数理解析研究所講究録
536 巻 1984 年 230-248

ON THE WAVEFORMS OF VAN DER POL OSCILLATOR WITH LARGE NONLINEARITY

Masami KURAMITSU

倉光正己

Department of Electrical Engineering, Kyoto University
Yoshida, Sakyo, Kyoto 606, Japan.

ABSTRACT

It is well known that the periodic solution of van der Pol equation $\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0$ $(\dot{}=d/dt)$ varies as follows as μ increases. When $0<\mu<<1$ x shows the almost sinusoidal waveform with a period 2π. The distortion from a sine wave increases, however, markedly as well as the period with increase of μ. On the other hand, the author has introduced the new concept of "Averaged Potential", and pointed out that an active element produces approximately rectangular waveforms when the rational ratio frequencies are applied to it. This paper shows that the waveform of van der Pol oscillator with large μ can be explained by the above mentioned function of an active element.

1. INTRODUCTION

The self-excited oscillation is one of the typical phenomena in nonlinear electrical circuits. Van der Pol introduced a differential equation with a nonlinear damping term, i.e., so-called van der pol equation $\ddot{x} - \mu(1-x^2)\dot{x} + x = 0$ $(\dot{}=d/dt)$ which describes a negative resistance oscillator. Since his study [1], a number of works have been devoted to van der Pol equation [2],[3],[4]. Now it is well known theoretically as well as numerically that there exists an unique orbitally stable periodic solution for each value of μ and that the solution varies as follows with increase of μ:

(1) For $0<\mu<<1$, the resulting waveform of x is nearly sinusoidal, and

the period is nearly 2π (angular frequency $\omega=1$).
(2) As μ increases, the distortion from a sine wave increases as well as the period T. For $\mu>>1$, the waveform is known as the relaxation oscillation.
(3) In spite of the above mentioned change of the waveform, the maximum value of x (amplitude) is nearly equal to 2 for each value of μ.

Recently, the author has shown a new approach to the analysis of almost-linear and almost-lossless oscillator with many degrees of freedom based on new concept of "Averaged Potential" [5],[6]. Moreover, using this concept, he has pointed out that an active element has the function to make a rectangular waveform as well as possible [7].

This paper is an attempt to explain that the waveform of van der Pol oscillator with large μ is produced by the above mentioned function of an active element.

2. AVERAGED POTENTIAL AND AVERAGED EQUATIONS

In this section we introduce a new concept of the "averaged potential" and summarize the fundamental of the new method to analyze many degrees of freedom oscillators.

Brayton and Moser showed that a system of differential equations for complete RLC-networks, which are composed of K inductors, J capacitors, and N resistors, can be written in the special form [8]

$$
\begin{aligned}
L_k(i_k)\frac{di_k}{dt} &= \frac{\partial P(i,v)}{\partial i_k} \qquad (k = 1,\ldots,K) \\
C_j(v_j)\frac{dv_j}{dt} &= -\frac{\partial P(i,v)}{\partial v_j} \qquad (j = 1,\ldots,J)
\end{aligned}
\tag{1}
$$

$$i = (i_1,i_2,\ldots,i_K), \qquad v = (v_1,v_2,\ldots,v_J)$$

where i_k represent the current in the inductor L_k and v_j the voltage across the capacitor C_j. The function P(i,v), "mixed potential function", is constructed as follows:

$$P(i,v) = -F(i) + G(v) + H(i,v) \tag{2}$$

where F(i) and G(v) are the "current potential" and the "voltage potential" of the network, respectively. The current (voltage) potential of the network is given as the sum of the current (voltage) potential of resistors in series with L (conductors in parallel with

C). Those potentials are defined by integrals along the characteristic of the resistor and the conductor, i.e.,

$$F(i) = \int v(i)di, \quad G(v) = \int i(v)dv \tag{3}$$

where the directions of i and v are assigned in the conventional way.

H(i,v) is determined by connection of the inductors and the capacitors and takes the form

$$\sum_{k=1}^{K} \sum_{j=1}^{J} \Gamma_{kj} v_j i_k \tag{4}$$

where $\Gamma_{kj} = 0, +1, -1$.

Let us consider van der Pol oscillators with many degrees of freedom. Therefore we assume that all L and C are linear and that all the resistors connected in series with L and all the conductors connected in parallel with C are small, i.e., F(i) and G(v) are small. Under these conditions, the system is almost-linear and almost-lossless. Hence the averaging method can be applicable to (1) as follows.

We write i_k and v_j in (1) in the same form as the generating system of (1), i.e.,

$$\begin{aligned} i_k(t) &= \sum_{m=1}^{M} d_{mk} r_m(t) \cos \phi_m(t) \\ v_j(t) &= \sum_{m=1}^{M} d'_{mj} r_m(t) \sin \phi_m(t) \end{aligned} \tag{5}$$

where M represents the degrees of freedom of the system, and d_{mk} and d'_{mj} are eigen vectors of the generating system.

If there exists a resonant relation between some pair of the natural or mode frequencies of the generating system n_m, i.e., their ratio is a simple rational number, a synchronization of their frequencies will take place. However the frequencies themselves will not be effected by the synchronization, then $\phi_m(t)$ can be assumed as

$$\phi_m(t) = n_m t + \theta_m(t) \tag{6}$$

If there exists no complete but an almost resonant relation between some pair of n_m, those n_m are slightly modified to ω_m for generating a synchronism. For describing this synchronization, $\phi_m(t)$ is assumed as

$$\phi_m(t) = \omega_m t + \theta_m(t) \tag{7}$$

If there is no resonant relation among n_m, $\phi_m(t)$ can be again be assumed as (6). When F(i) and G(v) are sufficiently small, $r_m(t)$ and $\theta_m(t)$ in (5) through (7) are taken to be slowly varying functions of time t.

Upon substituing (2) through (7) into (1) and applying the averaging method to it, we obtain [5],[6]

$$\dot{r}_\ell = \lim_{T\to\infty} \frac{1}{T} \int_0^T \{ \frac{-1}{I_\ell} \frac{\partial}{\partial r_\ell} (F(r,\theta,t) + G(r,\theta,t))\}dt$$
$$r_\ell^2(\dot{\theta}_\ell + \omega_\ell - n_\ell) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \{ \frac{-1}{I_\ell} \frac{\partial}{\partial \theta_\ell} (F(r,\theta,t) + G(r,\theta,t))\}dt \quad (8)$$

$(\ell = 1,\ldots,M)$, $(. = d/dt)$

Here we define the "Averaged Potential" U as follows:

$$U(r,\theta) = U(r_1,\ldots,r_M,\theta_1,\ldots,\theta_M)$$
$$= \lim_{T\to\infty} \frac{1}{T} \int_0^T \{F(r,\theta,t) + G(r,\theta,t)\}dt$$
$$= \{ \overline{F} + \overline{G} \} \quad (9)$$

U is the time average of the dissipation function, which is associated with the total loss in the resistors and conductors of the system.

Interchanging the integration with respect to t and the partial derivatives with respect to r_ℓ and θ_ℓ, (8) is written as

$$\dot{r}_\ell = -\frac{1}{I_\ell}\frac{\partial U}{\partial r_\ell}$$
$$r_\ell^2 (\dot{\theta}_\ell + \omega_\ell - n_\ell) = -\frac{1}{I_\ell}\frac{\partial U}{\partial \theta_\ell} \quad (10)$$

Especially, in case of the complete resonance, owing to $\omega_\ell = n_\ell$, (10) is reduced to

$$\dot{r}_\ell = -\frac{1}{I_\ell}\frac{\partial U}{\partial r_\ell}$$
$$r_\ell^2\dot{\theta}_\ell = -\frac{1}{I_\ell}\frac{\partial U}{\partial \theta_\ell} \quad (11)$$

Further, in case of no resonance among all n_m, owing to the fact that U depends only on r, (10) is reduced to

$$\dot{r}_\ell = -\frac{1}{I_\ell}\frac{\partial U(r)}{\partial r_\ell}$$
$$\dot{\theta}_\ell = 0 \qquad (12)$$

By observing (11) and (12), we know that these equations take the form of the so-called gradient system. Hence the following important result can be stated: *An isolated (local) minimal point of the averaged potential U is the asymptotically stabel equilibrium point of the system, and vice versa.* This statement implies that all the stable nonresonant and completely resonant oscillations of the system are obtained by finding all the minimal points of U.

As mentioned before, the method based upon the averaged potential yields the equivalent averaged equations to those derived by the conventional averaging method. However the former method has various advantages which could be summarized as follows:

(i) It enables us to understand the behavior of the system in a nice physical manner, i.e., the oscillatory behavior varies so as to minimize the time average of the loss in the system, and it is finally settled at a point of minimal loss. Hereafter, we call this property the "principle of minimal loss". This physical understanding is very much effective for analyzing or even synthesizing complicated phenomena in many degrees of freedom systems.

(ii) The process of mathematical calculation for constructing the averaged potential and for finding its minimal points is much less tedious than that for deriving equations by the averaging method and for obtaining the stable equilibrium points.

In case of the synchronism brought about by almost-resonant modes, the equations of (10) are not in gradient form in terms of the one function U. However they still give us a clear physical sense as to the oscillatory behavior: U is minimized with respect the amplitude r_ℓ under constraints among ω_ℓ, n_ℓ and the phase angle θ_ℓ.

For more details of the fundamental of the averaged potential, refer to Ref. [5] and [6].

This method was successfully applied to the analysis of nonresonant multimode oscillations in a ladder and a square array of van der Pol oscillators coupled by inductors [9], [5].

3. MODE LOCKING IN AN OSCILLATOR WITH RATIONAL RATIO FREQUENCIES [7]

The mode locking (synchronization) is one of the most interesting aspects in oscillators with many degrees of freedom. Let us consider the most typical and the simplest oscillator composed of one active element g(v) and M resonators as shown in Fig.1.

When no resonance occurs among the mode frequencies $n_m = 1/\sqrt{L_m C_m}$, the averaged potential U is merely the function of the amplitudes r_m, and its minimal point can be found easily. On the contrary, if the mode frequencies have the simple rational ratios, a mode locking (synchronization) among them may occur, and U gets to have many additional terms containing the phase angles θ_m. In this case, deriving the averaging potential U and finding its minimal points need tedious calculation.

For overcoming this difficulty some considerations are needed. As the oscillator has only one active element, we can rewrite the averaged potential U in the form

$$U(r,\theta) = \lim_{T\to\infty} \frac{1}{T} \int_0^T G\{ \sum_{m=1}^{M} r_m \sin(\omega_m t + \theta_m)\} dt$$

$$= \int_{-\infty}^{\infty} G(v) W(v;r,\theta) dt \qquad (13)$$

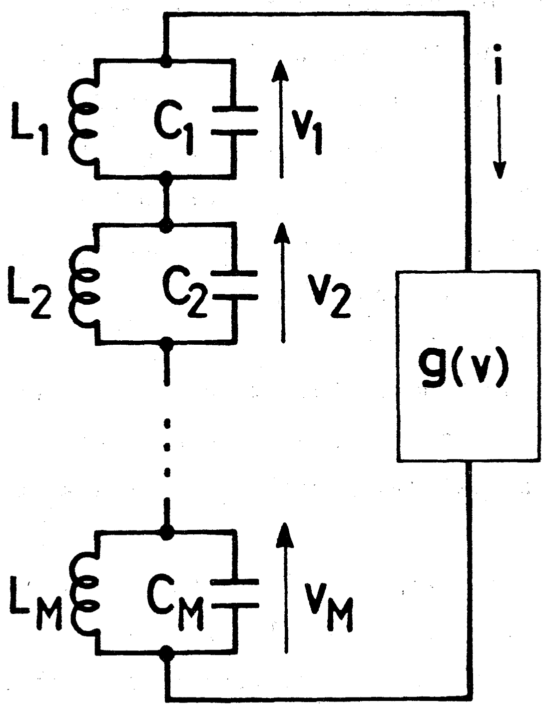


Fig.1. Multimode oscillator with one active element.

where W is a sort of probability density function of v(t). That is, wdv represents the probability that the value of v(t) lies between v and v + dv during the oscillatory behavior. As mentioned in Sec.2, the oscillation varies so as to decrease U with the lapse of the time t. Noting that G(v) depends on the active element and that W depends on the waveform of v(t), we can consider in the following manner. An ideal waveform to minimize U is the rectangular one with amplitude v_0(a minimal point of G(v)) which is composed of infinite number of frequency (harmonic) components. If only M components are available due to M resonators as in the oscillator of Fig.1, we can expect the waveform keeping the value of v at v_0 as long as possible, i.e., an approximately rectangular one.

The voltage-current characteristic of the active element is assumed to be described by

$$i = g(v) = \mu(-v + v^3/3) \tag{14}$$

where μ is considered to be small. Then, the voltage potential is given by

$$G(v) = \int g(v)dv = \mu(- v^2/2 + v^4/12) \tag{15}$$

The minimal points of G(v) are given by $v_0 = \pm\sqrt{3}$.

By using the active element with the characteristic (14) with μ = 0.1 and by putting the ratios of the mode frequencies in 1:3:...:(2M-1) in Fig.1, actual waveforms in stable synchronization are obtained. Figure 2 shows the change of the waveforms as M increases from 1 to 6. The upper wave in each figure shows the total voltage v across the active element, and the lower ones show its harmonic components v_m. It is readily observed that, as M increases, v approaches to the ideal rectangular wave of amplitude $\sqrt{3}$. Other types of synchronized oscillations are found stably in the same oscillator, when $M \geq 3$. The author has found the way how all the stable synchronized oscillations are obtained, and it is confirmed that all the waveforms of them keep the minimal value $\sqrt{3}$ as long as possible.

When the mode frequencies are not in simple rational ratios, but are almost near to them, a synchronization of them will also take place. However, the waveforms of the synchronized oscillations in this case produce a distortion from those in Fig.2. (See Fig.10.)

4. VAN DER POL OSCILLATOR

In this section the fundamental of van der Pol oscillation is summarized. First, when the parameter μ is small, the analysis of van der Pol oscillator is shown using the averaged potential mentioned in Sec.2. Secondly, when μ is extremely large, the analysis of so-called relaxation oscillation is shown using the discontinuous theory introduced by van der Pol. Finally, for values of μ lying between those two limits, we analyze the oscillator using the harmonic balance

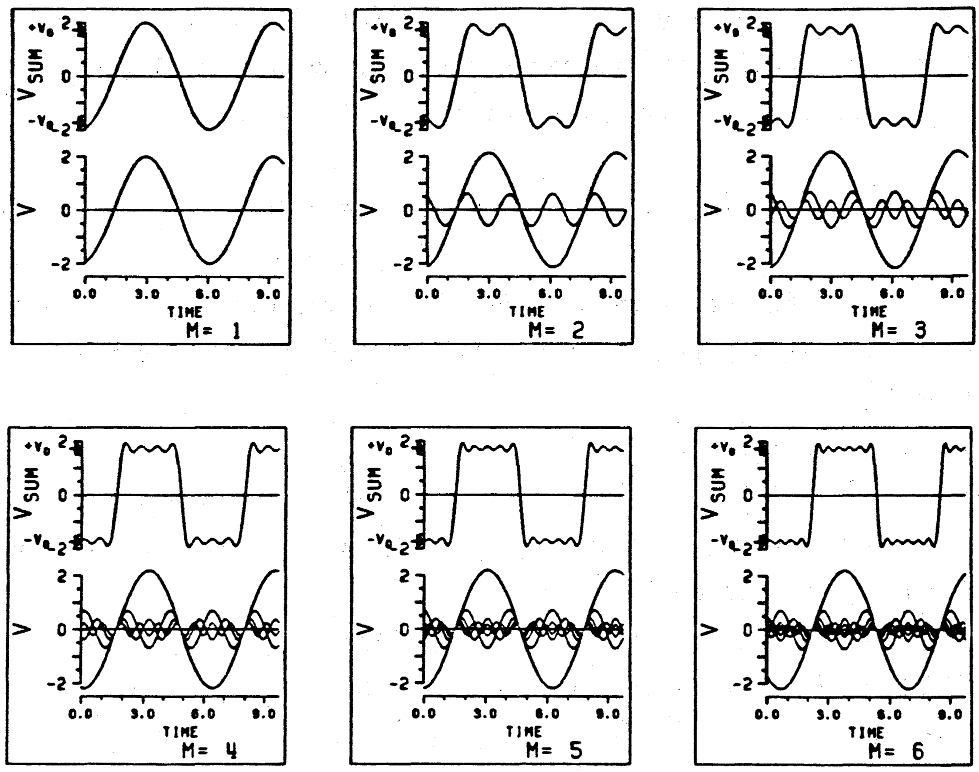


Fig.2. Change of the waveforms of the synchronized oscillations as M increases (cubic nonlinearity with μ = 0.1).

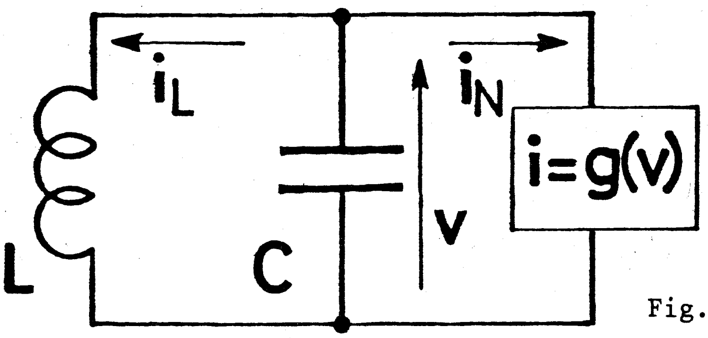


Fig.3. Van der Pol oscillator.

method (Galerkin procedure). From this result, it is shown that a many degrees of freedom oscillator with small nonlinearity can produce the same waveform as was produced by van der Pol oscillator with large nonlinearity.

4.1 The Almost Sinusoidal Oscillation

Van der Pol oscillator is shown in Fig.3, where the voltage-current characteristic is described by (14),i.e.,

$$i = g(v) = \mu(- v + v^3/3)$$

Without loss of generality, we can set L=C=1, hence, $\omega = 1/\sqrt{LC} = 1$. Then, we obtain the van der Pol equation

$$\ddot{v} - \mu(1 - v^2)\dot{v} + v = 0 \quad (\dot{} = d/dt) \tag{16}$$

When μ is small, the waveform of v is almost sinusoidal. Therefore we can use the averaged potential U to the oscillator. The solution can be written by

$$v = r \cos(t + \theta) \tag{17}$$

The voltage potential of (14) is given by (15). Upon substituting (17) into (15) and using (9), the averaged potential U of the circuit in Fig.3 is calculated to be

$$U = \mu(- r^2 + r^4/8)/4 \tag{18}$$

The minimal value of (18) is given by r=2, which coincides with the well-known result.

4.2. The Relaxation Oscillation [1] - [4]

Upon integrating (16) with respect to t, and introducing $\int v dt = \mu z$, (16) can be reduced in the usual way to the following first order equation

$$\frac{dv}{dz} = \mu^2 \frac{F(v) - z}{v} \tag{19}$$

where

$$F(v) = \int(1 - v^2)dv = v - v^3/3 \tag{20}$$

If μ is made extremely large, the field direction would be nearly vertical at all points except those very near the characteristic curve

$$z = F(v) = v - v^3/3 \tag{21}$$

Hence the solution curve of (19) approaches the shape ABA'B' in Fig.4. It is to be noted that the maximum values of v are equal to 2 at points A and A'. The waveform is shown in Fig.5.

4.3. Analysis of van der Pol Oscillator Using the harmonic Balance Method

For values of μ lying between two limits mentioned in 4.1 and 4.2, the waveforms of periodic solutions have the distortion from a sine wave. They can be, however, approximated well by several harmonics. Hence, we will apply the harmonic balance method (Galerkin procedure) to those oscillations.

Let us consider the generalized oscillation composed of one active element, whose characteristic is given by i=g(v), and the linear LC network, whose admittance is given by $Y(\omega)$, as shown in Fig.6.

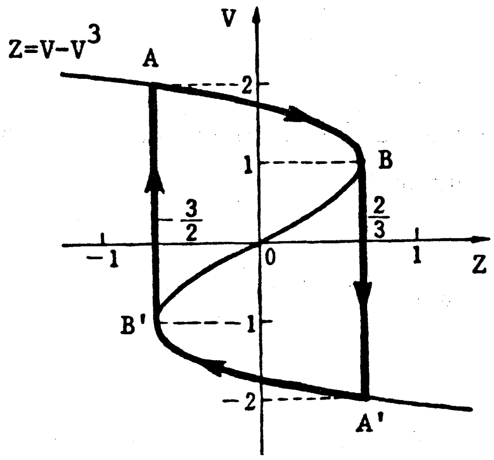


Fig.4. Phase plane of (19).

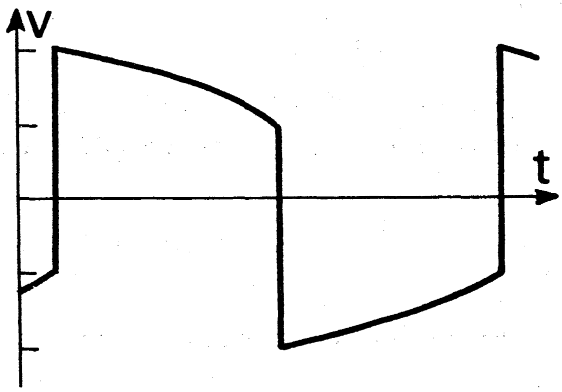


Fig.5. Relaxation oscillation.

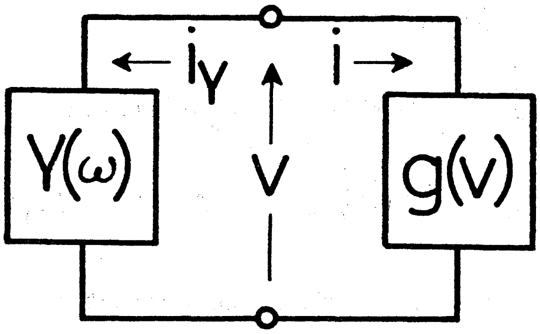


Fig.6. Generalized oscillator.

In order to clarity the physical meaning of the harmonic balance method, we apply this method not to differential equations but to the circuit in Fig.6. Then the following relations must by satisfied

$$i = g(v)$$
$$i_Y = Y(\omega)v \tag{22}$$
$$i = -i_Y$$

Now, we write v in the steady state as follows

$$v = \sum_{m=1}^{M} v_m = \sum_{m=1}^{M} r_m \sin \phi_m \tag{23}$$

where

$$\phi_m = \omega_m t + \theta_m = m\omega t + \theta_m$$

Upon substituting (23) into g(v), and expanding it, following equation is obtained in general,

$$i = g(\sum_{m=1}^{M} v_m) = \mu \sum_{m=1}^{M} \{g_{sm}(r,\theta)\sin\phi_m + g_{cm}(r,\theta)\cos\phi_m\} + \text{other freq.}$$
$$= \sum_{m=1}^{M} \{g_{sm} + jg_{cm}\} v_m/r_m + \text{other freq.} \tag{24}$$

where

$$j = \sqrt{-1}$$

For example, if g(v) is represented by a cubic function (14), and if we represent v by two odd harmonic components,

$$v = r_1\sin(\omega t + \theta_1) + r_3\sin(3\omega t + \theta_3)$$

the following relation is obtained.

$$i = \mu\{-1 + (r_1^2 + 2r_3^2 - r_1r_3\cos\theta)/4\}r_1\sin\phi_1$$
$$- \mu(r_1r_3\sin\theta)/4\ r_1\cos\phi_1$$
$$+ \mu\{-1 + (2r_1^2 + r_3^2 - r_1^3\cos\theta/3r_3)/4\}r_3\sin\phi_3$$
$$+ \mu(r_1^3\sin\theta/12r_3)r_3\cos\phi_3, \qquad (\theta = \theta_3 - 3\theta_1) \tag{25}$$

From (24) and (25), we can see that the current i has not only the same angle component as the voltage v but also the component leading or lagging v by $\pi/2$. In other words, *the nonlinear conductance g(v)* *behaves as if it is admittance, when the voltage containing the higher harmonics is applied to it.*

On the other hand, $Y(\omega)$ is admittance of linear network. Hence, when the voltage (23) is applied to it, the current i_Y can be written by

$$i_Y = \sum_{m=1}^{M} Y(\omega_m)v_m \tag{26}$$

Consequently, the harmonic balance in $i=-i_Y$ gives the following equations

$$\begin{aligned} \mu g_{sm}(r,\theta) &= -r_m \operatorname{Re} Y(\omega_m) = 0 \\ \mu g_{cm}(r,\theta) &= -r_m \operatorname{Im} Y(\omega_m) \qquad (m = 1,\ldots,M) \end{aligned} \tag{27}$$

where Re and Im denote the real and imaginary part of a complex number, respectively. Solving (27) gives r_m's and θ_m's as well as ω. It is worth noting that g_{sm}, g_{cm} in (27) can be derived from one function as follows. First, we define the extended averaged function U for large μ by substituting (23) into (15) and averaging it. Then we can write

$$\frac{\partial G(v)}{\partial r_i} = \sum_{m=1}^{M} \frac{\partial G(v)}{\partial v_m}\frac{\partial v_m}{\partial r_i} = g(v)\sin\phi_i$$

where v is denoted by (23). Therefore we obtain

$$\mu g_{si} = 2\overline{g(r,\theta)\sin\phi_i} = 2\frac{\partial}{\partial r_i}\overline{G(v)} = 2\frac{\partial U(r,\theta)}{\partial r_i} \tag{28}$$

Similarly

$$\mu g_{ci} = \frac{2}{r_i}\frac{\partial U(r,\theta)}{\partial \theta_i} \tag{29}$$

Now we apply equation (27) to van der Pol equation with large nonlinearity $\mu=\mu_s$ by putting

$$Y(\omega) = Y_s(\omega) = j\omega C + 1/j\omega L = j(\omega^2 - n^2)/\omega = j(\omega^2 - 1)\omega \tag{30}$$

We consider two cases where the solution is approximated by two and three frequency components, respectively.

$$v = r_1 \sin\omega t + r_3\sin(3\omega t + \theta_3)$$

$$v = r_1 \sin \omega t + r_3 \sin(3\omega t + \theta_3) + r_5 \sin(5\omega t + \theta_5) \tag{31}$$

where we put $\theta_1=0$, without loss of generality.

These results are illustrated in Fig.7 and 8. The broken line in the figure of period T shows the exact one obtained by Urabe [4]. It is easily seen that two and three components are fairly good approximation when $\mu_s<1.5$ and $\mu_s<2.0$, respectively. For larger μ_s, however, more higher harmonics must be considered.

Next, we apply the same method to the oscillator in Fig.1 composed of the active element (14) with small nonlinearity $\mu=\mu_M$ and M resonators with rational ratio frequencies 1:3:...:(2M-1). $Y(\omega)$ is represented as follows.

$$Y(\omega) = Y_M(\omega) = \{ \sum_{m=1}^{M} 1/Y_m(\omega)\}^{-1}$$

$$Y_m(\omega) = j(\omega^2 - n_m^2)/\omega = j \{\omega^2 - (2m-1)^2 n_1^2\} \tag{32}$$

When $\omega=\omega_m \doteqdot n_m$, we can put

$$Y_M(\omega_m) \doteqdot 2jC_m(\omega_m - n_m) \tag{33}$$

It is easily seen that the equation (27) has the same solution for these two oscillators if the following relations are satisfied.

$$Y_s(\omega_m)/\mu_s = Y_M(\omega_m)/\mu_M \qquad (m = 1,3,\ldots,(2M-1)) \tag{34}$$

In other words, we can construct a many degrees of freedom oscillator with small μ so as to produce approximately the same waveform as van der Pol oscillator with large μ. For example, by putting μ_s =1.5, μ_M =0.1 and M=3 in (34) the resonant frequencies of resonators are determined as follows.

$$n_1 = 0.89 \quad , \quad n_3 = 2.58 \quad , \quad n_5 = 4.29 \tag{35}$$

Figure 9 and 10 show the waveforms of these two oscillators, which agree approximately.

As mentioned in Sec.2 and 3, the waveform of the many degrees of freedom oscillator in uncomplete resonance is produced by the function of an active element to make the rectangular one under some constraints. Therefore, the waveform of van der Pol oscillator with large μ can also be considered as a product of the same function of the active element.

Another problem that the increase of period T with increase of μ can be explained as follows. As shown in Fig.10, the higher harmonics

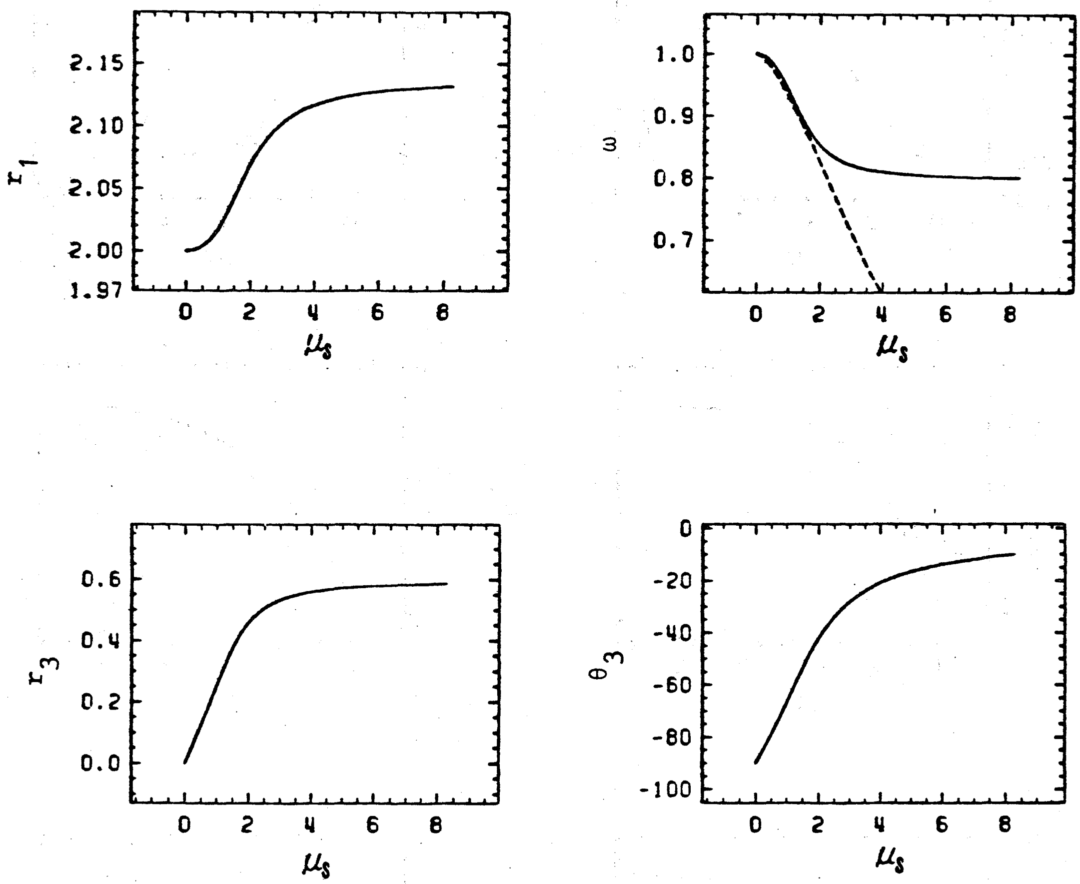


Fig.7. Characteristic of the oscillation approximated by two frequency components.

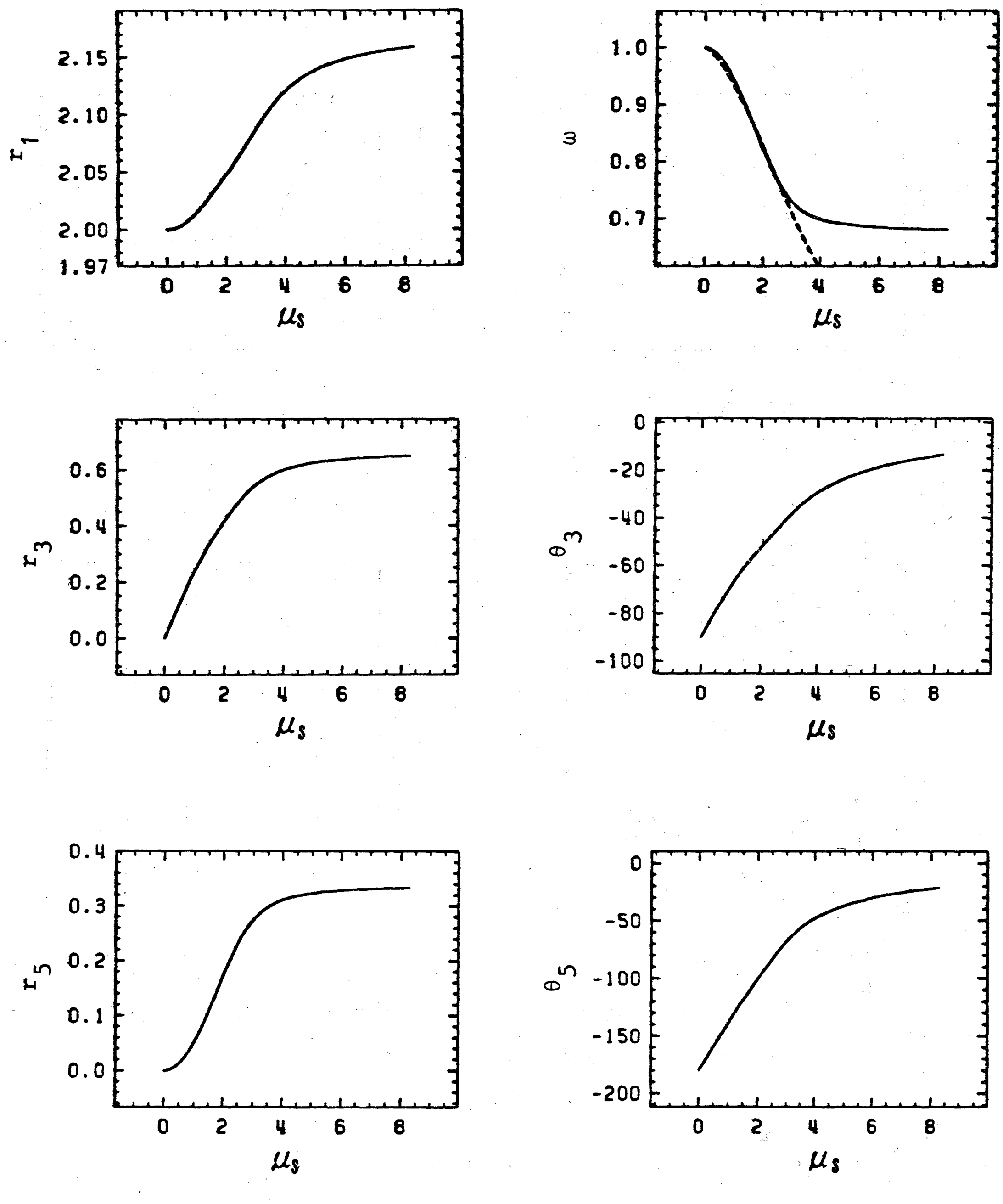


Fig.8. Characteristic of the oscillation approximated by three frequency components.

v_m in the many degrees of freedom oscillator are supplied by the corresponding resonators, respectively. On the contrary, in van der Pol oscillator, all harmonics have to be produced by the large current i in one resonator with n=1. In order to make the higher harmonic components v_m large enough, their frequencies have to approach to resonant frequency n=1, i.e., the fundamental harmonic ω must decrease. We can also show that these tendency satisfies the constraints for the phase angles θ_m.

Obviously, the result that the maximum amplitude of van der Pol oscillator is nearly equal to 2 for any value of μ depends on the fact that it is exactly equal to 2 for both limits $\mu \to 0$, $\mu \to \infty$. As mentioned in Sec.4.1 and 4.2, however, these results are due to quite different physical meanings, respectively. It is to be noted that this agreement is caused by the characteristic of the cubic curve (14).

4.4. One Prediction and Its Proof

In order to show the advantage of the consideration in Sec.4.3, let us consider the oscillator in Fig.1 with rational ratio frequencies and large μ. In van der Pol equation, the period T decreases so as to increase the higher harmonics as μ increases. The oscillator in Fig.1, however, has many resonators which produce the higher harmonics. Therefore, we can predict that the period T does not decrease so much in spite of increase of μ, because higher harmonics can be supplied by corresponding resonators.

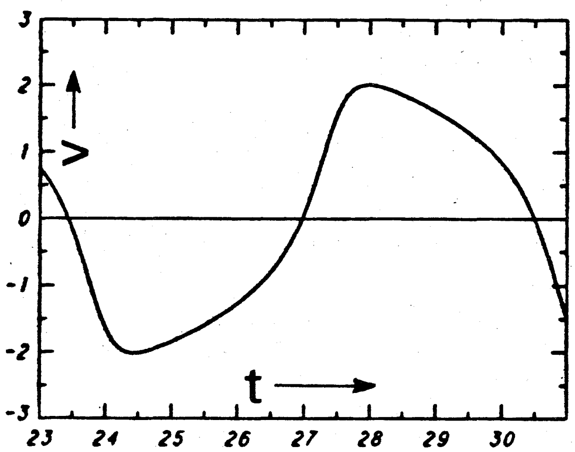


Fig.9. Waveform of van der Pol oscillator (μ = 1.5)

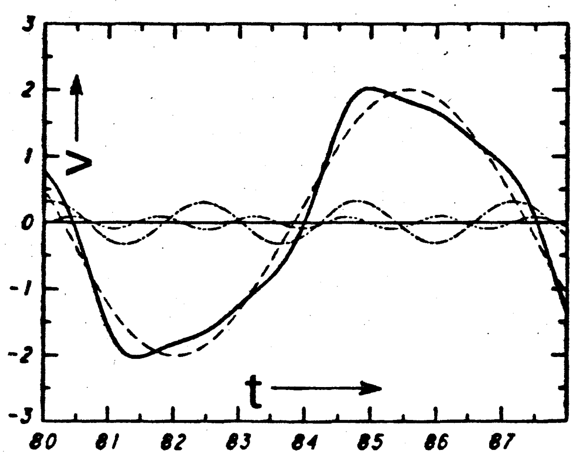
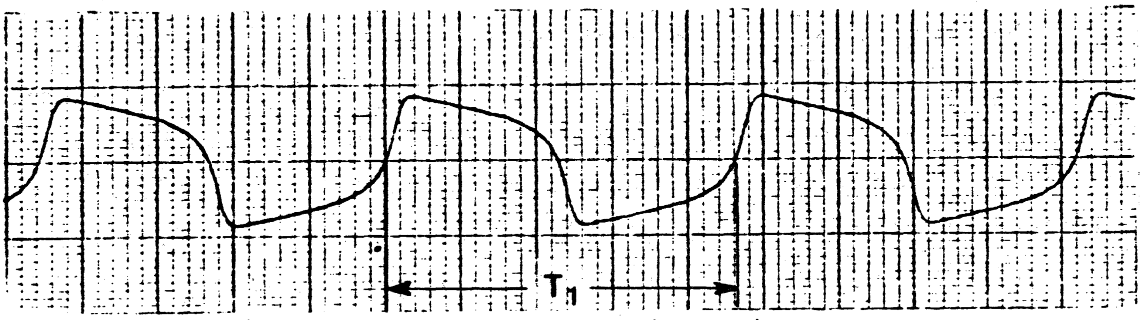


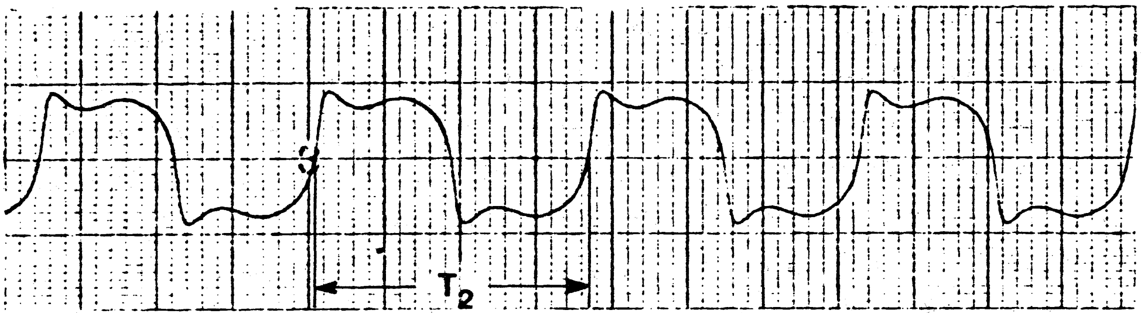
Fig.10. Waveform of oscillator with three degrees of freedom (μ=0.1,n_1=o.89,n_3=2.58,n_5=4.29).

We consider one, two, and three degrees of freedom oscillators with odd harmonics 1, 3, and 5. Figure 11 shows their waveforms for μ=3.0. In fact, the increase of the period T from 2π is reduced as the degree of freedom increases. Hence, our prediction is confirmed.

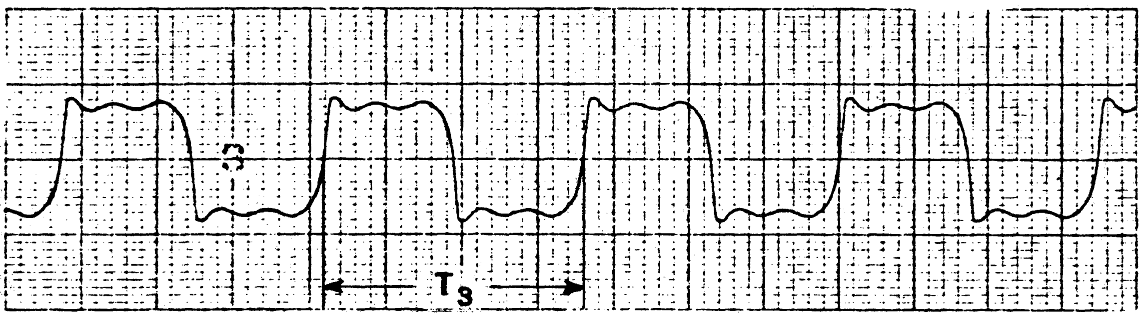
The whole aspects for larger μ, however, are more complicated. Another paper on the details of these phenomena will appear in near future.



(a) Van der Pol oscillator (n_1 = 1).



(b) Oscillator with two resonators (n_1=1, n_3=3).



(c) Oscillator with three resonators (n_1=1, n_3=3, n_5=5).

Fig.11. Waveforms of oscillators with large nonlinearity μ = 3.0 (by analog computer).

5. CONCLUSION

In this paper we introduced a new method for the analysis of many degrees of freedom oscillator based on a new concept of the averaged potential. From this concept, we also showed that an active element makes approximately rectangular waveforms when the frequencies applied to it are exactly or nearly in the rational ratios.

Using these considerations and the harmonic balance method (extended averaged potential), we showed that we can construct many degrees of freedom oscillators with small nonlinearity which produce approximately the same waveform as van der Pol oscillator with large nonlinearity produces. Therefore, we could explain that the waveform of van der Pol oscillator is due to the function of the active element to make a rectangular wave.

The author wishes to express his thanks to Mr. F. Takase for his cooperation to develop the averaged potential method. The author also would like to thank Prof. Y. Nishikawa for his helpful comments and former students for their excellent works.

REFERENCES

1. Pol, B. van der: On Relaxation Oscillations II, Phil. Mag.,2,pp.978-992 (1926).
2. Stoker, J. J.: Nonlinear Vibrations, Interscience Publishers, Inc., New York, 1950.
3. Cunningham, W. J.: Introduction to Nonlinear Analysis, McGraw-Hill Book Company, New York, 1958.
4. Urabe, M.: Nonlinear Autonomous Oscillations, Academic Press, New York, 1967.
5. Kuramitsu, M. and Takase, F.: A New Approach for the Analysis of Multimode Oscillators Based on the Averaged Potential, Proc. 1981 IEEE ISCAS, 3, pp.943-946 (1981).
6. Kuramitsu, M. and Takase, F.: Analytical Method for Multimode Oscillator Using the Averaged Potential, Electronics and Communication in Japan, scripta pub. com., 66, no.4, pp.10-19 (1983).

7. Kuramitsu, M., Takase, F. and Nitta, H.: Mode Locking in a Multimode Oscillator with Rational Ratio Frequencies: Wave-Shaping by Mode Locking, Proc. 1983 IEEE ISCAS, 3, pp.1153-1156 (1983).

8. Brayton, R. K. and Moser, J. K.: A Theory of Nonlinear Networks, I and II, Quart. Appl. Math., 22, pp.1-33 and 81- 104 (1964).

9. Kuramitsu, M. and Takase, F.: Analysis of Ladder Oscillators Using the Averaged Potential, Electronics and Communication in Japan, scripta pub. com., 67, NO.2 (1984).