

台が有限な分布の位置母数の同時推定

慶応大工 赤井 豊秋 (Toyoaki Akai)

Abstract

Suppose that components of a random vector $X = (X_1, \dots, X_p)'$ ($p \geq 3$) are independently distributed, each having a probability density which is positive on a finite interval, is symmetric about ϑ_i and has the same variance. In estimation of the location vector $\vartheta = (\vartheta_1, \dots, \vartheta_p)'$ under the squared loss function explicit estimators which dominate X are obtained by using integration by parts to evaluate the risk function. Further, explicit dominating estimators are given when the distributions of components are mixture of two uniform distributions. For the loss function $L(\hat{\vartheta}, \vartheta) = \|\hat{\vartheta} - \vartheta\|^2$ such an estimator is also given when the distributions of components of X are uniform distributions.

1 Introduction.

Stein (1955) and Brown (1966) proved that the best invariant estimator of the location vector of three or more dimensions are inadmissible, and there has been considerable interest in how to improve it. James and Stein (1961) presented an explicit estimator $\{1 - (p-2)/\|X\|^2\}X$, which is better than X under squared error loss if X has a normal distribution with covariance matrix I , the identity matrix. They also showed that the estimator

$$(1.1) \quad \delta_1(X) = \left\{ 1 - \frac{b}{a + \|X\|^2} \right\} X,$$

is better than X , without the normality assumption, for sufficiently small b and sufficiently large a . They did not, however, determine explicitly the values of these constants.

When $X = (X_1, \dots, X_p)'$ is an observed value from a spherically symmetric p -dimensional distribution, explicit estimators of a location vector which dominate X are given. See Brandwein and Strawderman (1980) and the papers in their references. Shinozaki (1984) obtained similar results in the case where components of X are independent, identically and symmetrically distributed p random variables, by applying integration by parts to three typical distributions; uniform, double exponential and t . Since Stein (1973) used integration by parts for estimating the location parameter of the normal distribution, it has been shown to apply to simultaneous estimation problems in general continuous exponential family by many authors. See Hudson (1978).

In this paper, we assume that components of $X = (X_1, \dots, X_p)'$ are independently distributed, each having a probability density which is positive on a finite interval, is symmetric about ϑ_i , the center of the interval, and has the same variance, and we estimate the location vector $\vartheta = (\vartheta_1, \dots, \vartheta_p)'$ by the estimator δ_1 of (1.1) under the squared error loss function.

Berger (1978) showed some results for losses which are polynomial in the coordinates of $(\vartheta - \vartheta)$ for the normal case, and Brandwein and Strawderman (1980) for the spherically symmetric distribution when the loss is a nondecreasing concave function of quadratic loss. Here we also study a special form of Berger's loss function for the uniform distribution.

In Section 2, some sufficient conditions on the constants a and b for the estimator δ_1 to dominate X are given. Further the constants in another

estimator

$$(1.2) \quad \delta_2(X) = \left\{ 1 - \frac{b(I-B)}{\alpha + X(I-B)X} \right\} X,$$

where B is $p \times p$ projection matrix and I is the identity matrix, are determined. In Section 3, the results in Section 2 are applied to three distributions. The distributions are the truncated normal, the parabola and the cusp shaped distributions which are defined by (3.1), (3.3) and (3.5), respectively. The cusp shaped distribution is the distribution of the best invariant estimator of the location parameter of the uniform distribution.

In Section 4, the values of a and b for the estimator δ_1 to dominate X are given when the distributions of components are mixture of two uniform distributions with a common center.

In Section 5, we give sufficient conditions on the constants a and b for the estimator δ_1 to dominate X under the loss $L(\delta_1, \vartheta) = \|\delta_1 - \vartheta\|^4$ when the distributions of components of X are uniform distributions.

2. Estimation of location parameters of the distributions with finite support

Let X_i ($i=1, \dots, p$), components of $X = (X_1, \dots, X_p)'$, be p independent random variables from a probability density of the form

$$(2.1) \quad f_i(x_i - \vartheta_i) = \begin{cases} f_i(|x_i - \vartheta_i|) > 0, & \text{if } |x_i - \vartheta_i| \leq c_i, \\ 0, & \text{otherwise,} \end{cases}$$

which are assumed to have the same variance V . Set $Z_i = X_i - \vartheta_i$, and $EZ_i = 0$.

In estimating the location vector $\vartheta = (\vartheta_1, \dots, \vartheta_p)'$ by $\delta_1(X) = (\delta_{11}(X), \dots, \delta_{1p}(X))'$ under the squared loss function

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$$\|\delta_1(X) - \vartheta\|^2 = \sum_{i=1}^p (\delta_{1i}(X) - \vartheta_i)^2,$$

some sufficient conditions on the constants a and b of (1.1) are given such as the risk $R(\delta_1, \vartheta) = E_X \|\delta_1 - \vartheta\|^2$ is uniformly smaller than $R(X, \vartheta)$. E_X denotes the expectation with respect to X .

Theorem 2.1. Let X_i have a probability density of the form (2.1). Assume the following conditions be satisfied for $i=1, \dots, p$:

$$(2.2) \quad EZ_i^4 < \infty,$$

and there exists a constant $d_i > 0$ such that

$$(2.3) \quad \left| \int_{-c_i}^z \int_{-c_i}^y \left(\int_{-c_i}^u t f_i(t) dt / V + f_i(u) \right) du dy \right| \leq d_i f_i(z),$$

where $V = \int_{-c_i}^{c_i} z^2 f_i(z) dz$. Then the risk of δ_1 is uniformly smaller than that of X if $a \geq 6 \sum_{i=1}^p d_i / p$ and $0 < b \leq 2(p-2)V$.

Proof.

$$(2.4) \quad R(X, \vartheta) - R(\delta_1, \vartheta) = 2bE \left\{ \sum_{i=1}^p \frac{X_i(X_i - \vartheta_i)}{a + \|X\|^2} - \frac{b\|X\|^2}{2V(a + \|X\|^2)^2} \right\}.$$

The conditional expectation of a term of the summation in (2.4) equals

$$I \equiv E_{X_i} \left\{ \frac{X_i(X_i - \vartheta_i)}{a + \|X\|^2} \right\} = E_{Z_i} \left\{ \frac{Z_i(Z_i + \vartheta_i)}{a + \|Z + \vartheta\|^2} \right\}$$

Integration by parts gives

$$(2.5) \quad I = - \int_{-c_i}^{c_i} g_1(z_i) \int_{-c_i}^{z_i} y f_i(y) dy dz_i \\ = VE_{Z_i} \{g_1(Z_i)\} + I_1,$$

where

$$(2.6) \quad g_1(z_i) = \frac{1}{a + \|z + \vartheta\|^2} - \frac{2(z_i + \vartheta_i)^2}{(a + \|z + \vartheta\|^2)^2}$$

and

$$I_1 = -V \int_{-c_i}^{c_i} g_1(z_i) \left(\int_{-c_i}^{z_i} y f_i(y) dy / V + f_i(z_i) \right) dz_i.$$

Integration by parts is applied to I_1 twice, then

$$(2.7) \quad I_1 = 6V \int_{-c_i}^{c_i} g_2(z_i) \left\{ \int_{-c_i}^{z_i} \int_{-c_i}^y \left(\int_{-c_i}^u t f_i(t) dt / V + f_i(u) \right) du dy \right\} dz_i,$$

where

$$g_2(z_i) = \frac{1}{(a + \|z + \vartheta\|^2)^2} - \frac{8(z_i + \vartheta_i)^2}{(a + \|z + \vartheta\|^2)^3} + \frac{8(z_i + \vartheta_i)^4}{(a + \|z + \vartheta\|^2)^4}.$$

Note that $\int_{-c_i}^y \left(\int_{-c_i}^u t f_i(t) dt / V + f_i(u) \right) du$ is an odd function and that $\int_{-c_i}^z \int_{-c_i}^y \left(\int_{-c_i}^u t f_i(t) dt / V + f_i(u) \right) du dy$ is an even function.

The inequality $|g_2(z_i)| \leq \frac{1}{(a + \|z + \vartheta\|^2)^2}$ and the condition (2.3) show that

$$(2.8) \quad I_1 \geq -6d_i VE_{X_i} \left\{ \frac{1}{(a + \|X\|^2)^2} \right\},$$

and from the last expression of (2.5) and (2.8),

$$I \geq VE_{X_i} \left\{ \frac{1}{a + \|X\|^2} - \frac{2X_i^2}{(a + \|X\|^2)^2} - \frac{6d_i}{(a + \|X\|^2)^2} \right\}.$$

Hence from (2.4),

$$\begin{aligned} & R(X, \vartheta) - R(\delta_1, \vartheta) \\ & \geq 2b VE \left[\frac{ap + (p-2)\|X\|^2 - 6 \sum_{i=1}^p d_i}{(a + \|X\|^2)^2} - \frac{b\|X\|^2}{2V(a + \|X\|^2)^2} \right], \end{aligned}$$

which is nonnegative if $a \geq 6 \sum_{i=1}^p d_i / p$ and $0 < b \leq 2(p-2)V$.

Remark 2.1. Note that $-\int_{-\infty}^y t f_i(t) dt / V = f_i(y)$ in the normal case. It is shown, by using integration by parts also, that

$$-\int_{-c_i}^{c_i} \int_{-c_i}^z \int_{-c_i}^y \left(\int_{-c_i}^u t f_i(t) dt / V + f_i(u) \right) du dy dz = \{EZ_i^4 - 3(EZ_i^2)^2\} / 6EZ_i^2.$$

Under a stronger condition an alternative sufficient condition is obtained.

Theorem 2.2. Let f_i satisfy the conditions of Theorem 2.1 and

$$(2.9) \quad \int_{-c_i}^z \int_{-c_i}^y \left(\int_{-c_i}^u t f_i(t) dt / V + f_i(u) \right) dx dy \geq 0, \quad |z| \leq c_i.$$

Then the risk of δ_1 is uniformly smaller than that of X if $\alpha \geq 24(2 - \sqrt{2}) \max_{1 \leq i \leq p} d_i / p$ and $0 < b \leq 2(p - 2)V$.

Proof. The inequality

$$g_2(z_i) \geq -4(2 - \sqrt{2})(z_i + \vartheta_i)^2 / (\alpha + \|z + \vartheta\|^2)^3,$$

(2.7) of Theorem 2.1 and the condition (2.9) give

$$(2.10) \quad I_1 \geq -24(2 - \sqrt{2}) V E_{X_i} \{d_i X_i^2 / (\alpha + \|X\|^2)^3\}.$$

Therefore from the last expression of (2.5) of Theorem 2.1 and (2.10), $R(X, \vartheta) - R(\delta_1, \vartheta) \geq 0$ if the conditions on α and b are satisfied.

In Theorem 2.1 and 2.2 it is shown that the estimator δ_1 , which pulls X_i 's towards the origin, dominates X . Here the estimator δ_2 of (1.2), which pulls the estimators towards a sub-space spanned by B , is considered, and some sufficient conditions on the constants α and b of (1.2) are given such as $R(X, \vartheta) - R(\delta_2, \vartheta)$ is nonnegative.

Theorem 2.3. Let f_i satisfy the conditions of Theorem 2.1. Then the risk of δ_2 is uniformly smaller than that of X if

$$\alpha \geq \frac{6 \left\{ 8 \sum_{i=1}^p b_{ii} d_i + \sum_{i=1}^p (1 - b_{ii})^2 d_i \right\}}{\sum_{i=1}^p (1 - b_{ii})}$$

and

$$0 < b \leq 2 \left(p - \sum_{i=1}^p b_{ii} - 2 \right) V,$$

where $B = (b_{ij})$.

Proof. Theorem 2.3 can be proved in a similar way as Theorem 2.1 and the proof is omitted.

Theorem 2.4. Let f_i satisfy the conditions of Theorem 2.2. Then the risk of δ_2 is uniformly smaller than that of X if

$$a \geq 6 \sum_{i=1}^p (1 - b_{ii})^2 d_i / \sum_{i=1}^p (1 - b_{ii})$$

and

$$0 < b \leq 2(p - \sum_{i=1}^p b_{ii} - 2)V.$$

Corollary 2.5. Let f_i satisfy the conditions of Theorem 2.2. Assume that the diagonal elements of B are equal. Then the risk of δ_2 is uniformly smaller than that of X if $a \geq 24(2 - \sqrt{2}) \max_i d_i / p$ and $0 < b \leq 2(p(1 - b_{ii}) - 2)V$.

Remark 2.2. If $B = (b_{ij})$, $b_{ij} = 1/p$ for all i and j , the estimator δ_2 pulls the estimators towards their average $\bar{X} = (X_1 + \dots + X_p)/p$.

3. Examples

In this section Theorem 2.1 and Theorem 2.2 are applied to the truncated normal, the parabola and the cusp shaped distributions to obtain the estimator δ_1 which dominates X .

Example 3.1. Suppose that Z_i has the common density of the form

$$(3.1) \quad f_i(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} I_{[-c,c]}(z) / (\Phi(c) - \Phi(-c)),$$

where Φ is a c.d.f. of standard normal distribution and $I_{[-c,c]}(z)$ is an indicator function of the interval $[-c,c]$. (i.e. $I_{[-c,c]}(z) = 1$ if $|z| \leq c$, and $I_{[-c,c]}(z) = 0$ if $|z| > c$.)

Then, integration by parts shows that

$$\begin{aligned} p_1(z_i) &\equiv -\int_{-c}^{z_i} \int_{-c}^y \left(\int_{-c}^u t f_1(t) dt / V + f_1(u) \right) dudy \\ &= \frac{e^{-c^2/2}}{(2\Phi(c) - 1)V} q_1(z_i) f_1(z_i), \end{aligned}$$

where

$$\begin{aligned} q_1(z_i) &= \{2cz_i\Phi(z_i) + 2c^2\Phi(-c) - (z_i^2 - c^2)(2\Phi(c) - 1) / 2 \\ &\quad - c(z_i + c)\} e^{z_i^2/2} + 2c(1 - e^{(z_i^2 - c^2)/2}) / \sqrt{2\pi}. \end{aligned}$$

Note that $q_1(z_i) \leq 0$ for $0 \leq z_i \leq c$. The inequality $\Phi(z_i) \geq (2\Phi(c) - 1)z_i/2c + 1/2$ gives

$$-q_1(z_i) \leq (-k_1 z_i^2/2 + c^2 k_1/2 + k_2) e^{z_i^2/2} - 2c/\sqrt{2\pi} \quad \text{for } 0 \leq z_i \leq c,$$

where $k_1 = 2\Phi(c) - 1$ and $k_2 = 2ce^{-c^2/2}/\sqrt{2\pi}$, and

$$(3.2) \quad -p_1(z_i) \leq (k_1 e^{-1+k_2/k_1} - k_2) f_1(z_i) / k_1 V.$$

By Theorem 2.2 and (3.2), the risk of δ_1 is uniformly smaller than that of X if

$$a \geq a_0, \quad \text{where } a_0 = 24(2 - \sqrt{2})(k_1 e^{-1+k_2/k_1} - k_2) / k_1 p V$$

and

$$0 < b \leq 2(p - 2)V.$$

Table 3.1 gives the value of a_0 for $c = 1$; $c = 2$ and $c = \infty$.

Table 3.1 a_0 : the lower bound of a

c	1	2	∞
a_0	$6.40V/p$ ($V=0.7559$)	$5.52V/p$ ($V=0.8214$)	$24(2-\sqrt{2})/ep$ ($V=1$)

Example 3.2. Suppose that Z_i has the common density of the form

$$(3.3) \quad f_2(z) = \frac{3}{4c^2} (c^2 - z^2) I_{[-c, c]}(z).$$

Without loss of generality we set $c = 1$. Then

$$(3.4) \quad -\int_{-1}^{z_i} \int_{-1}^y \left(\int_{-1}^u t f_2(t) dt / V + f_2(u) \right) dudy = -\frac{1}{24} (z_i^2 - 1)^2 f_2(z_i) \\ \geq -f_2(z_i) / 24.$$

According to Theorem 2.2 and (3.4), $R(X, \vartheta) - R(\delta_1, \vartheta)$ is nonnegative if $\alpha \geq 5(2 - \sqrt{2})V/p$ and $0 < b \leq 2(p - 2)V$.

Example 3.3. Suppose that Z_i has the common density of the form

$$(3.5) \quad f_3(z) = \frac{k+1}{2c^{k+1}} (c - |z|)^k I_{[-c, c]}(z),$$

where $k \geq 0$. Assume $c = 1$. Then

$$p_3(z_i) \equiv -\int_{-1}^{z_i} \int_{-1}^y \left(\int_{-1}^u t f_3(t) dt / V + f_3(u) \right) dudy \\ = \frac{(1 - z_i)^2}{2(k+1)(k+2)(k+4)} q_3(z_i) f_3(z_i), \quad \text{for } 0 \leq z_i \leq 1,$$

where $q_3(z_i) = -(k+1)(k+2)z_i^2 + (k+2)(k-2)z_i + k - 2$. To evaluate $p_3(z_i)$, we consider the following two cases: $0 \leq k \leq 2$ and $k > 2$.

Case 1: $0 \leq k \leq 2$. Noting that $q_3(z_i) \leq 0$, we can show that

$$(3.6) \quad -p_3(z_i) \leq l_1(l_2 + l_3 l_4^{1/2}) V f_3(z_i),$$

where

$$l_1 = (k+3) / 2048(k+1)^4(k+2)^2(k+4), \\ l_2 = -27k^6 - 252k^5 - 620k^4 + 640k^3 + 4992k^2 + 7168k + 1024, \\ l_3 = 9k^4 + 62k^3 + 72k^2 - 96k + 128,$$

and

$$l_4 = (k+2)^2(5k-4)^2 - 16k(k-2)(k+1)(k+2).$$

Therefore from Theorem 2.2 and (3.6), $R(X, \vartheta) - R(\delta_1, \vartheta) \geq 0$ if $\alpha \geq 24(2 - \sqrt{2})l_1(l_2 + l_3 l_4^{1/2})V/p$ and $0 < b \leq 2(p - 2)V$.

Case 2: $k > 2$. Noting that

$$q_3(z_i) \begin{cases} \geq 0, & \text{if } 0 \leq z_i \leq \frac{(k+2)(k-2) + \{(k+2)^2(k-2)^2 + 4(k-2)(k+1)(k+2)\}^{1/2}}{2(k+1)(k+2)}, \\ < 0, & \text{otherwise,} \end{cases}$$

we get

$$(3.7) \quad |p_3(z_i)| \leq l_1(|l_2| + l_3 l_4^{1/2}) V f_3(z_i).$$

By Theorem 2.1 and (3.7), $R(X, \vartheta) - R(\delta_1, \vartheta) \geq 0$ if $a \geq 6l_1(|l_2| + l_3 l_4^{1/2}) V$ and $0 < b \leq 2(p-2)V$.

Remark 3.1. Note that the best invariant estimator of the location parameter of the uniform distribution has the probability density (3.5). According to Example 3.3, the estimator δ_1 has a smaller risk than the best invariant one.

4. Estimation of location parameters for the mixture of uniform distributions.

Suppose that components of X have mixture of distributions with the same variance. Let their densities satisfy the conditions (2.2) and (2.3) of Theorem 2.1. In this case, sufficient conditions on the constants a and b of the estimator δ_1 in (1.1) to dominate X are obtained by applying Theorem 2.1. Here we study the case where Z_i has the common density which is a mixture of two uniform distributions, i.e. the density is

$$(4.1) \quad f(z) = \frac{\alpha}{2c_1} I_{[-c_1, c_1]}(z) + \frac{1-\alpha}{2c_2} I_{[-c_2, c_2]}(z),$$

where $0 \leq \alpha < 1$ and $c_1 \leq c_2$.

Theorem 4.1. Let Z_i have the common density of the form (4.1). Then the risk of δ_1 is uniformly smaller than that of X if

$$a \geq \max \{6c_0 + 3(2 - \sqrt{2})c_2^2/p, 3c_0 + (9c_0^2 + 60c_0(2 - \sqrt{2})c_2^2)^{1/2}\}$$

and

$$0 < b \leq 2(p-2)V,$$

where

$$c_0 = \max \left\{ \frac{\alpha(1-\alpha)c_1c_2(c_2-c_1)(V_{c_2}-V_{c_1})}{2((1-\alpha)c_1 + \alpha c_2)V}, \frac{\alpha(c_2-c_1)^2(V_{c_2}-V_{c_1})}{2V} \right\}$$

$$V_{c_i} = \int_{-c_i}^{c_i} \frac{z^2}{2c_i} dz, \text{ and } V = \int_{-c_i}^{c_i} z^2 f(z) dz = \alpha V_{c_1} + (1 - \alpha) V_{c_2}.$$

Note. It is easily seen that

$$c_0 = \begin{cases} \frac{\alpha(1-\alpha)c_1c_2(c_2-c_1)(V_{c_2}-V_{c_1})}{2((1-\alpha)c_1+\alpha c_2)V}, & \text{if } \alpha \leq \frac{c_1^2}{c_2(c_2-c_1)+c_1^2}, \\ \frac{\alpha(c_2-c_1)^2(V_{c_2}-V_{c_1})}{2V}, & \text{if } \alpha > \frac{c_1^2}{c_2(c_2-c_1)+c_1^2}. \end{cases}$$

Proof. From (2.4) of Theorem 2.1, the conditional expectation of a term of the summation in $\{R(X, \vartheta) - R(\delta_1, \vartheta)\}/2b$ can be written as

$$\begin{aligned} L &\equiv E_{X_i} \left\{ \frac{X_i(X_i - \vartheta_i)}{\alpha + \|X\|^2} \right\} \\ &= \alpha \int_{-c_1}^{c_1} \frac{z_i(z_i + \vartheta_i)}{\alpha + \|z + \vartheta\|^2} \frac{1}{2c_1} dz_i + (1 - \alpha) \int_{-c_2}^{c_2} \frac{z_i(z_i + \vartheta_i)}{\alpha + \|z + \vartheta\|^2} \frac{1}{2c_2} dz_i \end{aligned}$$

Integration by parts shows that

$$\begin{aligned} (4.2) \quad &\int_{-c_j}^{z_i} \int_{-c_j}^y \left(\int_{-c_j}^u t dt / 2c_j V_{c_j} + 1/2c_j \right) du dy \\ &= \frac{1}{16c_j^3} (z_i^2 - c_j^2)^2 \quad j = 1, 2. \end{aligned}$$

The last expression of (2.5) of Theorem 2.1 and (4.2) give

$$\begin{aligned} (4.3) \quad L &\geq \alpha V_{c_1} \int_{-c_1}^{c_1} \frac{t_1(z_i)}{2c_1} dz_i + (1 - \alpha) V_{c_2} \int_{-c_2}^{c_2} \frac{t_1(z_i)}{2c_2} dz_i \\ &= VE_{Z_i} \{t_1(Z_i)\} + L_1, \end{aligned}$$

where

$$\begin{aligned} t_1(z_i) &= \frac{1}{\alpha + \|z + \vartheta\|^2} - \frac{2(z_i + \vartheta_i)^2}{(\alpha + \|z + \vartheta\|^2)^2} - \frac{3(2 - \sqrt{2})c_2^2(z_i + \vartheta_i)^2}{(\alpha + \|z + \vartheta\|^2)^3}, \\ L_1 &= V \int_{-c_2}^{c_2} t_1(z_i) (\tilde{f}(z_i) - f(z_i)) dz_i, \end{aligned}$$

and

$$\tilde{f}(z) = \frac{\alpha V_{c_1}}{2c_1 V} I_{[-c_1, c_1]}(z) + \frac{(1 - \alpha) V_{c_2}}{2c_2 V} I_{[-c_2, c_2]}(z).$$

Note that $\tilde{f}(z) - f(z) > 0$ if $c_1 < |z| \leq c_2$, and $\tilde{f}(z) - f(z) < 0$ if $|z| \leq c_1$.

Integration by parts is applied to L_1 twice, then

$$L_1 = -6V \int_{-c_2}^{c_2} t_2(z_i) \tilde{g}(z_i) dz_i,$$

where

$$\begin{aligned} t_2(z_i) = & \frac{1}{(a + \|z + \vartheta\|^2)^2} + \frac{(2 - \sqrt{2})c_2^2 - 8(z_i + \vartheta_i)^2}{(a + \|z + \vartheta\|^2)^3} \\ & + \frac{8(z_i + \vartheta_i)^4 - 15(2 - \sqrt{2})c_2^2(z_i + \vartheta_i)^2}{(a + \|z + \vartheta\|^2)^4} \\ & + \frac{24(2 - \sqrt{2})c_2^2(z_i + \vartheta_i)^2}{(a + \|z + \vartheta\|^2)^5}, \end{aligned}$$

and

$$\begin{aligned} \tilde{g}(z_i) = & (1 - \alpha)(V_{c_2}/V - 1)z_i^2/4c_2 - (1 - \alpha)(V_{c_2}/V - 1)|z_i|/2 \\ & + (1 - \alpha)(V_{c_2}/V - 1)c_2/4, \quad \text{if } c_1 < |z_i| \leq c_2, \\ = & \alpha(V_{c_1}/V - 1)z_i^2/4c_1 + (1 - \alpha)(V_{c_2}/V - 1)z_i^2/4c_2 \\ & + (1 - \alpha)(V_{c_2}/V - 1)(c_2 - c_1)/4, \quad \text{if } |z_i| \leq c_1. \end{aligned}$$

Note that $\tilde{g}(z_i)$ is positive, even and unimodal on $(-c_2, c_2)$ and $f(z_i)$ is a step function of the same property. Thus, $\tilde{g}(z_i)/f(z_i)$ has the maximum value when $z_i = 0$ or $z_i = c_1$, and

$$(4.4) \quad \tilde{g}(z_i) \leq c_0 f(z_i).$$

The inequalities (4.4) and

$$t_2(z_i) \leq 1/(a + \|z + \vartheta\|^2)^2 - 10(2 - \sqrt{2})c_2^2/(a + \|z + \vartheta\|^2)^3,$$

give

$$(4.5) \quad L_1 \geq -6c_0 V E_{X_i} \left\{ \frac{1}{(a + \|X\|^2)^2} + \frac{10(2 - \sqrt{2})c_2^2}{(a + \|X\|^2)^3} \right\}.$$

Therefore from (4.3) and (4.5),

$$\begin{aligned} L \geq & V E_{X_i} \left\{ \frac{1}{a + \|X\|^2} - \frac{2X_i^2 + 6c_0}{(a + \|X\|^2)^2} \right. \\ & \left. - \frac{3(2 - \sqrt{2})c_2^2 X_i^2 + 60(2 - \sqrt{2})c_0 c_2^2}{(a + \|X\|^2)^3} \right\}. \end{aligned}$$

Hence

$$R(X, \vartheta) - R(\delta_1, \vartheta) \geq 2bVE \left\{ \frac{\alpha p + (p-2)\|X\|^2 - 6c_0 p - 3(2-\sqrt{2})c_2^2}{(\alpha + \|X\|^2)^2} + \frac{3(2-\sqrt{2})\alpha c_2^2 - 60(2-\sqrt{2})c_0 c_2^2 p}{(\alpha + \|X\|^2)^3} - \frac{b\|X\|^2}{2V(\alpha + \|X\|^2)^2} \right\},$$

which is nonnegative if

$$\alpha \geq \max \{ 6c_0 + 3(2-\sqrt{2})c_2^2/p, 3c_0 + (9c_0^2 + 60c_0(2-\sqrt{2})c_2^2)^{1/2} \},$$

and

$$0 < b \leq 2(p-2)V.$$

Table 4.1 shows the lower bound of a/V for $p=3$; $c_2/c_1 = 1.5, 2, 2.5$ and $\alpha = 0.2, 0.4, 0.6, 0.8$. The lower bound of a/V becomes large when c_2/c_1 or α is large.

Table 4.1 The lower bound of a/V

α c_2/c_1	0.2	0.4	0.6	0.8
1.5	2.63	3.82	4.84	7.51
2	3.91	6.61	12.15	23.93
2.5	4.55	9.42	19.01	44.58

Remark 4.1 Theorem 4.1 can be extended to the mixture of n -uniform distributions.

5. Estimation of location parameters for the uniform distribution under the

loss $\|\delta_1 - \vartheta\|^4$

In this section the constants a and b of (1.1) are given such as the risk of the estimator δ_1 is uniformly smaller than that of X with respect to the loss

$$(5.1) \quad \|\delta_1 - \vartheta\|^4 = \left\{ \sum_{i=1}^p (\delta_{1i} - \vartheta_i)^2 \right\}^2.$$

for the uniform distribution.

Following Berger's notation (1978), let

$$\gamma(X) = (\gamma_1(X), \dots, \gamma_p(X))' = \delta_1(X) - X$$

and

$$c_{n,i} = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{if } i < 0 \text{ or } i > n \text{ or } i \text{ is odd,} \\ \sum_{j=2}^{n-1} j c_{j-1, i-2}, & \text{if } 2 \leq i \leq n. \end{cases}$$

If $g : R^1 \rightarrow R^1$ is n times differentiable, let

$$g^{(j)}(z) = \frac{d^j}{dz^j} g(z), \quad 0 \leq j \leq n, \quad (g^{(0)}(z) = g(z)).$$

If $h : R^p \rightarrow R^1$ is a function with sufficient order derivatives, let

$$h^{i(k), j(l)}(z) = \frac{\partial^{(k+l)}}{\partial z_i^k \partial z_j^l} h(z).$$

The following lemma is useful in carrying out integration by parts for the loss $\|\delta_1 - \vartheta\|^4$.

Lemma 5.1. Let Z have the density of the form

$$f(z) = \begin{cases} f(|z|), & \text{if } |z| \leq c, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that $g(z)$ is a real-valued n times continuously differentiable function and that

$$\int_{-c}^c |g^{(n-j)}(z)| |z|^j f_{n-j}(z) dz < \infty \quad \text{for } 0 \leq j \leq n,$$

where

$$f_i(z) = \int_{-z}^z -y f_{i-1}(y) dy \quad \text{for } 1 \leq i \leq n \quad \text{and } f_0(z) = f(z).$$

Then

$$E\{g(Z)Z^n\} = \int_{-c}^c \sum_{i=0}^n c_{n,i} g^{(n-i)}(z) f_{[n-i/2]}(z) dz.$$

where $[\]$ denotes Gauss' symbol.

Proof. This lemma can be proved by induction on n along the same line as in the proof of Lemma 1 in Berger (1978).

Theorem 5.2. Let Z_i ($i=1, \dots, p$) be p independent random variables from a uniform distribution on $(-c, c)$. Then the risk of the estimator δ_1 is uniformly smaller than that of X with respect to the loss given by (5.1) if

$$a \geq \frac{p + 107/30}{p + 4/5} b + \frac{331}{20} V$$

and

$$0 < b \leq 2 \frac{p + 4/5}{p + 2} (p - 2) V.$$

Proof.

$$\|\delta_1 - \vartheta\|^4 = \left\{ \sum_{i=1}^p (\gamma_i(x) + x_i - \vartheta_i)^2 \right\}^2 = \left\{ \sum_{i=1}^p (\gamma_i(z + \vartheta) + z_i)^2 \right\}^2.$$

Therefore

$$\begin{aligned} (5.2) \quad \Delta(\vartheta) &= E\|\delta_1 - \vartheta\|^4 - E\|X - \vartheta\|^4 \\ &= E\left\{ 4 \sum_{i,j} \gamma_i \gamma_j Z_i Z_j + \sum_{i,j} \gamma_i^2 \gamma_j^2 + 4 \sum_{i,j} \gamma_i Z_i Z_j^2 \right. \\ &\quad \left. + 2 \sum_{i,j} \gamma_i^2 Z_j^2 + 4 \sum_{i,j} \gamma_i \gamma_j^2 Z_i \right\}. \end{aligned}$$

Using Lemma 5.1 to evaluate the conditional expectation of a term of each summation in (5.2), we get

$$\begin{aligned} E_{Z_i} \{ \gamma_i^2 Z_i^2 \} &= 2 \int_{-c}^c \{ (\gamma_i^{i(1)})^2 + \gamma_i \gamma_i^{i(2)} \} f_2(z_i) dz_i + \int_{-c}^c \gamma_i^2 f_1(z_i) dz_i, \\ E_{Z_i, Z_j} \{ \gamma_i \gamma_j Z_i Z_j \} &= \int_{-c}^c \int_{-c}^c \{ \gamma_i^{i(1),j(1)} \gamma_j + \gamma_i^{i(1)} \gamma_j^{j(1)} \\ &\quad + \gamma_i^{j(1)} \gamma_j^{i(1)} + \gamma_i \gamma_j^{i(1),j(1)} \} f_1(z_i) f_1(z_j) dz_i dz_j, \\ (5.3) \quad E_{Z_i} \{ \gamma_i Z_i^3 \} &= \int_{-c}^c \gamma_i^{i(3)} f_3(z_i) dz_i + 3 \int_{-c}^c \gamma_i^{i(1)} f_2(z_i) dz_i, \\ E_{Z_i, Z_j} \{ \gamma_i Z_i Z_j^2 \} &= \int_{-c}^c \int_{-c}^c \gamma_i^{i(1),j(2)} f_1(z_i) f_2(z_j) dz_i dz_j \\ &\quad + \int_{-c}^c \int_{-c}^c \gamma_i^{i(1)} f_1(z_i) f_1(z_j) dz_i dz_j, \\ E_{Z_j} \{ \gamma_i^2 Z_j^2 \} &= 2 \int_{-c}^c \{ (\gamma_i^{j(1)})^2 + \gamma_i \gamma_i^{j(2)} \} f_2(z_j) dz_j + \int_{-c}^c \gamma_i^2 f_1(z_j) dz_j. \end{aligned}$$

and

$$E_{Z_i}\{\gamma_i \gamma_j^2 Z_i\} = \int_{-c}^c \{\gamma_i^{i(1)} \gamma_j^2 + 2\gamma_i \gamma_j \gamma_j^{i(1)}\} f_1(z_i) dz_i,$$

where

$$\begin{aligned} \gamma_i(z + \vartheta) &= -\frac{b(z_i + \vartheta_i)}{a + \|z + \vartheta\|^2}, \quad f_1(z_i) = \frac{1}{4c}(-z_i^2 + c^2), \\ f_2(z_i) &= \frac{1}{16c}(z_i^2 - c^2)^2, \quad \text{and} \quad f_3(z_i) = \frac{1}{96c}(-z_i^2 + c^2)^3. \end{aligned}$$

Similarly in the proof of Theorem 2.1, a straightforward calculation of each term in (5.3) gives

$$(5.4) \quad \Delta(\vartheta) \leq E \left\{ \frac{b}{a + \|X\|^2} \left\{ m_0 + \frac{m_1 \|X\|^2 + m_2}{a + \|X\|^2} + \frac{m_3 \|X\|^2 + m_4}{(a + \|X\|^2)^2} \right. \right. \\ \left. \left. + \frac{m_5 \|X\|^4 + m_6 \|X\|^2 + m_7}{(a + \|X\|^2)^3} + \frac{m_8 \|X\|^2}{(a + \|X\|^2)^4} + \frac{m_9 \|X\|^4}{(a + \|X\|^2)^5} \right\} \right\},$$

where

$$\begin{aligned} m_0 &= -4p(p + 4/5)c^4/9, \quad m_1 = 2(p + 2)c^2b/3 + 8(p + 4/5)c^4/9, \\ m_2 &= 4p(p + 107/30)c^4b/9 + 61p(p + 89/61)c^6/135, \\ m_3 &= -4(p + 2)c^2b^2/3 - 176(p + 19/11)c^4b/45 - 128(p + 2)c^6/135, \\ m_4 &= pc^4b^2 + 4p(p - 1/10)c^6b/3 + 3p(p - 1)c^8/5, \\ m_5 &= b^3 + 8c^2b^2 + 64c^4b/5 + 128c^6/45, \\ m_6 &= 21(p - 4/7)c^4b^2/5 + 1492(p - 1477/1492)c^6b/75 + 186(p - 1)c^8/25, \\ m_7 &= 3p(p - 1)c^8b/2, \quad m_8 = 352(p - 1)c^8b, \quad \text{and} \quad m_9 = 480c^8b. \end{aligned}$$

Define

$$\begin{aligned} v(y, a, b) &= m_0 + \frac{m_1 y + m_2}{a + y} + \frac{m_3 y + m_4}{(a + y)^2} + \frac{m_5 y^2 + m_6 y + m_7}{(a + y)^3} \\ &\quad + \frac{m_8 y}{(a + y)^4} + \frac{m_9 y^2}{(a + y)^5}. \end{aligned}$$

It is clear from (5.4) that $\Delta(\vartheta) \leq 0$ if a and b can be chosen so that $v(y, a, b) \leq 0$ for all $0 \leq y \leq \infty$. To evaluate $v(y, a, b)$, we decompose $v(y, a, b)$ as

$$v(y, a, b) \equiv v_1(y, a, b) + v_2(y, a, b),$$

where

$$v_1(y, a, b) = m_0 + \frac{m_1 y + m_2}{a + y} + \frac{m_4}{(a + y)^2} + \frac{m'_6 y + m_7}{(a + y)^3} + \frac{m_8 y}{(a + y)^4},$$

$$v_2(y, a, b) = \frac{m_3 y}{(a + y)^2} + \frac{m_5 y^2 + m''_6 y}{(a + y)^3} + \frac{m_9 y^2}{(a + y)^5},$$

$$m'_6 = 3(p - 2/5)c^4 b^2 + 1492(p - 1477/1492)c^6 b/45 + 186(p - 1)c^8/25,$$

and

$$m''_6 = 6(p - 1)c^4 b^2/5.$$

It is straightforward to verify that $v_1(y, a, b) \leq 0$ and $v_2(y, a, b) \leq 0$ if the conditions on a and b are satisfied. The proof is completed.

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