

A Note on The Invariant Holonomic System

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INTRODUCTION. In this report, I shall prove a theorem which is an interpretation of the main result of [S]. I have studied in [S] the holonomic system which governs an invariant spherical hyperfunction (= ISH) on the tangent space of a symmetric space. In particular, if the tangent space in question satisfies a condition introduced there, I have shown that there exists no non-zero singular ISH on it (cf. Theorem 5.2 in [S]). This result is interpreted in terms of  $\underline{D}$ -modules. This will be shown in Theorem 1 of this report.

§1. Let  $\mathfrak{g}$  be a complex semisimple Lie algebra and let  $\sigma$  be its complex linear involution. Then we obtain a direct sum decomposition  $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ , where  $\mathfrak{h} = \{X \in \mathfrak{g}; \sigma X = X\}$  and  $\mathfrak{q} = \{X \in \mathfrak{g}; \sigma X = -X\}$ . Let  $S(\mathfrak{q})$  be the symmetric algebra over  $\mathfrak{q}$ . If  $\mathfrak{q}^*$  is the dual of  $\mathfrak{q}$ , then  $S(\mathfrak{q})$  is regarded as the polynomial algebra on  $\mathfrak{q}^*$ . On the other hand, a constant coefficient differential operator  $\partial(P)$  on  $\mathfrak{g}$  is associated with every element  $P \in S(\mathfrak{q})$ . For example, if  $A \in \mathfrak{q}$ , then  $\partial(A)$  is the vector field on  $\mathfrak{g}$  defined by  $(\partial(A)f)(X) = \frac{d}{dt} f(X + tA)|_{t=0}$ . Let  $H$  be a connected Lie group with Lie algebra  $\mathfrak{h}$ . We may assume that  $H$  acts on  $\mathfrak{q}$  as  $\mathfrak{h}$

does. Let  $I(\mathfrak{g})$  be the subalgebra of  $S(\mathfrak{g})$  consisting of  $H$ -invariant elements. Then  $\underline{N}(\mathfrak{g}) = \{X \in \mathfrak{g}; P(X) = P(0) \text{ for all } P \in I(\mathfrak{g})\}$  is called the nilpotent subvariety of  $\mathfrak{g}$ . An element  $X$  of  $\mathfrak{g}$  is contained in  $\underline{N}(\mathfrak{g})$  if and only if  $\text{ad}_{\mathfrak{g}}(X)$  is a nilpotent matrix. Take a Cartan subspace  $\mathfrak{a}$  of  $\mathfrak{g}$ . Then, by definition, almost all elements of  $\mathfrak{g}$  are  $H$ -conjugate to those of  $\mathfrak{a}$ . Let  $\mathfrak{g}'$  be the totality of  $\mathfrak{g}$ -regular semisimple elements of  $\mathfrak{g}$  and put  $\mathfrak{g}_s = \mathfrak{g} - \mathfrak{g}'$ . There is an  $H$ -invariant homogeneous polynomial  $D(X)$  on  $\mathfrak{g}$  called the discriminant such that  $\mathfrak{g}_s = \{X \in \mathfrak{g}; D(X) = 0\}$ .

Since the notations introduced above are not familiar to non-experts in this area, it seems worthwhile to give an example. Take  $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$  and let  $\sigma$  be the involution of  $\mathfrak{g}$  defined by  $\sigma X = -{}^t X$ . In this case,  $\mathfrak{h} = \mathfrak{so}(n, \mathbb{C})$  (= the totality of skew-symmetric matrices with trace zero) and  $\mathfrak{g} = \{X \in \mathfrak{g}; {}^t X = X\}$ . We may take  $H = SO(n, \mathbb{C})$ . In this case, the totality of diagonal matrices with trace zero is one of Cartan subspaces of  $\mathfrak{g}$ . So we take this as  $\mathfrak{a}$ . As to the algebra  $I(\mathfrak{g})$ , it is a polynomial algebra with  $(n-1)$  generators  $P_2, \dots, P_n$  which are defined as follows :

$$\det(t + X) = t^n + P_2(X)t^{n-2} + \dots + P_n(X).$$

Here we have identified  $\mathfrak{g}^*$  with  $\mathfrak{g}$  by the Killing form  $B(X, Y) = \text{tr}(XY)$  ( $X, Y \in \mathfrak{g}$ ). Let  $D(X)$  be the discriminant of  $\det(t + X)$ . Then  $\mathfrak{g}_s = \{X \in \mathfrak{g}; D(X) = 0\}$ . By definition, an element  $X$  of  $\mathfrak{g}$  is contained in  $\mathfrak{g}_s$  if and only if at least two eigenvalues of  $X$  coincide. Last the nilpotent subvariety  $\underline{N}(\mathfrak{g})$  consists of symmetric nilpotent matrices.

§2. Return to the general situation.

As usual, let  $\underline{\underline{O}}_{\mathfrak{g}}$  and  $\underline{\underline{D}}_{\mathfrak{g}}$  be the sheaf of holomorphic functions on  $\mathfrak{g}$  and that of differential operators with coefficients in  $\underline{\underline{O}}_{\mathfrak{g}}$ , respectively.

For any  $A \in \mathfrak{h}$ , define a vector field  $\tau(A)$  on  $\mathfrak{g}$  by  $(\tau(A)f)(X) = \frac{d}{dt} f(X + t[X, A])|_{t=0}$ . A function  $f(X)$  on  $\mathfrak{g}$  is invariant if  $f(h \cdot X) = f(X)$  for any  $h \in H$ . In this case, it follows that  $(\tau(A)f)(X) = 0$  for any  $A \in \mathfrak{h}$ . Noting this, we define that a function  $f(X)$  on an open subset  $U$  of  $\mathfrak{g}$  is locally invariant if  $\tau(A)f = 0$  for any  $A \in \mathfrak{h}$ .

For any  $\lambda \in \mathfrak{g}^*$ , we define the sheaf  $\underline{\underline{J}}_{\lambda}$  of left ideals of  $\underline{\underline{D}}_{\mathfrak{g}}$  as follows:

$$\underline{\underline{J}}_{\lambda} = \sum_{P \in I(\mathfrak{g})} \underline{\underline{D}}_{\mathfrak{g}}(\partial(P) - P(\lambda)) + \sum_{A \in \mathfrak{h}} \underline{\underline{D}}_{\mathfrak{g}}\tau(A)$$

Let  $\underline{\underline{M}}_{\lambda} = \underline{\underline{D}}_{\mathfrak{g}} / \underline{\underline{J}}_{\lambda}$  be the sheaf of left  $\underline{\underline{D}}_{\mathfrak{g}}$ -modules associated to  $\underline{\underline{J}}_{\lambda}$ . It is known that  $\underline{\underline{M}}_{\lambda}$  is coherent as the sheaf of left  $\underline{\underline{D}}_{\mathfrak{g}}$ -modules. Let  $u$  be the canonical generator of  $\underline{\underline{M}}_{\lambda}$ . Then  $\underline{\underline{M}}_{\lambda} = \underline{\underline{D}}_{\mathfrak{g}}u$  and  $u$  satisfies the system of differential equations

$$(1) \quad \begin{cases} (\partial(P) - P(\lambda))u = 0 & \text{for all } P \in I(\mathfrak{g}) \\ \tau(A)u = 0 & \text{for all } A \in \mathfrak{h}. \end{cases}$$

By definition,  $u$  is regarded as a locally invariant function on  $\mathfrak{g}$  which is a joint eigenfunction of the differential operators  $\partial(P)$  ( $\forall P \in I(\mathfrak{g})$ ).

Since  $\mathfrak{g}$  is an affine space, the cotangent bundle  $T^*\mathfrak{g}$  is identified with  $\mathfrak{g} \times \mathfrak{g}^* \simeq \mathfrak{g} \times \mathfrak{g}$  (by the Killing form). Then it

follows from the definition that  $\text{Ch}(\underline{M}_\lambda)$  is contained in the analytic subset

$$\Lambda = \{ (X, \Xi) \in \underline{g} \times \underline{g} ; [X, \Xi] = 0, \Xi \in \underline{N}(\underline{g}) \}$$

of  $T^*\underline{g}$ . Since  $\dim \Lambda = \dim \underline{g}$ , we find that  $\underline{M}_\lambda$  is holonomic in the sense of Sato-Kashiwara.

The purpose of this report is to prove the next theorem.

Theorem 1. Assume the following. There is a real semisimple Lie algebra  $\underline{g}_0$  and its maximal compact subalgebra  $\underline{k}_0$  such that  $\underline{g}_0$  is a normal real form of  $\underline{g}$  and that  $\underline{h}$  is a complexification of  $\underline{k}_0$ . Then, for any  $\lambda \in \underline{g}^*$ ,  $\underline{M}_\lambda$  has no non-trivial coherent quotient  $\underline{L}$  such that  $\text{Supp } \underline{L} \subseteq \underline{g}_s$ .

The assumption in Theorem 1 holds for the case where  $\underline{g} = \underline{sl}(n, \mathbb{C})$  and  $\underline{h} = \underline{so}(n, \mathbb{C})$ .

This connects with Theorem 6.7.2 in [HK]. Hotta and Kashiwara mentioned that Theorem 6.7.2 in [HK] is shown by an argument similar to a result of Harish-Chandra in [H]. I will give in §4 a proof of Theorem 1 which is an analogue of Theorem 6.7.2 in [HK].

§3. Let  $X$  be a complex manifold of  $\dim = n$  and let  $Y$  be a closed submanifold of  $X$  with  $\dim Y = n-k$ . As usual, let  $\underline{D}_X$  denote the sheaf of differential operators on  $X$ .

Let us consider the algebraic local cohomology  $\underline{H}_{[Y]}^k(\underline{D}_X)$  which is a sheaf of  $\underline{D}_X$ -modules. This sheaf is usually denoted

by  $\underline{B}_{Y|X}$ . If  $k = 0$ , this is nothing but the structure sheaf  $\underline{O}_X$ . But if  $k \neq 0$ , this is not coherent over  $\underline{O}_X$ . In spite of this, it is known that  $\underline{B}_{Y|X}$  has the structure of left  $\underline{D}_X$ -modules.

It is easy to describe  $\underline{B}_{Y|X}$  by using local coordinate systems. Let  $x = (x_1, \dots, x_n)$  be a local coordinate system on an open subset  $U$  of  $X$ . Assume that  $Y \cap U = \{x \in U; x_1 = \dots = x_k = 0\}$ . Then, by definition,  $\underline{B}_{Y|X}$  coincides with

$$\underline{D}_X / \sum_{i=1}^k \underline{D}_X x_i + \sum_{i=k+1}^n \underline{D}_X \frac{\partial}{\partial x_i}.$$

Note that  $\underline{B}_{Y|X}$  is an example of holonomic systems and that  $\text{Ch}(\underline{B}_{Y|X})$  coincides with  $T_Y^*X$  which is the conormal bundle of  $Y$ . The next lemma concerns the converse of this property.

Lemma 2 ([K]). Let  $\underline{M}$  be a holonomic system on  $X$ . Assume that  $\text{Ch}(\underline{M})$  coincides with  $T_Y^*X$ . Then  $\underline{M}$  is isomorphic to the direct sum of a finite number of copies of  $\underline{B}_{Y|X}$ .

Let  $\underline{M}$  be a regular holonomic system on  $X - Y$ . Assume that  $\underline{M}$  is extendable to  $X$ , namely that there exists a holonomic system  $\tilde{\underline{M}}$  on  $X$  such that  $\tilde{\underline{M}}|_{(X-Y)} \simeq \underline{M}$  as sheaves of left  $\underline{D}_{X-Y}$ -modules.

Definition 3 ([HK]). A holonomic system  $\underline{M}'$  on  $X$  is a minimal extension of  $\underline{M}$  if the following conditions hold:

- (i)  $\underline{M}'|_{(X-Y)} \simeq \underline{M}$  as  $\underline{D}_{X-Y}$ -modules.

(ii) Let  $\underline{L}$  be a holonomic system on  $X$ . Assume that  $\underline{L}$  is a subquotient of  $\underline{M}'$  and that  $\text{Supp}(\underline{L}) \subset Y$ . Then  $\underline{L} = 0$ .

Note that minimal extensions of  $\underline{M}$  are unique up to isomorphism. So we denote by  ${}^{\pi}\underline{M}$  the minimal extension of  $\underline{M}$ .

§4. Proof of Theorem 1. Let  $\underline{L}$  be a coherent sheaf of  $\underline{D}_X$ -modules. Assume that  $\underline{L}$  is a quotient of  $\underline{M}_\lambda$  and that  $\text{Supp}(\underline{L})$  is contained in  $\mathfrak{q}_s$ .

Let  $\underline{C}_1, \dots, \underline{C}_r$  be the totality of mutually distinct  $H$ -orbits of  $\underline{N}(\mathfrak{q})$ . For each  $i$ , let  $\Lambda_i$  be the closure of the set  $\{(X, Y) \in \Lambda; Y \in \underline{C}_i\}$  and put  $Y_i = \{X \in \mathfrak{q}; (X, Y) \in \underline{C}_i \text{ for some } Y \in \underline{N}(\mathfrak{q})\}$ . Then each  $Y_i$  is a closed analytic subset of  $\mathfrak{q}$ . Since  $\text{Ch}(\underline{M}_\lambda) \subset \Lambda$ , it follows from the definition that  $\text{Supp}(\underline{L})$  coincides with the union  $Z = \bigcup_{i \in A} Y_i$  for some subset  $A$  of  $\{1, \dots, r\}$ . The assumption of  $\underline{L}$  implies that each  $Y_i$  ( $i \in A$ ) is contained in  $\mathfrak{q}_s$ . Since  $Z$  is a locally closed, there exists an  $i \in A$  such that  $Y_i$  contains an open subset of  $Z$ . Take an open subset  $U$  of  $\mathfrak{q}$  such that  $U \cap Y_i$  is a closed submanifold of  $U$ . We may assume from the first that  $\underline{L}|U \neq 0$ . For the sake of simplicity, put  $\underline{L}' = \underline{L}|U$  and  $Y = U \cap Y_i$ . Then it follows that  $\text{Ch}(\underline{L}') = T^*Y$ . Hence, by means of Lemma 2, we find that  $\underline{L}' \simeq \bigoplus^d \underline{B}_{Y|U}$  for some integer  $d$ . Then

$$(2) \quad \underline{\text{Hom}}_{\underline{D}_U}(\underline{M}_\lambda|U, \underline{L}') \simeq \bigoplus^d \underline{\text{Hom}}_{\underline{D}_U}(\underline{M}_\lambda|U, \underline{B}_{Y|U}).$$

Now we recall the proof of Theorem 5.2 in [S]. By an

argument similar to that, we obtain the next lemma.

Lemma 4. Retain the above notation and the assumption in Theorem 1. Then  $\underline{\text{Hom}}_{\underline{D}_U}(\underline{M}_\lambda|_U, \underline{B}_Y|_U) = 0$ .

This lemma combined with (2) implies that  $\underline{L}$  is not a quotient of  $\underline{M}_\lambda$ . This contradicts the assumption and the theorem is proved.

§5. I proposed in [S] the next conjecture.

Conjecture I. Retain the notation and the assumption in Theorem 1. Then  $\underline{M}_\lambda$  is a minimal extension of  $\underline{M}_\lambda|_{\mathfrak{g}'}$ .

In virtue of Theorem 1, Conjecture I is reduced to the next one.

Conjecture I'. Retain the same situation in Conjecture I. Let  $\underline{L}$  be a holonomic system on  $\mathfrak{g}$ . Assume that  $\underline{L}$  is a subsheaf of  $\underline{M}_\lambda$  and that  $\text{Supp}(\underline{L}) \subset \mathfrak{g}_S$ . Then  $\underline{L} = 0$ .

I have no idea to prove this conjecture at present. But I have some examples which agree Conjecture I'. For example, Conjecture I' is true for the cases  $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{sl}(n, \mathbb{C}), \mathfrak{so}(n, \mathbb{C}))$ ,  $n = 2, 3$ . The case  $n = 2$  is easy to prove but the case  $n = 3$  is slightly complicated to prove.

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