

Normal Surfaces and Intersection Theory

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In this note we develop geometry of normal surfaces by using the intersection theory introduced by Mumford [4]. We shall study the contraction criterion, the projection formula, the Noether formula, the vanishing theorem, the minimal model, the Miyaoka inequality, etc. Details will be discussed elsewhere.

Notation

A surface will mean an irreducible reduced compact complex space of dimension 2. A divisor will mean a Weil divisor (i.e., a linear combination of irreducible curves) unless otherwise specified. We use "birational morphism" instead of bimeromorphic morphism.

$Y$  : a normal surface

$X$  : a resolution of singularities of  $Y$

$\text{Div}(Y)$  : the group of divisors on  $Y$

An element of  $\text{Div}(Y, \mathbb{Q}) = \text{Div}(Y) \otimes \mathbb{Q}$  is called a  $\mathbb{Q}$ -divisor.

Given a  $\mathbb{Q}$ -divisor  $D = \sum \alpha_i C_i$  where the  $C_i$  are irreducible curves and  $\alpha_i \in \mathbb{Q}$ , we write as

$$[D] = \sum [\alpha_i] C_i \quad ([\alpha] \text{ is the greatest integer } \leq \alpha)$$

$$\{D\} = \sum \{\alpha_i\} C_i \quad (\{\alpha\} \text{ is the least integer } \geq \alpha)$$

1. Contraction criterion

Let  $Y$  be a normal surface. The intersection pairing  $\text{Div}(Y) \times \text{Div}(Y) \rightarrow \mathbb{Q}$  is defined as follows ([4]). Let  $\pi: X \rightarrow Y$  be a resolution of singularities and let  $A = \bigcup E_i$  denote the exceptional set of  $\pi$ . For a divisor  $D$  on  $Y$  we define the inverse image  $\pi^*D$  as

$$\pi^*D = \bar{D} + \sum \alpha_i E_i$$

where  $\bar{D}$  is the strict transform of  $D$  and the rational numbers  $\alpha_i$  are uniquely determined by the equations:  $\bar{D}E_j + \sum \alpha_i E_i E_j = 0$  for all  $j$ . For two divisors  $D$  and  $D'$  the intersection number  $DD'$  is defined to be the rational number  $(\pi^*D)(\pi^*D')$ .

A divisor  $D$  on  $Y$  is numerically equivalent to zero, denoted by  $D \approx 0$ , if  $DC = 0$  for all curves  $C$  on  $Y$ . Two divisors  $D$  and  $D'$  are numerically equivalent,  $D \approx D'$ , if  $D - D' \approx 0$ . Set  $N(Y, \mathbb{Q}) = (\text{Div}(Y) / \approx) \otimes \mathbb{Q}$ . The Picard number  $\rho(Y)$  of  $Y$  is the rank of the  $\mathbb{Q}$ -vector space  $N(Y, \mathbb{Q})$ . We have the equality:  $\rho(Y) = \rho(X) - \rho(\pi)$  where  $\rho(\pi)$  is the number of irreducible components of  $A$ .

The following is the normal surface version of the Grauert's contraction criterion theorem.

Theorem (1.1)(Contraction Criterion). Let  $C_1, \dots, C_k$  be irreducible curves on a normal surface  $Y$ . Then the union  $\bigcup C_i$  can be contracted to normal points if and only if the intersection matrix  $(C_i C_j)$  is negative definite.

Proof. By definition  $\pi^*C_i = \bar{C}_i + Z_i$  with  $\text{Supp}(Z_i) \subset A$ . Let  $G = \sum \alpha_i \bar{C}_i + Z$  be a  $\mathbb{Q}$ -divisor on  $X$  such that  $\text{Supp}(Z) \subset A$ . Write  $G = \pi^*(\sum \alpha_i C_i) + Z'$  where  $Z' = Z - \sum \alpha_i Z_i$ . We find that  $G^2 = (\sum \alpha_i C_i)^2 + Z'^2$ . As a consequence the Grauert's theorem applied to  $X$  proves the assertion. Q.E.D.

We now consider a birational morphism  $f: Y' \rightarrow Y$  between normal surfaces  $Y'$  and  $Y$ . We denote by  $A_f$  the exceptional set of  $f$ . Write  $A_f = \cup C_i$ . For a divisor  $D$  on  $Y$  the inverse image  $f^*D$  is also defined. As a corollary of the above criterion, we can write as

$$(1.2) \quad f^*D = \bar{D} + \sum \beta_i C_i$$

where  $\bar{D}$  is the strict transform of  $D$  by  $f$  and the rational numbers  $\beta_i$  are determined by the equations:  $\bar{D}C_j + \sum \beta_i C_i C_j = 0$  for all  $j$ . We can prove that  $\rho(Y') = \rho(Y) + \rho(f)$ .

Definition (1.3). Let  $D$  be a  $\mathbb{Q}$ -divisor on a normal surface. We say that  $D$  is nef (numerically effective) if  $DC \geq 0$  for all curves  $C$  on  $Y$  and that  $D$  is pseudo effective if  $DP \geq 0$  for all nef divisors  $P$  on  $Y$ .

## 2. Projection formula

A coherent sheaf  $F$  on  $Y$  is reflexive if  $F^{\vee\vee} \simeq F$  where  $F^\vee$  is the dual sheaf  $\text{Hom}(F, \mathcal{O}_Y)$ . A reflexive sheaf of rank one is called a divisorial sheaf. Set  $Y_0 = Y \setminus \text{Sing } Y$  with the inclusion

$i:Y_0 \rightarrow Y$ . A coherent sheaf  $F$  on  $Y_0$  is said to be extendible if it extends to a coherent sheaf on  $Y$ . It is proved by Serre (Ann.Inst.Fourier 16) that if  $F$  is an extendible reflexive sheaf on  $Y_0$ , then  $i_*F$  is a reflexive sheaf on  $Y$ , which is unique as a reflexive extension of  $F$ .

For a divisor  $D$  on  $Y$  the invertible sheaf  $O(D|_{Y_0})$  on  $Y_0$  is extendible. Indeed the coherent sheaf  $\pi_*O(\bar{D})$  is an extension. It follows that the sheaf  $i_*O(D|_{Y_0})$  is a divisorial sheaf on  $Y$ . We denote it by  $O(D)$ . Clearly  $i_*i^*O(D) \simeq O(D)$ . When  $Y$  is Moisézon, every divisorial sheaf is defined by a divisor. For a  $\mathbb{Q}$ -divisor  $D$  we understand that  $O(D) = O([D])$ . Two  $\mathbb{Q}$ -divisors  $D$  and  $D'$  are linearly equivalent, denoted by  $D \sim D'$ , if the difference  $D - D'$  is a principal divisor of a non-zero meromorphic function. We have the equivalence:  $D \sim D' \iff$  (i)  $D - D'$  is integral, (ii)  $O(D) \simeq O(D')$ .

The following result connects the cohomological invariants of  $Y$  with those of  $X$ .

Theorem(2.1)(Projection Formula). Let  $D$  be a  $\mathbb{Q}$ -divisor on a normal surface  $Y$ . Let  $\pi:X \rightarrow Y$  be a resolution. Then

$$\pi_*O(\pi^*D) \simeq O(D).$$

Outline of Proof. It is sufficient to consider the local situation. Let  $(V,y)$  be a normal surface singularity with a resolution  $\pi:U \rightarrow V$ . As before let  $A = \bigcup E_i$  denote the exceptional set of  $\pi$ . There is an exact sequence originated by Laufer:

$$0 \rightarrow H^0(U, O(\pi^*D)) \rightarrow H^0(U \setminus A, O(\pi^*D)) \rightarrow H^1_C(U, O(\pi^*D)).$$

Since  $H^0(U \setminus A, \mathcal{O}(\pi^*D)) \simeq H^0(V \setminus Y, \mathcal{O}(D)) \simeq H^0(V, \mathcal{O}(D))$ , the assertion follows from the vanishing:  $H_C^1(U, \mathcal{O}(\pi^*D)) = 0$ . By duality we have  $H_C^1(U, \mathcal{O}(\pi^*D)) \simeq H^1(U, \mathcal{O}(K + \{-\pi^*D\}))$  where the  $K$  is a canonical divisor of  $U$ . So we can complete the proof by the following

Theorem (2.2) (Local Vanishing Theorem). Let  $D$  be a  $\mathbb{Q}$ -divisor on  $U$ . Suppose that  $DE_j \geq 0$  for all  $j$ . Then

$$R^1 \pi_* \mathcal{O}(K + \{D\}) = 0.$$

Remark. In the algebraic context, Theorems (2.1) and (2.2) hold in all characteristics.

Theorem (2.3) (Generalized Projection Formula). Let  $f: Y' \rightarrow Y$  be a birational morphism of normal surfaces. Let  $D$  be a  $\mathbb{Q}$ -divisor on  $Y$  and let  $Z$  be an effective  $\mathbb{Q}$ -divisor supported on the exceptional set  $A_f$ . Then

$$f_* \mathcal{O}(f^*D + Z) \simeq \mathcal{O}(D).$$

3.  $\mathbb{Q}$ -divisor  $\Delta$ 

We study the inverse image of a canonical divisor. Let  $(V, y)$  be a normal surface singularity with a resolution  $\pi: U \rightarrow V$ . Let  $A = \cup E_i$  denote the exceptional set. If  $K$  is a canonical divisor of  $U$ , then  $K_V = \pi_* K$  is a canonical divisor of  $V$ . Now define a  $\mathbb{Q}$ -divisor  $\Delta = \sum \delta_i E_i$  by the equations:  $K E_i + \sum \delta_i E_i E_j = 0$  for all  $j$ . We infer from the definition in Sect. 1 that

$$(3.1) \quad \pi^* K_V = K + \Delta.$$

When  $\pi$  is the minimal resolution in the sense that there is no exceptional curve of the first kind in  $A$ , it can be shown that  $\Delta \geq 0$  and that  $\Delta = 0 \iff y$  is a rational double point. We introduce the following numerical invariants:

$$(3.2) \quad \begin{aligned} h(y) &= \dim R^1 \pi_* \mathcal{O}_U && \text{(the genus)} \\ \mu(y) &= e(A) + \Delta^2 - 1 + 12h(y) && \text{(the Milnor number)} \end{aligned}$$

where  $e(A)$  is the Euler number of  $A$ . Note that  $\mu(y) \in \mathbb{Q}$ .

Example (3.3). Let us examine the case in which the weighted dual graph of the exceptional set of the minimal good resolution is a star and only the central curve (if exists) may have positive genus. This is the case if the singularity has a good  $\mathbb{C}^*$ -action (Orlik-Wagreich).

(a) cyclic quotient singularity.

The weighted dual graph is a chain of  $\mathbb{P}^1$ 's.

$$\begin{array}{ccccc} E_1 & & E_2 & & E_n \\ & \circ & \text{---} & \circ & \text{---} & \circ \\ & -a_1 & & -a_2 & & -a_n \end{array}$$

Define  $d/e = [a_1, \dots, a_n] = a_1 - \frac{1}{a_2 - \frac{1}{\dots - \frac{1}{a_n}}}$

Consider the equations:  $X_{k+1} = a_k X_k - X_{k-1}$ . Let  $\{c_{ik}\}$ ,  $\{c'_{ik}\}$  be two solutions as

$$c_0 = d, c_1 = e, \dots, \text{ then } c_n = 1, c_{n+1} = 0,$$

$$c'_0 = 0, c'_1 = 1, \dots, \text{ then } c'_n = e', c'_{n+1} = d \quad (ee' = 1 \pmod{d}).$$

We have  $c_k \geq 0$ ,  $c'_k \geq 0$ . By a calculation (cf. Knöller, Math. Ann. 213),

$$(3.4) \quad \Delta = \sum (1 - (c_k + c'_k)/d) E_k$$

$$\mu = n + 4 - \nu - (e + e' + 2)/d$$

where  $\nu$  is the multiplicity of  $y$ , which is equal to  $\sum (a_i - 2) + 2$ .

(b) a star with a central curve  $E_0$  with genus  $g$ . There are finite number of branches of chains of  $\mathbb{P}^1$ 's,  $E_{ij}$   $j=1, \dots, n_i$ . Let  $E_0^2 = -a_0$ ,  $E_{ij}^2 = -a_{ij}$  ( $a_{ij} \geq 2$ ). Define  $d_i/e_i = [a_{i1}, \dots, a_{in_i}]$ ,  $\{c_{ik}\}$ ,  $\{c'_{ik}\}$  as above. The negative definiteness of the intersection matrix implies  $a_0 - \sum e_i/d_i > 0$ . With these notation we get

$$(3.5) \quad \Delta = \sum (1 - ((1 - \delta_0) c_{ik} + c'_{ik})/d_i) E_{ik} + \delta_0 E_0,$$

where

$$\delta_0 = 1 + \frac{\sum (1 - 1/d_i) + 2g - 2}{a_0 - \sum e_i/d_i}.$$

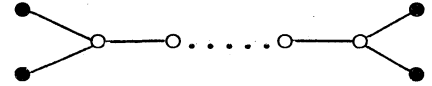
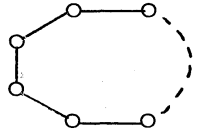
Example (3.6). Assume  $\pi$  is the minimal good resolution. Those singularities having the property:  $\delta_i \leq 1$  for all  $i$ , have been classified by K. Watanabe (Math. Ann. 250) and by Y. Kawamata (in somewhat different context, Lecture Notes in Math. 732,

Springer). We give the list. Here  $\circ$  denotes a non-singular rational curve and  $\bullet$  denotes a non-singular rational curve with self-intesection  $-2$ . Cf. Wagreich (Topology 11).

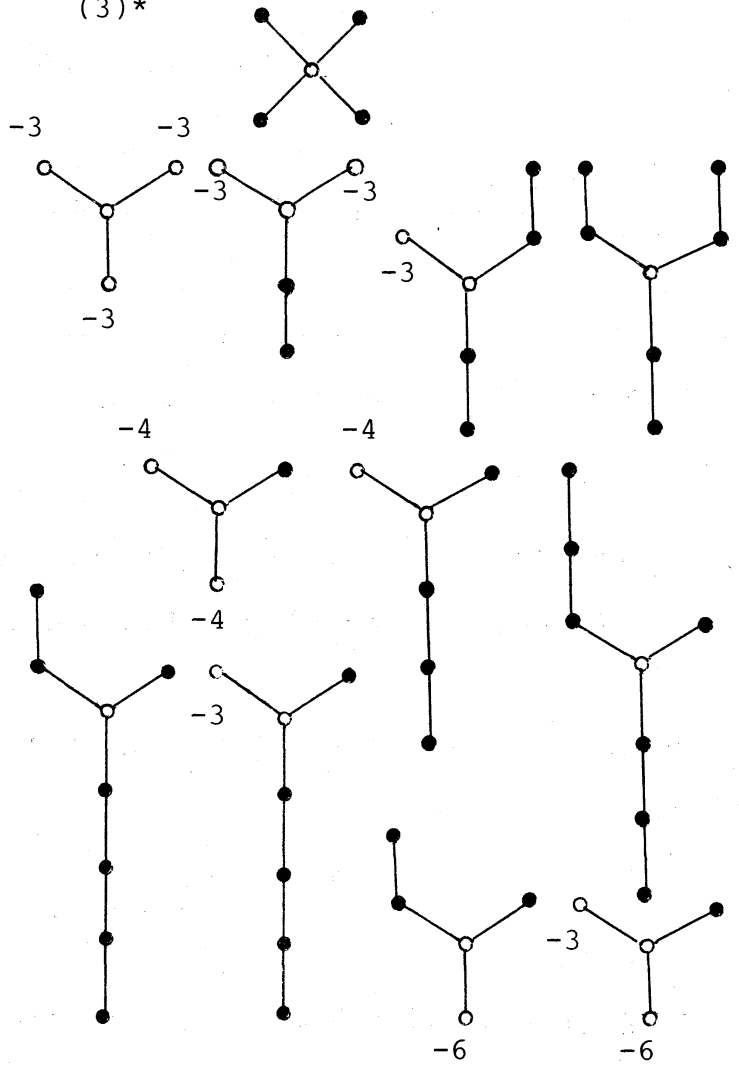
Table (3.7)(Singularities with  $\delta_i \leq 1$  for all  $i$ ).

(1) smooth point (1)\* quotient singularities

(2) cusp singularities (2)\*




(3) simple elliptic singularities (3)\*



(quotients of simple elliptic singularities)



The proof proceeds as follows. It turns out that  $\pi$  coincides with the minimal resolution except the case  -1. If  $A$  is a single curve, it is either  $\mathbb{P}^1$  or an elliptic curve. We consider the case in which  $A$  has more than one component. Assume  $y$  is not a rational double point. If  $A'$  is a proper subset of  $A$ , letting  $\Delta'$  be the  $\mathbb{Q}$ -divisor associated to  $A'$ , then we must have  $\Delta > \Delta'$ . We infer from this that every component of  $A$  is  $\mathbb{P}^1$ . Next one shows that  $A$  is a star except the cases (2), (2)\*. In case  $A$  is a chain, every coefficient of  $\Delta$  is less than one (cf.(3.4)). In case  $A$  is a star with a central curve, we deduce from (3.5) that  $\delta_0 \leq 1 \iff \sum(1-1/d_i) \leq 2$ . Looking in the coefficients of  $\Delta$  the condition  $\delta_0 \leq 1$  implies that all other coefficients are less than one. The inequality  $\sum(1-1/d_i) \leq 2$  has finite possibilities of  $d_i$ , which correspond to the cases (1)\* and (3)\*:

$$(1)^* \quad (2,2,d), (2,3,3), (2,3,4), (2,3,5)$$

$$(3)^* \quad (2,2,2,2), (3,3,3), (2,4,4), (2,3,6).$$

Remark. The above singularities (3) and (3)\* have appeared as ball cusp singularities ([2]). The case (1)\*  $\iff \delta_i < 1$  for all  $i$ .

#### 4. Noether formula, vanishing theorem

We come back to study normal surfaces. Let  $Y$  be a normal surface and let  $\pi: X \rightarrow Y$  be a resolution with  $A$  the exceptional set. For the sake of simplicity, we assume that  $X$  has a canonical divisor  $K$ . This is the case if  $X$  is projective, or equivalently if  $Y$  is Moisëzon. In general we have to deal with the canonical line bundle. For this argument, we refer to [6].

Since  $\pi_*K$  becomes a canonical divisor of  $Y$ , we denote it by  $K_Y$ . If  $\text{Sing } Y = \cup y_i$ , let  $\Delta_i$  be the  $\mathbb{Q}$ -divisor associated to  $y_i$  supported on  $A_i = \pi^{-1}(y_i)$ . Write  $\Delta = \sum \Delta_i$ . By (3.1) we have  $\pi^*K_Y = K + \Delta$  and hence

$$(4.1) \quad K_Y^2 = K^2 - \Delta^2.$$

Theorem (4.2) (Noether Formula). Let  $Y$  be a normal surface. Then

$$\chi(O_Y) = \frac{1}{12}(K_Y^2 + e(Y) + \sum \mu(y_i))$$

where  $e(Y)$  is the Euler number of  $Y$ .

Proof. Recall the Noether formula for  $X$ :

$$\chi(O_X) = \frac{1}{12}(K^2 + e(X)).$$

We have the following relations of Euler numbers:

$$e(X) = e(X \setminus A) + e(A) = e(Y \setminus \text{Sing } Y) + e(A) = e(Y) + \sum (e(A_i) - 1).$$

On the other hand  $\chi(O_Y) = \chi(O_X) + \dim R^1 \pi_* O_X$ . Combining these with (4.1) and the definition of  $\mu$ , we get the required result.

Q.E.D.

For the Riemann-Roch formula for divisorial sheaves, see [1]. We shall state the vanishing results.

Theorem (4.3) (Generalized Ramanujam Vanishing Theorem).

Let  $Y$  be a normal Moisëzon surface. Let  $D$  be a nef  $\mathbb{Q}$ -divisor with  $D^2 > 0$  on  $Y$ . Then

$$H^i(Y, O(K_Y + \{D\})) = 0 \quad \text{for } i > 0.$$

Proof. This follows from the corresponding vanishing theorem for  $X$ , combined with the local vanishing theorem and the projection formula (for details see [6]). Q.E.D.

The local vanishing theorem can be generalized as follows.

Theorem (4.4). Let  $f: Y \rightarrow Y'$  be a birational morphism of normal surfaces. If  $D$  is a relatively nef  $\mathbb{Q}$ -divisor on  $Y$ , then

$$R^1 f_* \mathcal{O}(K_Y + \{D\}) = 0.$$

Corollary. In particular we have  $R^1 f_* \mathcal{O}(K_Y) = 0$ , which is a generalization of the Grauert-Riemenschneider's vanishing theorem.

## 5. Minimal model

Let  $Y$  be a normal surface and  $D$  a divisor on  $Y$ . For every positive integer  $m$  we infer from the projection formula that  $\dim H^0(Y, \mathcal{O}(mD)) = \dim H^0(X, \mathcal{O}(m\pi^*D))$ . We define the  $D$ -dimension of  $Y$ , denoted by  $\kappa(D, Y)$ , to be  $\kappa(\pi^*D, X)$ .

Definition.

$$P_m(Y) = \dim H^0(Y, \mathcal{O}(mK_Y)) \quad (\text{the arithmetic } m\text{-genus})$$

$$\kappa(Y) = \kappa(K_Y, Y) \quad (\text{the arithmetic Kodaira dimension})$$

Let  $(Y, D)$  be a pair of a normal surface  $Y$  and a  $\mathbb{Q}$ -divisor  $D$  on  $Y$ . Such a pair is called a normal pair. We say that  $(Y, D)$

is (relatively) minimal if  $Y$  contains no irreducible curves  $C$  with  $DC < 0$ ,  $C^2 < 0$ . A birational morphism  $f: (Y, D) \rightarrow (Y', D')$  is a birational morphism  $f: Y \rightarrow Y'$  satisfying  $f_*D = D'$ . Write as  $D = f_*D' + R$  where  $\text{Supp}(R) \subset A_f$ . We say that  $f$  is totally discrepant if every irreducible component of  $A_f$  appears in  $R$  with positive coefficient. Given a normal pair  $(Y, D)$ , a minimal normal pair  $(Y', D')$  is called its minimal model if there is a totally discrepant birational morphism  $f: (Y, D) \rightarrow (Y', D')$ . In this case, by the projection formula (2.3) we get  $H^0(Y, \mathcal{O}(mD)) \cong H^0(Y', \mathcal{O}(mD'))$  for every positive integer  $m$ , hence  $\kappa(D, Y) = \kappa(D', Y')$ .

Theorem (5.1). Every normal pair has a minimal model. Furthermore, if  $D$  is pseudo effective, then  $(Y, D)$  admits a unique minimal model  $(Y', D')$  and  $D'$  is nef.

Proof. Let  $(Y, D)$  be a normal pair. Suppose it is not minimal. Then it contains an irreducible curve  $C$  with  $DC < 0$ ,  $C^2 < 0$ . Let  $\varphi: Y \rightarrow Y_1$  be the contraction of  $C$ . If we put  $D_1 = \varphi_*D$ , by (1.2) we find that  $D = \varphi_*D_1 + (DC/C^2)C$ . It follows from the hypothesis that  $D > \varphi_*D_1$ . Note that  $\rho(Y_1) = \rho(Y) - 1$ . Thus by a finite number of successive such contractions we arrive at a minimal model (for the latter assertion see [6]). Q.E.D.

Corollary (Zariski Decomposition). Let  $(Y, D)$  be a normal pair. Suppose  $D$  is pseudo effective. Let  $(Y', D'; f)$  be its minimal model. If we write  $P = f_*D'$ , then the decomposition

$$D = P + N$$

satisfies the following properties: (i)  $P$  is nef, (ii)  $N$  is

effective and  $\text{Supp}(N)$  is contracted by  $f$ . Furthermore, such decomposition is unique.

We talk of a pair  $(X, K+D)$  where  $X$  is a smooth surface and  $D$  is a reduced curve with normal crossings. If  $(Y, K_Y+B)$  is its minimal model, then  $Y$  has only quotient singularities (cf. [8]). Indeed, write as  $K+D=f^*(K_Y+B)+R$ ,  $\Delta=\Delta^+-\Delta^-$ , then  $D+\Delta^-=\Delta^++f^*B+R$ . Since  $f$  is totally discrepant, every coefficient of  $\Delta^+<1$ .

For normal surfaces a birational morphism  $f:Y \rightarrow Y'$  is totally discrepant if  $f:(Y, K_Y) \rightarrow (Y', K_{Y'})$  is totally discrepant in the above sense. In this case we have  $P_m(Y)=P_m(Y')$  for  $m>0$  and  $\kappa(Y)=\kappa(Y')$ . We say that  $Y$  is minimal if the pair  $(Y, K_Y)$  is minimal. Also  $Y'$  is a minimal model of  $Y$  if (i)  $Y'$  is minimal, (ii) there is a totally discrepant birational morphism  $f:Y \rightarrow Y'$ . Theorem (5.1) asserts that every normal surface has a minimal model. We are thus reduced to study minimal normal surfaces. If  $Y$  is minimal, then either (i)  $K_Y$  is not pseudo effective, or (ii)  $K_Y$  is nef. For further discussions and classification theory, we refer to [7] (for the Gorenstein case see [5]).

Example (5.2). Let  $B$  be a non-singular curve of genus  $g \geq 2$ . Let  $X=\mathbb{P}(E)$  be a ruled surface defined by a rank 2 vector bundle  $E$  on  $B$ . Suppose  $E$  is normalized as in the book of Hartshorne. Set  $\mathfrak{e}=\det E$ ,  $e=-\deg \mathfrak{e}$ . There is a base section  $b$  with  $b^2=-e$ . Suppose  $e>0$ . Let  $\pi:X \rightarrow Y$  be the contraction of  $b$ . Since  $\rho(Y)=1$ ,  $Y$  is of course minimal. We have  $\pi^*K_Y=K+\Delta=((2g-2-e)/e)b+p^*(k+\mathfrak{e})$  where  $p:X \rightarrow B$  is the projection map and  $k$  denotes a canonical divisor of  $B$ . It follows that  $K_Y^2=(2g-2)^2/e \geq 0$  and  $e(Y)=3-2g < 0$ .

There occur three cases: (i)  $K_Y$  is nef (if  $e < 2g-2$ ), (ii)  $K_Y \cong 0$  (if  $e = 2g-2$ ), (iii)  $-K_Y$  is nef (if  $e > 2g-2$ ).

Finally we mention about the Miyaoka inequality. We recall the following recent result (Miyaoka [3]): Let  $X$  be a smooth projective surface and  $D$  a divisor having normal crossings on  $X$ . Suppose  $K+D$  is pseudo effective and let  $K+D=P+N$  be the Zariski decomposition. Then

$$(5.3) \quad (K+D)^2 - \frac{1}{4}N^2 \leq 3e(X \setminus D).$$

We deal with normal surfaces whose singularities are contained in Table (3.7). Notice that there  $\{(2), (3)\}$  are elliptic singularities and  $\{(1)^*, (2)^*, (3)^*\}$  are rational singularities. We want to point out two facts.

$$(5.4) \quad (i) \quad \kappa(Y) \geq 0 \text{ if and only if } K_Y \text{ is pseudo effective.}$$

(ii) If  $K_Y$  is nef, then

$$\frac{3}{2} \# \text{ rat.Sing } Y + 3 \# \text{ ellip.Sing } Y + K_Y^2 \leq 3e(Y).$$

In particular, we have  $e(Y) \geq 0$ .

We show (ii). Let  $\pi: X \rightarrow Y$  be the minimal resolution. As noticed in Example (3.6), the exceptional set  $A = \cup E_i$  has normal crossings. If we write  $D = \sum E_i$ , then  $D - \Delta \geq 0$ . The pseudo effectiveness of  $K_Y$  implies that of  $K+D$ . Clearly,  $e(X \setminus D) = e(Y) - \# \text{ Sing } Y$ . On the other hand  $(K+D)^2 = (K+\Delta)^2 + (D-\Delta)^2$  and

$$(D-\Delta)^2 = (K+D)D - \Delta(D-\Delta) \geq (K+D)D = -2 \# \text{ rat.Sing } Y.$$

If  $K_Y$  is nef, we get  $P = \pi^* K_Y$  and so  $N = D - \Delta$ . By (5.3) we get (ii).

When  $Y$  has worse singularities, this is not necessarily the case. For instance in Example (5.2), if  $e < 2g - 2$ , then  $K_Y$  is nef and  $\kappa(Y) = 2$ , but  $e(Y) < 0$ .

In the case of quotient singularities, a more precise result can be found in [3].

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