})_0$

This is a Scubert cycle of codimension six.

From this and Lemma 1.4.2 and 1.4.3, we say that our local freeness conditions (at the present moment) are applied for Stein or projective manifolds up to codimension < 6.

§2. Residue conditions

Recall that our construction of holomorphic vector bundles start with a datum $\underline{D}=(\bar{X}^1, N_1; \underline{e}^0, \underline{e}^1)$ as in Definition 1.1.1. Then letting $E_{\bar{X}}$ be the direct image sheaf in question and assuming that $E_{\bar{X}}$ is locally free, the following two procedures may be our basis for investigations of global properties of the bundle $E_{\bar{X}}$:

(*-1) To stratify the codimension two subvariety $\bar{X}^2$ suitably and to attach a suitable open neighborhood to each stratum, and

(*-2) to attach a suitable frame of $E_{\bar{X}}$ in the neighborhood of each stratum.

Now the residue condition in the title is spoken in terms of the frames as in (*-2) and concerns explicit determinations of the characteristic classes in the sense of Atiyah [ ]; we may regard the validity of the condition as a basic factor in our procedures (*-1) and (*-2)

1, First we recall very quickly the theory of Atiyah on the characteristic classes([1]) in a convenient form for our explicit computations from now on: Let M be a complex manifold and E_M a holomorphic bundle over M. Moreover, let $\underline{N} = \{N_\lambda\}_{\lambda\in\Lambda}$ be an open covering of M such that $E_M|N_\lambda$ is trivial for each $\lambda\in\Lambda$. Fixing a frame $\underline{e}_\lambda$ of $E_M|N_\lambda$, we let $h_{\lambda\mu}$ denote the transition matrix for (e_λ, e_μ); $\lambda,\mu\in\Lambda$. Letting $\underline{e}$ denote the collection $\{\underline{e}_\lambda ; \lambda\in\Lambda\}$, we have an element (Atiyah class) $\theta = \theta(\underline{N},\underline{e}) \in Z^1(\underline{N}, \mathrm{End}(E_M)\otimes\Omega^1_M)$, where, for each $(\lambda,\mu)\in\Lambda\times\Lambda$, the component $\theta_{\lambda\mu}$ of θ is as follows:

(2.1.1) $\theta_{\lambda\mu} = dh_{\lambda\mu}\cdot h^{-1}_{\lambda\mu} \quad (\in \Gamma(N_\lambda\cap N_\mu, \mathrm{End}(E_M)\otimes\Omega^1_M))$.

(Recall that this is the obstruction for the existence of holomorphic connection for E_M ([1]). Also we recall the following (p.191,[1]): let $\tilde{\theta}$ be the element of $H^1(M,\mathrm{End}(E_M))$ which is defined by θ. Then, by means of Dolbeaut isomorphism, $\tilde{\theta}$ corresponds to the curvature form $\Theta\in H^{1,1}(M,\mathrm{End}(E_M))$ of a suitable connection form of E_M.)

Next let I_p be a polynomial of degree $p: M_r(\mathbb{C}) \longrightarrow \mathbb{C}$, which is invariant by the adjoint action of $GL_r(\mathbb{C})$, and elt $\tilde{I}_p$ be the corresponding $\mathcal{O}_M$-morphism:

(2.1.2) $$\underbrace{\mathrm{End}(E_M)\times\cdots\times\mathrm{End}(E_M)}_{p} \longrightarrow \mathcal{O}_M$$

Then, one can attach to θ an element $\omega^p = \omega^p(\underline{N},\underline{e}) \in Z^p(\underline{N},\Omega^p_M)$ by means of the map $\tilde{I}_{p,0}$ as follows:

$$\begin{tikzcd}
Z^1(N, \mathrm{End}(E_M)\otimes\Omega^1_M) \ni \theta \arrow[rr, "\cup^p"] \arrow[dr, "\bar{I}_p"'] & \arrow[d, phantom, "\circlearrowright"] & Z^p(N, \mathrm{End}(E_M)^{\otimes p}\otimes(\Omega^1_M)^{\otimes p}) \ni \cup^p\theta \arrow[dl, "I_p\otimes\Lambda^p"] \\
& Z^p(N, \Omega^p_M) \ni \omega^p = \bar{I}_{p,0}(\theta) &
\end{tikzcd}$$

where $\cup^p$ and $\wedge^p$ are the cup and exterior products. Then the characteristic class of Atiyah([1]) is defined to be the element $\tilde{\omega}^p \in H^p(M, \Omega^p_M)$, which is determined by $\omega^p = \bar{I}_{p,0}(\theta)$ $\in Z^p(\underline{N}, \Omega^p_M)$, with the basic invariant polynomial $I_{p,0}$ of degree p.

In our context, the pair $(\underline{N}, \underline{e})$ will be a basic datum for investigations of global structures of E_M, and the element $\omega = \omega(\underline{N}, \underline{e}) \in Z^1(\underline{N}, \Omega^1_M \otimes \mathrm{End}(E_M))$ may be a most basic invariant of the pair $(\underline{N}, \underline{e})$ from the view point of de Rham complexes.

2. Now let us return to our original situation: Assume that E_X is locally free and that there is a stratification S of X in such a manner that

(2.2.1-1) $S^0 = X - X^1$ and $S^1 := X^1 - X^2$ are elements of $\underline{S}$ (and so the codimension two subvariety $\overline{X}^2$ is the union of strata of $\underline{S}$),

and, for each stratum S of $\underline{S}$, we have:

(2.2.1-2) there is an open neighborhood N_S of S in X such that $E_X|N_S$ is trivial.

We fix a frame $\underline{e}_S$ of $E_X|N_S$ for each $S \in \underline{S}$, and we write $\underline{N}_S$ and $\underline{e}_S$ for the collections $\{N_S; S \in \underline{S}\}$ and $\{\underline{e}_S; S \quad \underline{S}\}$.

Henceforth in §2 we assume that $\overline{X}$ is smooth. Then we have the element $\theta = \theta(\underline{N}_{\underline{S}}, \underline{e}_{\underline{S}}) \in Z^1(\underline{N}_{\underline{S}}, End(E_{\overline{X}}) \otimes \Omega^1_{\overline{X}})$ and the characteristic element $\omega^p = \omega^p(\underline{N}_{\underline{S}}, \underline{e}_{\underline{S}}) \in Z^p(\underline{N}_{\underline{S}}, \Omega^p_{\overline{X}})$. Now take an element $\underline{U} = (S^0, \ldots, S^p) \in Nv^p_{\underline{S}}$, where $\operatorname{codim}_{\overline{X}} S^i = i\ (0 \leqq i \leqq p)$ and let $i_{\underline{U}}$ be the inclusion: $\underline{N}_{\underline{U}} := \{N_{S^j}\}_{j=0}^p \hookrightarrow \underline{N}_{\underline{S}}$ and we set:

(2.2.2) $\quad \omega^p_{\underline{U}} := i^*_{\underline{U}} \omega^p \in Z^p(\underline{N}_{\underline{U}}, \Omega^p_{\overline{X}}) \ (\simeq \Gamma(\bigcap_{j=0}^p N_{S^j}, \Omega^p_{\overline{X}}))$.

Now we introduce a condition for this element $\omega^p_{\underline{U}}$, which concerns the boundary behaviors of $\omega^p_{\underline{U}}$ around the main body S^p of $\underline{U}$: To formulate this condition, let $N_{S^j|p}$ denote $S^j \cap S^p$ $(0 \leqq j \leqq p)$ and we set $N_{\underline{U}|p} = \{N_{S^j|p}\}_{j=0}^p$. Remark that $\bigcap_{j=0}^p N_{S^j} = N_{S^j|p}$ and we may regard:

(2.2.3) $\quad \omega^p_{\underline{U}} \in Z^p(N_{\underline{U}|p}, \Omega^p_{\overline{X}}) \ (\simeq \Gamma(\bigcap_{j=0}^p N_{S^j|p}, \Omega^p_{\overline{X}}))$.

Next define a subset $\underline{N}'_{\underline{U}|p} = \{N_{S^j|p}\}_{j=0}^{p-1}$ of $\underline{N}_{\underline{U}|p}$, and for an abelian sheaf $\underset{\sim}{F}$ over N_{S^p}, we define a relative cochain complex $C^{\cdot}_{S^p}(\underline{N}_{\underline{U}|p}, \underset{\sim}{F})$ by the following exact sequence:

(2.2.4) $\quad 0 \longrightarrow C^{\cdot}_{S^p}(\underline{N}_{\underline{U}|p}, \underset{\sim}{F}) \longrightarrow C^{\cdot}(\underline{N}_{\underline{U}|p}, \underset{\sim}{F}) \longrightarrow C^{\cdot}(\underline{N}'_{\underline{U}|p}, \underset{\sim}{F}) \rightarrow 0$

Remark that $N_{\underline{U}|p}$ consists of $p+1$-elements, and we have:

(2.2.5)
$$\begin{array}{ccc} C^p_{S^p}(\underline{N}_{\underline{U}|p}, \underset{\sim}{F}) & \simeq & C^p(\underline{N}_{\underline{U}}, \underset{\sim}{F}) \\ \cup\!\!\!\!\downarrow & & \cup\!\!\!\!\downarrow \\ Z^p_{S^p}(\underline{N}_{\underline{U}|p}, \underset{\sim}{F}) & \simeq & Z^p(\underline{N}_{\underline{U}}, \underset{\sim}{F}) \end{array}$$

and we may regard:

(2.2.6) $\quad \omega^{p}_{\underline{U}} \in Z^{p}_{SP}(N_{\underline{U}|p}, \Omega^{p}_{X})$.

Now assume that there are elements $\underset{\sim}{f}=(f_1,..,f_p) \in \Gamma(N_{SP}, \underset{\sim}{O}_X)$ such that

(2.2.7-1) $\quad S^p(\subset N_{SP})$ is the (set theoretical) locus of $\underline{f}$

and

(2.2.7-2) $\quad f_j (1 \leqq j \leqq p)$ does not vanish in $N_{U|p} := \bigcap^{p}_{i=0} N_{Sj|p}$.

Then setting $\mathrm{dlog}\ \underline{f} = \mathrm{dlog}\ f_1 \wedge \cdots \wedge \mathrm{dlog}\ f_p \in Z^{p}_{SP}(N_{U|p}, \Omega^{p}_{X})$ we make:

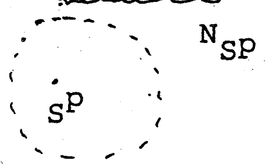
Definition 2.1. We say that $\omega^{p}_{\underline{U}}$ satisfies <u>residue condition with respect to $\underline{f}$,</u> if one can write:

(2.2.8) $\quad \omega^{p}_{\underline{U}} = a\ \mathrm{dlog}\ f + \delta\omega^{p-1}_{U}$, with an element $a \in \underset{\sim}{C}$ and an element $\omega^{p-1}_{\underline{U}} \in C^{q-1}_{S^P}(N_{U|p}, \Omega^{p}_{X})$.

If N_{SP} is a sufficiently small neighborhood of S^p (in X), the residue condition (2.2.8) concerns a boundary behavior of the differential form $\omega^{p}_{\underline{U}}$ around the main part S^p of U.

We like to take the residue condition just above as our basis for determination of the characteristic class of $E_{\overline{X}}$ (in terms of de Rham complex). Here we make a simple remark on the condition (2.2.8):

Figure I



Letting Φ^p_X denote the (abelian) sheaf of d-closed holomorphic differential forms over X, we have the following diagram:

$$
(2.2.9)\quad \begin{array}{ccc}
H^p(\underline{N}_{U|p}, \Phi^p_X) \ni \mathrm{adlog}\, \underline{f} & \xrightarrow{\beta} & \mathbb{H}^{2p}(N_{U|p}, \Omega^p_X) \\
\Big\downarrow \partial & & \\
H^p(N_{\underline{U}|p}, \Omega^p_X) \ni \omega^p_{\underline{U}} & &
\end{array}
$$

where $\mathbb{H}$ denotes the symbol of 'hyper-cohomology'. Note that the two cohomology groups in the top line in (2.2.9) are of topological nature whilr the one in the last line is of complex analytic nature. Now letting $\omega^p_{\underline{U}}$ be the element of $H^p(N_{\underline{U}|p}, \Omega^p_X)$ which is determined by $\omega^p_{\underline{U}}$, the residue condition (2.2.8) insures:

(2.2.10) $\omega^p_U \quad \in \quad$ image of ∂ .

and insures that the complex analytic element $\omega^p_{\underline{U}}$ is endowed with a topological meaning: .

3. Now assume that our bundle $E_{\overline{X}}$ is of type (G), and let $\check{c}=(\tilde{c}_1,\ldots,\tilde{c}_{r+1}) \subset \Gamma(\overline{X}, O_{\overline{X}})$ be as in § 1.5. We assume the following 'generic condition' for $\underline{c}$:

(2.3.1-1) $\overline{X}^{j+1} := (c_{r+1},\ldots,c_{r+1-j})_{0,\mathrm{red}}$ $(0 \leq j \leq r)$ is of codimension $j+1$ (if $j+1 \leq \dim X$),

and we set:

(2.3.1-2) $\quad X^{j+1} = \overline{X}^{j+1} - \overline{X}^{j+2} \quad (0 \leq j \leq r)$

Also attach suitably an open neighborhood N_{j+1} of X^{j+1} to each j. Then setting

(2.3.1-3) $\quad Y = \bigcap_j (c_{r+1-j})_{0,red} \quad (0 \leq j \leq \min(\dim \bar{X} - 1, r))$,

we remark that $\{X^{j+1}\}_j \cup \{X^0\}$, $X^0 = \bar{X} - \bar{X}^1$, gives a stratification of $\bar{X} - \bar{Y}$. Letting $\tilde{e}^{r+1-j} = (e_1, \ldots, \hat{e}_{r+1-j}, \ldots, e_{r+1})$ be as in the end of § 1.3, we recall that $\tilde{e}^{r+1-j}$ is taken to be a frame of $E_{X|N_{j+1}}$. Setting $\underline{N} = \{N_{j+1}\}_j$ and $\underline{e} = (e_{r+1-j})$, we have the element $\omega^p(\underline{N}, \underline{e}) \in Z^p_{XP}(N_{\underline{U}|p}, \Omega^p_X)$ with $\underline{U} = (X^p, \ldots, X^0)$ $(p \leq \min(\dim X - 1, r))$. Then remarking that $(\tilde{c}_{r+1})_0 \cap \cdots \cap (\tilde{c}_{r+2-p})_0 = \bar{X}^p$ we have:

Theorem 2.1. *The element $\omega^p(\underline{N}, \underline{e})$ satisfies the residue condition with respect to $(\tilde{c}_{r+1-k})_{k=0}^{p-1}$.*

This is essentially very elementary, but requires some long computations. Details will be given in an another place.

References

1.M.F.ATiyah, Complex analytic connections in fiber bundles, Trans.Amer.Math.Soc.7.181-207(1957)

2.H.Grauert-G.Mullich M Vectorbundel vom rang 2 uber den n-dimensionalen komplex-projectiven Raum. Manuscripta.math. 16,75-100(1975).

3.R.Hartshorne, Algebraic vector bundles on projective spaces, a problem list. Topology, Vol.18, 117-128(1978)

4. S.Kleiman, Geometry on Grassmannians and applications to splitting bundles and smoothing cycles. Publ.Math.I.H.E.S. (1969)281-297

5. M.Maruyama, On a family of algebraic vector bundles, Kinokuniya, Tokyo,95-146(1973)

6. N.Sasakura, Cohomology with polynomial growth and completion theory, Publ.R.I.M.S, 171-352(1981)

7.___________, A type of comparison theorem in polynomial growth cohomology, Proc.of Japan Acad. 297-301(1983)

8. Y.T.Siu-G.Trautmann , Gap sheaves and extensions of coherent analytic sheaves, Lect.Note in Math.Vol.173

9. H.Schneider, Holomorphic vector bundles on P_n, Seminaraire Bourbaki 530(1978 ~ 1979)

10. A.N.Tjurin, The classification of vector bundles over algebraic curves of arbitrary genus., Izv.Acad.29,657-688(1965)

11. A.Weil, Generalization des fonctions abeliennes, J.Math. pures Appl.17,47-87(1938)