数理解析研究所講究録
538巻 1984年 185-197

Convergence rates in the empirical Bayes estimation under the uniform $U(0,\theta)$ and a location parameter family of gamma distributions.

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1. Introductions.

The empirical Bayes (EB) problem as treated here is called generalized empirical Bayes by H. Robbins(1983). This with the squared-error loss in the non-regular families of distributions is so far dealt with by R. Fox (1970, 1978) and by Y. Nogami(1983a, 1983b) with convergence rates. In Nogami(1983a), the EB estimators are exhibited with exact $n^{-2/3}$ rate of risk convergence in retracted distributions over $[\theta, \theta+1)$.

In this paper the author exhibits (in Sections 2 and 3) modified Fox's(1978) EB estimator for θ with exact $n^{-2/3}$ rate of risk convergence under the uniform distribution $U(0,\theta)$ where $\theta\varepsilon\,\Omega=(0,m)$ $(0<m<\infty)$. The author also shows that under the uniform $U(0,\theta)$ where $\theta\ \varepsilon\ (0,\infty)$ Fox's(1978) EB estimate for θ has near exact $n^{-2/3}$ rate of convergence (Section 4) and that under a location parameter family of gamma distributions Fox's(1978) EB estimator has a lower bound, a constant times $n^{-2/3}$, of risk convergence (Section 5).

As notational conventions we use the followings. A distribution function will also be used to denote the associated measure. The argument of a function will not be displayed sometimes and $\int g(t)\,d\mu(t)$ might be abbreviated as $\mu(g(t))$ or $\mu(g)$. [A] denotes the indicator function fo the event A. $\doteq$ means a defining property. Let $\vee$ and $\wedge$ denote the supremum and the infimum, respectively. Let $g]_a^b \doteq g(b)-g(a)$.

Let X be a random variable distributed according to cdf F_θ and the θ's are i.i.d. random variables distributed according to the unknown prior distribution G. Let $X_1, \ldots, X_n$ be n i.i.d. past observations with each X_i distributed according to the marginal cdf $K(x) = \int F_\theta(x)\, dG(\theta)$. Let X denote the (n+1)st observation X_{n+1} distributed according to $F_{\theta_{n+1}}$. The EB estimation problem is to estimate $\theta \doteq \theta_{n+1}$ by using all n+1 observations $\underset{\sim}{X} \doteq (X_1, \ldots, X_n, X)$. Let $\underset{\sim}{E}$ be the product measure on the space of $(X_1, \ldots, X_n, (X, \Theta))$, resulting from K^n and the joint distribution of (X, Θ). With $\underset{\sim}{X}=x$, let $\phi_G(x)$ denote the Bayes estimator vs G given by

$$\phi_G(x) = \int \theta\, f_\theta(x)\, dG(\theta) / \int f_\theta(x)\, dG(\theta) \tag{1.1}$$

where $f_\theta(x)$ is the pdf of X, conditionally on θ. The risk of an EB estimator t_n for θ is $R(t_n, G) = \underset{\sim}{E}((t_n(\underset{\sim}{X}) - \theta)^2)$ and the Bayes envelope is $R \doteq R(\phi_G, G) = \inf_\phi R(\phi, G)$. When $R(t_n, G)$ and R are both finite

$$(0<)\; R(t_n, G) - R = \underset{\sim}{E}(\phi_G(X) - t_n(\underset{\sim}{X}))^2. \tag{1.2}$$

We call an EB estimator asymptotically optimal (a. o.) when $R(t_n, G) - R \to 0$ as $n \to \infty$. We shall find convergence rates of (1.2).

Hereafter, let $\underset{\sim}{P}_x$ and E be the conditional product measure on the space of $(X_1, \ldots, X_n, (\Theta \mid x))$ given $X=x$, and the marginal probability measure of X, respectively.

2. An Upper Bound for $R(\phi_n, G) - R$ in $U(0, \theta)$ for $\theta \in \Omega = (0, m)$.

Let m be a positive finite number. Let $f_\theta(x) = \theta^{-1}[0<x<\theta]$ for $\theta \in (0, m)$. We shall introduce modified Fox's a.o. estimators and show that those have exact $n^{-2/3}$ rate of convergence for (1.2). To get an upperbound for (1.2) we use R. Singh's bound (1974, Lemma A.2). A lower bound for (1.2) will be obtained by using lim-inf procedure as appeared in Nogami(1983a).

Let $k(x)$ be the marginal pdf of X of the following form: for $x>0$,

$$k(x) = \int f_\theta(x)\, dG(\theta) = \int_x^m \theta^{-1}\, dG(\theta). \tag{2.1}$$

Let G be the prior on θ having the following assumptions:

(i) Assume that the first derivative k' exists and for any given h>0 and some positive number M, $\sup_{y\in(x-h,x+h)} |k'(y)| \leq M\ (<\infty)$

(ii) $E(k^{-2}(x)) \leq N\ (<\infty)$ where E denotes the expectation wrt the marginal distribution of X and N is some finite positive number.

Since $F_\theta(x)=\theta^{-1}x[0<x<\theta]+[x\geq\theta]$, $K(x)=xk(x)+G(x)$. Thus, from (1.1) we have the following Bayes response: when $k(x)>0$;

(2.2) $\quad \phi_G(x) = (1-G(x))/k(x) = x + \psi(x)$

where

(2.3) $\quad \psi(x) = (1-K(x))/k(x).$

We note that

(2.4) $\quad 0 < \psi(x) \leq m-x\ (\leq m).$

Let h be a positive number depending on n such that $0<h<1$ and $h\to 0$ as $n\to\infty$. We also let $K_n(y)=n^{-1}\Sigma_{j=1}^n[X_j\leq y]$ and $k_n(y)=h^{-1}K_n]_y^{y+h}=(nh)^{-1}\Sigma_{j=1}^n[y<X_j\leq y+h]$. We first estimate $\psi(x)$ by

(2.5) $\quad \psi_n(x) = \{(1-K_n(x))/k_n(x)\}\wedge(m-x),$

and furthermore estimate $\phi_G(x)$ by

(2.6) $\quad \phi_n(x) = x + \psi_n(x).$

Since we use Lemma A.2 of R. S. Singh(1974) to obtain a rate $O(n^{-2/3})$ for $R(\phi_n,G) - R$ with a choice of $h=n^{-1/3}$, we state it below without a proof.

Lemma 2.1. (Lemma A.2 of Singh(1974)) Let y, z and L be in $(-\infty,\infty)$ with $z\neq 0$ and $L>0$. If Y and Z are two real random variables, then for every $\gamma>0$

$$E\left(\left|\frac{y}{z}-\frac{Y}{Z}\right|\wedge L\right)^\gamma$$

$$\leq 2^{\gamma+(\gamma-1)^+}|z|^{-\gamma}\{E|y-Y|^\gamma + (|\tfrac{y}{z}|^\gamma + 2^{-(\gamma-1)^+}L^\gamma)E|z-Z|^\gamma\}$$

where E means the expectation wrt the joint distribution of (Y,Z) and $a^+ = a$ if $a>0$; $=0$ if $a<0$.

In (2.3) and (2.5), let $\psi(x)= v/w$ and $\psi_n(x) = (V/W)\Lambda(m-x)$. In view of (1.2) with t_n replaced by ϕ_n. and from above Lemma 2.1 and weakening the bound by $\psi(x) \le m$

$$(2.7) \qquad (0 \le\ ) R(\phi_n,G) - R \le E\ \underset{\sim}{P}_x(|\psi(x) - \psi_n(x)|\ \Lambda\ m)^2$$

$$\le\ E[m^2\Lambda\ \{8k^{-2}(x)\{\ \underset{\sim}{P}_x(K(x)-K_n(x))^2+ (3m^2/2)\underset{\sim}{P}_x(k(x)\ -k_n(x))^2\}\}].$$

But, $\underset{\sim}{P}_x(K_n(x))=K(x)$. Thus,

$$(2.8) \qquad \underset{\sim}{P}_x(K(x)-K_n(x))^2 = n^{-1}K(x)(1-K(x)) \le n^{-1}.$$

On the other hand, by c_r-inequality (Loève(1963, p. 155)),

$$(2.9) \qquad \underset{\sim}{P}_x(k(x)-k_n(x))^2 \le 2^{-1}\{Var_x(k_n(x))+ \{\underset{\sim}{P}_x(k_n(x))-k(x)\}^2\ \}$$

where $Var_x(Y)$ implies the variance of Y wrt the probability measure $\underset{\sim}{P}_x$. To get a bound of rhs(2.9) we shall show following two lemmas:

<u>Lemma 2.2.</u> For each x and for M in the assumption (i)

$$Var_x(k_n(x)) \le (nh)^{-1}\{(2m+1)M + k(x+h)\}.$$

<u>Proof</u>.) Let $Y_j = h^{-1}[x<X_j\le x+h]$. Since $k_n(x)=n^{-1}\Sigma_{j=1}^n Y_j$, Y_j's are independent and $\underset{\sim}{P}_x(Y_j)=h^{-1}K]_x^{x+h}$, we have

$$(2.10) \qquad Var_x(k_n(x)) \le n^{-1}\underset{\sim}{P}_x Y_j^2 \le (nh)^{-1}(h^{-1}K]_x^{x+h}).$$

Since $K(x)=x\ k(x) +G(x)$, we have $h^{-1}K]_x^{x+h}= x(h^{-1}k]_x^{x+h}) + k(x+h)+h^{-1}G]_x^{x+h}$. Since by Taylor's theorem $k(x+h)-k(x)=hk'(x+\Delta h)$ for some Δh such that $0<\Delta h<h$, assumption (i) gives us that $|h^{-1}k]_x^{x+h}|\le M$. Using $h^{-1}G]_x^{x+h}\le(x+h)(h^{-1}k]_{x+h}^x)$ $\le(x+h)M$, $x\le m$ and $h\le 1$ leads to

$$(2.10) \qquad h^{-1}K]_x^{x+h} \le (2m+1)M+k(x+h).$$

Applying the rhs(2.11) to the extreme rhs of (2.10) leads to the asserted bound of Lemma 2.2.

<u>Lemma 2.3</u>. For each x and for M in the assumption (i)

$$\{\underset{\sim}{P}_x(k_n(x))-k(x)\}^2 \le 4^{-1}h^2M^2$$

Proof.) By a change of variable from t to $z=h^{-1}(t-x)$ and applying a triangle inequality to the rhs of the inequality below we get

(2.12) $$|\underset{\sim}{P}_x(k_n(x)) - k(x)| = |h^{-1}\int_x^{x+h}k(t)\,dt - k(x)| \le \int_0^1|k(x+hz) - k(x)|\,dz.$$

Applying Taylor's theorem and assumption (i) and making square on both sides of (2.12) leads to the asserted bound.

In view of (2.7) through (2.9), applying the bounds in Lemmas 2.2 and 2.3, using the assumption (ii) twice and applying $(k(x+h)/k(x))\le 1$ gives following theorem:

Theorem 1. With M and N appeared in assumptions (i) and (ii),

$$(0\le)\ R(\phi_n,G) - R \le [6m^2\{2MN(m+1)+1\}+8N](nh)^{-1} + (3/2)m^2M^2Nh^2.$$

From above Theorem 1 we can see that with $h=n^{-1/3}$,

(2.13) $$(0<)\ R(\phi_n,G) - R\le O(n^{-2/3}).$$

3. A Lower Bound for $R(\phi_n,G)-R$ in $U(0,\theta)$ with $\theta=1$.

Throughout this section, we assume that G is the degenerate distribution at $\theta=1$. In this case, m=1. Then, $\phi_G(x)=[0<x<1]$. Letting $B=[(1-K_n(x))/k_n(x)\le 1]$ and $\zeta_n(x)=1-x-\{(1-K_n(x))/k_n(x)\}$ we obtain by (1.2)

(3.1) $$R(\phi_n,G)-R \ge E(\underset{\sim}{P}_x(\zeta_n^2(x)B)[0<x<1-h]).$$

Let $u=\Sigma_1^n [X_j\le x]$ and $v=\Sigma_{j=1}^n[x<X_j\le x+h]$. Then, $\underset{\sim}{P}_x u=nx$, $Var_x(u)=nx(1-x)$, $\underset{\sim}{P}_x v= nh$ and $Var_x(v)=nh(1-h)$. Letting

(3.2) $$Y=(u-nx)/\sqrt{nx(1-x)} \quad\text{and}\quad Z=(v-nh)/\sqrt{nh(1-h)},$$

we obtain

$$(3.3) \qquad \sqrt{nh}\,\zeta_n(x) = \frac{(1-x)\sqrt{1-h}\,Z + h^{\frac{1}{2}}\sqrt{x(1-x)}\,Y}{(nh)^{-\frac{1}{2}}\sqrt{1-h}\,Z + 1}.$$

Let $\overset{D}{\to}$ denote convergence in distribution. Also, N(c,d) denotes the normal distribution with mean c and variance d. To get a lower bound for $R(\phi_n,G)-R$ (Theorem 2) we use (3.1) and the fact that for fixed x $\sqrt{nh}\,\zeta_n(x)B \overset{D}{\to} N(0,(1-x)^2)$. We then apply a convergence theorem (cf. Loéve(1963) 11.4, A(i)):

$$(3.4) \qquad \text{If } U_n \overset{D}{\to} U, \text{ then } \underline{\lim}\ EU_n^2 \geq E\,U^2.$$

We shall first prepare Lemma 3.1 to prove above convergence in distribution (Lemma 3.2) for the proof of forthcoming Theorem 2.

<u>Lemma 3.1.</u> If h is a function of n such that $h\to 0$ as $n\to\infty$, then letting A^c be the complement of a set A we obtain that for $0<x<1$,

$$(3.5) \qquad P_{\underset{\sim}{x}}\, B^c \to 0 \qquad \text{as } n\to\infty.$$

<u>Proof.)</u> Let $W_j = 1-[X_j\leq x]-h^{-1}[x<X_j\leq x+h]$. Then,

$$(3.6) \qquad P_{\underset{\sim}{x}}\, B^c = P_{\underset{\sim}{x}}[1-K_n(x) > k_n(x)] = P_{\underset{\sim}{x}}[\Sigma_{j=1}^n W_j > 0].$$

Since $P_x\, W_j = -x[0<x<1-h]-(h^{-1}-1)(1-x)[1-h\leq x<1]$ and $-1\leq W_j \leq 1$ for all j, it follows by Hoeffding's bound(1963, p.16, Theorem 2) applied to the lhs of the second inequality below that

$$\begin{aligned}(3.6) &= P_{\underset{\sim}{x}}[\Sigma_{j=1}^n(W_j - P_{\underset{\sim}{x}}W_j) > -n\, P_{\underset{\sim}{x}}W_j] \\ &\leq P_{\underset{\sim}{x}}[\Sigma_j(W_j - P_x W_j) > nx[0<x<1-h]] \\ &\leq \exp\{-\tfrac{1}{2}nx[0<x<1-h]\}\end{aligned}$$

which goes to zero as $n\to\infty$.

<u>Lemma 3.2.</u> When h is a function of n such that $h\to 0$ and $(nh)^{\frac{1}{2}}\to\infty$ as $n\to\infty$, for $0<x<1$

$$(3.7) \qquad \sqrt{nh}\ \zeta_n(x)\ B \overset{D}{\to} N(0,(1-x)^2).$$

Proof.) Let $\overset{P}{\to}$ denote convergence in probability. Since $h^{\frac{1}{2}}Y \overset{P}{\to} 0$ as $n\to\infty$ and $(nh)^{-\frac{1}{2}}\sqrt{1-h}\ Z \overset{P}{\to} 0$ as $(nh)^{\frac{1}{2}}\to\infty$ and $n\to\infty$ and since by Lemma 3.1 $P_{\underset{\sim}{x}} B \to 1$ as $n\to\infty$, we will obtain the asserted lemma by applying Slutsky's theorem (R. J. Serfling (1980),p.19).

Theorem 2. For any $\varepsilon>0$ and $h>0$ such that $nh\to\infty$ as $n\to\infty$,

(3.8) $$R(\phi_n,G) - R \ge \{3^{-1}(1-h^3) - \varepsilon\}(nh)^{-1}.$$

Proof.) Applying a convergence theorem (3.4) to (3.7) gives $\underline{\lim}\ P_{\underset{\sim}{x}}((nh)\ \zeta_n^2(x)\ B) \ge (1-x)^2$. Thus, by Fatou's lemma

$$\underline{\lim}\ E\{P_{\underset{\sim}{x}}((nh)\zeta_n^2(x)\ B)[0<x<1-h]\} \ge E\{\underline{\lim}\ P_{\underset{\sim}{x}}((nh)\zeta_n^2(x)B)[0<x<1-h]\}$$
$$\ge \int_0^{1-h} (1-x)^2dx = 3^{-1}(1-h^3).$$

Finally, (3.1) and the definition of lim-inf gives us (3.8).

Therefore, from Theorems 1 and 2 we obtain that with $h=n^{-1/3}(<\frac{1}{2})$ and for some constants c_0 and c_1

(3.9) $$c_0\ n^{-2/3} \le R(\phi_n,\ G) - R \le c_1\ n^{-2/3}.$$

In the next section we will mension about upper and lower bounds of (1.2) for Fox's(1978) EB estimates under $U(0,\theta)$ with $\theta\varepsilon\Omega=(0,\infty)$.

4. Fox's(1978) EB estimates under $U(0,\theta)$ with $\theta\varepsilon(0,\infty)$.

In this section we assume $\theta\varepsilon(0,\infty)$. In (2.2), R. Fox(1978) estimates $\psi(x)$ by

$$\psi_n^*(x) = \{\{(1-K_n(x))/k_n(x)\} \wedge a_n(x)\}[x\ge h]$$

where for each n $a_n(x)$ is a bounded function in x and for each x $a_n(x)\to\infty$ as $n\to\infty$. (Note that $k_n(x)$ in Fox(1978) is defined by $h^{-1}\{K_n(x)-K_n(x-h)\}$.) Hence, Fox(1978) estimates $\phi_G(x)$ by

$$\phi_n^*(x) = x + \psi_n^*(x).$$

Instead of the boundedness of the range of θ, if we add extra assumption $G(\theta^2)<\infty$ on G, then in the similar fashions to those in Sections 2 and 3 we will get following upper bound:

Lemma 4.1. With $h=n^{-1/3}$ and $a_n^2 = \log n$,

$$R(\phi_n^*,G) - R \leq O(n^{-2/3}\log n).$$

For a lower bound if we define $B=[a_n(x) \geq (1-K_n(x))/k_n(x)]$ and notice that $R(\phi_n^*,G) - R \geq E(P_{\underset{\sim}{x}}(\zeta_n^2(x)B)[h\leq x<1])$, then according to the similar methods to prove Lemma 3.1 we can easily show that for each x $P_x B^c \to 0$ and therefore get the similar bound to that in Theorem 2 with ϕ_n there replaced by ϕ_n^*.

5. A Lower Bound for $R(\phi_n^*,G)$ -R in a Location Parameter Family of Gamma Distributions.

For $\theta\varepsilon\Omega=(-\infty,\infty)$, let $f_\theta(x) = (\Gamma(\alpha))^{-1}(x-\theta)^{\alpha-1}e^{-(x-\theta)}[x\geq\theta]$ where $\alpha\geq 1$ and Γ represents the gamma function. For this family, the marginal pdf of X is given by

(5.1) $$k(x)= (\Gamma(\alpha))^{-1}\int_{-\infty}^{x} (x-\theta)^{\alpha-1}e^{-(x-\theta)}\, dG(\theta).$$

By (1.1) and Lemma 4.1 of R. Fox(1978), for $k(x)>0$

(5.2) $$\phi_G(x) = x - \alpha\psi(x)$$

where

(5.3) $$\psi(x) = \int_{-\infty}^{x} e^{-(x-t)}\, dK(t)/k(x).$$

R. Fox has shown in Theorem 4.1(1978) that under the assumption $G(\theta^2)<+\infty$ there exist a. o. estimators for $\theta\varepsilon(-\infty,\infty)$. In this section we shall show that risk difference (1.2) for his estimates has a lower bound of order $(nh)^{-1}$.

Let $k_n(x)=(nh)^{-1}\Sigma_{j=1}^{n}[x-h<X_j\leq x]$. Let $a_n(x)$ be a bounded function of x such that for each x $a_n(x)\to\infty$ as $n\to\infty$. From the form (5.3) $\psi(x)$ in (5.2) is estimated by

$$\psi_n^*(x) = \min(a_n(x), \{\int_{-\infty}^{x} e^{-(x-t)}\, dK_n(x)\}/k_n(x)) \tag{5.4}$$

and hence $\phi_G(x)$ by

$$\phi_n^*(x) = x - \psi_n^*(x). \tag{5.5}$$

To get a lower bound we assume that G is the degenerate distribution at zero. Then, $\phi_G(x)\equiv 0$ for all x. Hence, letting $B=[a_n(x) \geq \int_{-\infty}^{x} e^{-(x-t)}dK_n(x)/k_n(x)]$ we have

$$R(\phi_n^*,G) - R \geq E\, \underset{\sim}{P}_x(\phi_n^{*2}(x)B). \tag{5.6}$$

where $\phi_n^*(x)$ in (5.5) is expressed as

$$\phi_n^*(x) = \frac{x\Sigma_{j=1}^{n}[x-h<X_j\leq x] - \alpha h\int_{-\infty}^{x}\Sigma_{j=1}^{n}[t<X_j\leq x]e^{-(x-t)}\, dt}{\Sigma_{j=1}^{n}[x-h<X_j\leq x]}\ . \tag{5.7}$$

Letting $v=\Sigma_{j=1}^{n}[x-h<X_j\leq x]$ and $u_t=\Sigma_{j=1}^{n}[t<X_j\leq x]$ we have by simple calculations $P_x(v) = nK]_{x-h}^{x}$, $\mathrm{Var}_x v = n\, K]_{x-h}^{x}(1- K]_{x-h}^{x})$, $P_x(u_t) = n\, K]_t^x$ and $\mathrm{Var}_x(u_t)= n\, K]_t^x(1- K]_t^x)$. Letting

$$Z = \frac{v - n\, K]_{x-h}^{x}}{\sqrt{n\, K]_{x-h}^{x}(1- K]_{x-h}^{x})}} \quad \text{and } Y_t = \frac{u_t - n\, K]_t^x}{\sqrt{n\, K]_t^x(1- K]_t^x)}}$$

we have the following equality:

$$\begin{aligned}&\sqrt{nh}\,(\phi_n^*(x) - \hat{\phi}_n(x))\\ &= \frac{x\sqrt{h^{-1}K]_{x-h}^{x}(1- K]_{x-h}^{x})}\, Z - \alpha\sqrt{h}\int_{-\infty}^{x}\sqrt{K]_t^x(1-K]_t^x)}\, Y_t e^{-(x-t)}\, dt}{(nh)^{-\frac{1}{2}}\sqrt{h^{-1}K]_{x-h}^{x}(1- K]_{x-h}^{x})}\, Z + h^{-1}K]_{x-h}^{x}}\end{aligned} \tag{5.8}$$

where

$$(5.9)\qquad \hat{\phi}_n(x) = \frac{xh^{-1}K]_{x-h}^{x} - \alpha \int_{-\infty}^{x} e^{-(x-t)}\, dK(t)}{(nh)^{-\frac{1}{2}}\sqrt{h^{-1}K]_{x-h}^{x}(1-K]_{x-h}^{x})}\, Z + h^{-1}K]_{x-h}^{x}}$$

To get a lower bound for $R(\phi_n^*,G)-R(\text{Theorem } 3)$ we use (5.6) and the facts that for fixed x $(nh)^{\frac{1}{2}}(k(x))^{\frac{1}{2}}(\phi_n^*(x) - \hat{\phi}_n(x))B \xrightarrow{D} N(0,x^2)$ (Lemma 5.2) and with $nh^3=O(1)$ $\sqrt{nh}\,\hat{\phi}_n(x)B \xrightarrow{P} \{-2^{-1}xg'(x;\alpha) + O(h)\}/g(x;\alpha)$ where

$$(5.10)\qquad g(x;\alpha) = x^{\alpha-1}e^{-x}/\Gamma(\alpha) \qquad \text{for } -\infty < x < \infty$$

(Lemma 5.3).

We then apply Slutsky's theorem and a derivation of the convergence theorem(3.4) : (cf. Nogami(1981, Theorem A))

$$(5.11)\qquad \text{If } U_n \xrightarrow{D} U, \text{ then } \underline{\lim}\, \text{Var}\, U_n \geq \text{Var}\, U.$$

We shall first prepare Lemma 5.1 to prove above two convergences in distribution (Lemmas 5.2 and 5.3) for the proof of forthcoming Theorem 3.

<u>Lemma 5.1.</u> For each x and $h>0$ such that $nh^2 \to \infty$ as $n\to\infty$,

$$(5.12)\qquad P_{\underset{\sim}{x}}B^c \to 0 \qquad \text{as } n \to \infty.$$

<u>Proof</u>.) Let $Y_j = \{[X_j \leq x] - \int_{-\infty}^{x} [X_j \leq t]e^{-(x-t)}\, dt - a_n(x)h^{-1}[x-h < X_j \leq x]\}$. Since $P_xY_j = \int_{-\infty}^{x} e^{-(x-t)}\, dK(t) - a_n(x)h^{-1}K]_{x-h}^{x}$, and since ,because $\phi_G(x)\equiv 0$ for all x, for each fixed x there always exists sufficiently large n such that $\psi(x)$ $(=\alpha^{-1}x) \leq 4^{-1}a_n(x)$, these applied to the rhs of the inequality below leads to

$$(5.13)\qquad P_{\underset{\sim}{x}}B^c = P_{\underset{\sim}{x}}[\, \Sigma_{j=1}^{n} Y_j > 0]$$
$$\leq P_{\underset{\sim}{x}}[\, \Sigma_{j=1}^{n}(Y_j - P_xY_j) > n\, a_n(x)(h^{-1}K]_{x-h}^{x} - 4^{-1}k(x))].$$

Since $h^{-1}K]_{x-h}^{x} \geq (1-h)k(x)$, we have for $h \leq \frac{1}{2}$

$$(5.14)\qquad \text{extreme rhs}(5.13) \leq P_{\underset{\sim}{x}}[\, \Sigma_{j=1}^{n}(Y_j - P_{\underset{\sim}{x}}Y_j) > 4^{-1}n\, a_n(x)\, k(x)].$$

Since $-1-a_n(x)h^{-1} \leq Y_j \leq 1$ for all j, by Hoeffding's inequality (1963, p.16, Theorem 2) we obtain

$$\text{rhs}(5.14) \le \exp\{-\{2n^2(4^{-1}a_n(x)k(x))^2\}/\{n(2+a_n(x)h^{-1})^2\}\}$$
$$\le \exp\{-2^{-1}nh^2(k(x))^2\}$$

which goes to zero when $nh^2 \to \infty$ as $n \to \infty$.

Lemma 5.2. For fixed x and $h > 0$ such that $h \to 0$ and $nh^2 \to \infty$ as $n \to \infty$,

$$(5.15) \qquad (nh)^{\frac{1}{2}}(k(x))^{\frac{1}{2}}(\phi_n^*(x)-\hat{\phi}_n(x))B \overset{D}{\to} N(0,x^2).$$

Proof.) Since $Z \overset{D}{\to} N(0,1)$ and $h^{-1}K]_{x-h}^{x} \to k(x)$ as $n \to \infty$, for h such that $(nh)^{\frac{1}{2}} \to \infty$ as $n \to \infty$,

$$(5.16) \qquad (nh)^{-\frac{1}{2}}\sqrt{h^{-1}K]_{x-h}^{x}(1-K]_{x-h}^{x})}\, Z \overset{p}{\to} 0.$$

Similarly, $h^{\frac{1}{2}}\sqrt{K]_t^x(1-K]_t^x)}\, Y_t \overset{p}{\to} 0$ and hence,

$$(5.17) \qquad \int_{-\infty}^{x} h^{\frac{1}{2}}\sqrt{K]_t^x(1-K]_t^x)}\, Y_t e^{-(x-t)}\, dt \overset{p}{\to} 0.$$

Thus, applying Slutsky's theorem (R.J. Serfling(1980, p.19)) by using Lemma 5.1, (5.16) and (5.17) and by the facts that $Z \overset{D}{\to} N(0,1)$ and $h^{-1}K]_{x-h}^{x} \to k(x)$ as $n \to \infty$, we get (5.15).

Lemma 5.3. With $h > 0$ such that $nh^3 = O(1)$ and $nh^2 \to \infty$ as $n \to \infty$,

$$(5.18) \qquad \sqrt{nh}\, \hat{\phi}_n(x)B \overset{p}{\to} \{-2^{-1}xg'(x;\alpha) + O(h)\}/g(x;\alpha)\}(=d_1(x)).$$

Proof.) Since G is degenerate at zero, $\phi_G(x) \equiv 0$ for all x and hence $\alpha\int_{-\infty}^{x} e^{-(x-t)}\, dK(t) = x\, k(x)$. But, in view of (5.10), $k(x) = g(x;\alpha)$ for $x > 0$; $=0$ otherwise. Hence,

$$(5.19) \qquad \sqrt{nh}\, \hat{\phi}_n(x) = \frac{\sqrt{nh}\, x\{h^{-1}K]_{x-h}^{x} - g(x;\alpha)\}}{(nh)^{-\frac{1}{2}}\sqrt{h^{-1}K]_{x-h}^{x}(1-K]_{x-h}^{x})}\, Z + h^{-1}K]_{x-h}^{x}} .$$

By a change of variable $h^{-1}(x-u)=v$ applied to the rhs of the second equality below,

$$h^{-1}K]_{x-h}^{x} - g(x;\alpha) = h^{-1}\int_{x-h}^{x} g(u;\alpha)\,du - g(x;\alpha)$$
$$= \int_0^1 (g(x-hv;\alpha) - g(x;\alpha))\,dv.$$

Using the second order Taylor expansion in (-hv) of g(x-hv) about x (see e.g. R. J. Serfling(1980, p. 45, Theorem C)) we have $g(x-hv;\alpha) = g(x;\alpha) - hv\, g'(x;\alpha) + o(h)$. Thus, $h^{-1}K]_{x-h}^{x} - g(x;\alpha) = -2^{-1}h\, g'(x;\alpha) + o(h)$. Applying this to the numerator in rhs(5.19), applying (5.16), Lemma 5.1 and the fact that $h^{-1}K]_{x-h}^{x} \to k(x)$ as $n \to \infty$ and using Slutsky's theorem gives us Lemma 5.3.

By Lemmas 5.2 and 5.3 and by again applying Slutsky's theorem we get

(5.20) $$(nh)^{\frac{1}{2}}(k(x))^{\frac{1}{2}}\, \phi_n^{*}(x)B \xrightarrow{D} N((k(x))^{\frac{1}{2}}d_1(x),\ x^2).$$

We are now ready to prove the following theorem:

<u>Theorem 3.</u> In h is a function of n such that as $n \to \infty$ $nh^2 \to \infty$, $h \to 0$ and $nh^3 = O(1)$, then for any ε such that $0 < \varepsilon < \alpha(\alpha+1)$ there exists $N < +\infty$ such that for all $n \geq N$

(5.21) $$R(\phi_n, G) - R \geq \{\alpha(\alpha+1) - \varepsilon\}(nh)^{-1}.$$

<u>Proof.)</u> By the fact that $k(x) \leq 1$ and by Fatou's theorem applied to the extreme lhs

(5.22) $$\underline{\lim}\, E\, \underset{\sim}{P}_x((nh)\, \phi_n^{*2}(x)B) \geq E\{\underline{\lim}\, \underset{\sim}{P}_x((nh)\, k(x)\, \phi_n^{*2}(x)B)\}$$
$$\geq E\{\underline{\lim}\, Var_x((nh)\, k(x)\, \phi_n^{*2}(x)B)\}$$
$$\geq E(x^2) = \alpha(\alpha+1)$$

where the third inequality follows from (5.20) and a convergence theorem (5.11). Therefore, by (5.6) and extreme rhs of (5.22), $\underline{\lim}\, (nh)(R(\phi_n^{*},G) - R) \geq \alpha(\alpha+1)$. Thus, the definition of lim-inf leads to the bound in (5.21).

Remark. Although Fox estimates (5.2) by (5.5) with (5.4), it may be natural to estimate (5.2) by (5.5) with $\psi_n^*(x)$ replaced by

$$(5.23) \qquad \psi_n(x) = \{ \int_{-\infty}^{x} e^{-(x-t)} k_n(t)\, dt / k'_n(x)\} [k_n(x) > 0].$$

However, for any of above estimates we had difficulty to get an upper bound for the risk convergence (1.2) by using Singh's Lemma A.2 because we cannot use the assumtion (ii) in a location parameter family of gamma distributions.

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