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Flow past two spheres located close to each other at small non-zero Reynolds number

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The problem of axisymmetric flow past two spheres at small non-zero Reynolds number(R) is considered. Assuming the location of the spheres in the inner region of expansion of the other, forces(in R) on two equal sized spheres are computed for an arbitrary seperation between their centers. The comparison between numerically determined forces and their difference(interaction), and those reported previously for large seperation is found promising and the case of small seperation is discussed in details.

Introduction

The study of steady laminar flow of an incompressible fluid past rigid spheres require the solution of the Navier-Stokes and continuity equations for a non-zero parameter R(the Reynolds number), subject to the prevailing boundary conditions. Such studies for the case of a single sphere for $R \geq 0$ have been done by Stokes (1851), Proudman & Pearson(1957) and Hamielec & Hoffman(1967); and for two spheres with R=0 and R << 1 respectively by Stimson & Jeffery(1926), Batchelor(1976), and Vassure & Cox(1977) and Kaneda & Ishii(1982). Both of the later studies deal with the asymptotic behaviour of the hydrodynamic forces on two spheres in the limit of large seperation between their centers. These works are based on the use of matched asymptotic expansions in the small Reynolds number for the treatment of non-linear term in the Navier-Stokes equations. In the former case spherical particles locate in the outer region of expansion, whereas in that of later one(Kaneda & Ishii) they lie in the inner region of expansion of each other.

It appears from the above discussions that the Navier-Stokes equations has so far proved insoluable for the problem of flow past two spheres with small seperation. The motivation of this study is therefore, to investigate these solutions for an arbitrary seperation between the spheres' center. Since a general study of such kind is a time consuming process, the present paper is devoted to the particular case of equal sized spheres with the assumptions that i) the spheres lie within each other's inner region of expansion, ii) Reynolds number is small but non-zero and iii) the flow is axisymmetric.

In this regard, the expression to obtain the drag force on one of the spheres in a uniform flow is known(Kaneda & Ishii). Utilizing this expression for the case of axisymmetric flow and getting the integrals involved in it in bipolar co-ordinates, forces are computed. Comparison of the numerically determined forces with those reported previously for large seperation is found in good agreement. Results are discussed in detail for the small seperation. Successful application of the approach to equal sized spheres leads to further investigation for the case of unequal spheres and the other related one

1. Flow past two spheres

Consider the streaming flow of an incompressible fluid past two solid spheres A and B(radii a and b). The local fluid motion satisfies the Navier-Stokes and continuity equations

$$\mu\nabla'.\nabla'\mathbf{u}'-\nabla'p' = \rho\mathbf{u}'.\nabla'\mathbf{u}', \qquad \nabla'.\mathbf{u}' = 0 \qquad (1.1)$$

With the exception of constant parameters such as μ and ρ, primed symbols are dimensional and unprimed symbols are dimensionless. For streaming flow with velocity $\mathbf{V}$(of magnitude V) in the direction of $\mathbf{e}$(a unit vector), the boundary conditions are

$$\mathbf{u}'= 0 \qquad \text{on } |\mathbf{r}'|= a \text{ and } |\mathbf{r}'-\bar{\mathbf{r}}'|= b,$$
$$\mathbf{u}'\to -V\mathbf{e} \qquad \text{as } r'\to \infty, \qquad (1.2)$$

$\mathbf{r}'$ and $\bar{\mathbf{r}}'$ are the position vectors of the spheres A and B respectively, and $r'=|\mathbf{r}'|$.

In terms of dimensionless quantities the previous equations of motion and the boudary conditions become

$$\nabla^2\mathbf{u} - \nabla p = R\ \mathbf{u}.\nabla\mathbf{u}, \qquad \nabla.\mathbf{u} = 0 \quad (R = \rho av/\mu), \qquad (1.3)$$

$$\begin{cases} \mathbf{u} = 0 & \text{on } r = 1,\ |\mathbf{r}-\bar{\mathbf{r}}'|= \lambda. \\ \mathbf{u} \to -\mathbf{e} & \text{at infinity} \end{cases} \qquad (1.4)$$

In these equations the characteristic length and the velocity are taken respectively a(the radius of the sphere A) and V.

The solutions of (1.3) with (1.4) for small R(R<<1) can easily be obtained by the use of the matched asymptotic expansion in the small Reynolds number. In this procedure, the non-linear field $(\mathbf{u},p)$ is expanded in the inner and outer regions(Brenner & Cox 1963), resulting in $(\mathbf{u}_0,p_0)$ and $(\mathbf{u}_1,p_1)$, with $(\mathbf{u}_0,p_0)$ the solution of (1.3) with (1.4) for R=0 and that of $(\mathbf{u}_1,p_1)$ upto the term of O(R).

2. Drag force (general case)

It is clear from the above discussion that the non-linear force F^s on s(s is A or B) due to the field(u,p) may be expanded as

$$F^s = \frac{F^s}{6\pi\mu a V} [\ F_0^s + R\ F_1^s + \text{---}\ O(R)] \qquad (2.1),$$

where F_0 and F_1 are the forces(dimensionless) due to the fields (u_0, p_0) and (u_1, p_1) respectively. It is to be noted that F_0 can be obtained by the analysis based on the Stokes equations, where as the determination of F_1 creates difficulty due to the pressence of non-linear term in the Navier-Stokes equations even for small R.

Kaneda & Ishii(1982), gave a formula for F_1, using solely the Stokes fields. It is given in the following form(details are given in their paper) for the first-order force $F_1{}^A$ on one of the spheres A in the streaming flow.

$$6\pi (F_1{}^A)_i = I_1 - I_2 - I_3 + I, \qquad (2.2)$$

where

$$I_1 = \lim_{L \to \infty} \int_{S_L} (u^*)_i\ (\tau_1)_{i,j}\ dS_j = 0, \qquad (2.3)$$

$$I_2 = \lim_{L \to \infty} \int_{S_L} (u_1)_i (\tau^*)_{i,j} dS_j \qquad (2.4)$$

$$I_3 = \lim_{L \to \infty} \int_{S_L} (u_0)_i (u_0)_j (u^*)_i dS_j, \qquad (2.5)$$

$$I = \lim_{L \to \infty} \int_{V_L} (u_0)_i (u_0)_j (e^*)_{i,j} dV, \qquad (2.6)$$

where the integral is taken over the volume V_L bounded by the spherical surface S_L of radious L and internally by the surfaces S_A and S_B of the spheres A and B.

dS has the direction of the outer normal to V_L bounded by S_A, S_B. τ_1, τ^* are the stress tensors due to the fields(u_1, p_1) and (u^*, p^*). $(e^*)_{i,j} = 1/2[(u^*)_{i,j} + (u^*)_{j,i}]$, is the dimensionless rate-of-strain tensor for the flow $\mathbf{u}^*$.

Using the following procedure, I(2.6) can be written as

$$I = I_3 + I_4, \tag{2.7}$$

where I_3 is given in(2.5) and I4 is

$$I_4 = \lim_{L \to \infty} \int_{V_L} u_0 \times (\nabla \times u_0) . \mathbf{u}^* \, dV. \tag{2.8}$$

$$(u_0)_i (u_0)_j (e^*)_{ij} = [(u_0)_i (u_0)_j (u^*)_{i,j}]_{,j} - [(u_0)_i (u_0)_{j,i} u^*_j]$$

Integrate both sides over the volume V_L, one can get

$$I = I_3 - \lim_{L \to \infty} \int_{V_L} (u_0)_i (u_0)_{j,i} u^*_j \, dV.$$

Use of the formulae

$$\left| \begin{array}{l} (\mathbf{u}.\nabla)\mathbf{u} = 1/2 \ \nabla \mathbf{u}^2 - \mathbf{u} \mathrm{x} (\nabla \mathrm{x} \mathbf{u}) \\ \nabla \mathbf{u}^2 . \mathbf{u} = \mathbf{u}^2 \nabla . \mathbf{u} - \mathbf{u}^2 . \mathbf{u}^* \\ \nabla . \mathbf{u}^* = 0 \end{array} \right.$$

gives I in the form

$$I = I_3 - \frac{1}{2} \lim_{L \to \infty} \int_{S_L} (u_0)_i \ (u_0)_j dS_j + \lim_{L \to \infty} \int_{V_L} \mathbf{u}_0 \mathrm{x} (\nabla \mathrm{x} \mathbf{u}_0) . \mathbf{u}^* \, dV.$$

The second integral in the right hand side vanishes(see Banner & Cox 1963) and we got

$$I = I_3 + I_4 (2.7)$$

3. Drag force (particular case)

In the present study, the flow is assumed axisymmetric past two equal sized spheres. For the axisymmetric flow, equation (2.9) may be written in bipolar co-ordinates (see Happel & Brenner p. 516) as follows

$$(F_1^A)_z = - I_2 + I_4, \tag{3.1}$$

where

$$I_2 = 3/8 \, (F_0)_z \, (F^*)_z, \tag{3.2}$$

$$I_4 = \frac{1}{3} \int_\beta^\alpha \int_{-1}^{1} \frac{1}{r^2} E^2(\psi_0) \, J(\psi_0, \psi^*) \, d\mu d\zeta, \tag{3.3}$$

(for the derivation see appendix)

where

$E^2 = r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$ (in cylindrical co-ordinates),

$J(\psi_0, \psi^*) = \frac{\partial \psi_0}{\partial \zeta} \frac{\partial \psi^*}{\partial \mu} - \frac{\partial \psi_0}{\partial \mu} \frac{\partial \psi^*}{\partial \zeta}$,

$\alpha > 0$ and $\beta < 0$ (for equal sphere $\beta = -\alpha$),

ψ_0 and ψ^* are the Stokes stream functions.

The Stokes stream functions appearing in the integrand of I_4 correspond to the following equations and boundary conditions.

$$E^4(\psi_0) = 0,$$

$$\psi_0 = \frac{\partial \psi_0}{\partial n} = 0 \quad \text{on } \alpha \text{ and } \beta, \qquad \psi_0 = \frac{1}{2} r^2 \quad \text{at infinity} \tag{3.4}$$

It is convenient to write the stream function ψ_0 in the form

$$\psi_0 = \frac{1}{2} r^2 - \bar{\psi} \tag{3.5}$$

and to formulate the problem in terms of $\bar{\psi}$ rather than ψ_0. It is clear that $\bar{\psi}$ satisfies the same differential equation (3.4) as ψ_0.

In addition, it must give rise to a vanishing velocity at infinity, and fulfil the conditions

$$\bar{\psi} = \frac{1}{2} r^2, \qquad \frac{\partial \bar{\psi}}{\partial n} = r \frac{\partial r}{\partial n} \quad \text{on } \alpha \text{ and } \beta. \tag{3.6}$$

Other Stokes stream function ψ^* satisfying the same equation as in (3.4) but with the following boundary conditions.

$$\left| \begin{array}{ll} \psi^* = \frac{1}{2} r^2, \qquad \frac{\partial \psi^*}{\partial n} = r \frac{\partial r}{\partial n} & \text{on } \alpha, \\ \psi^* = \frac{\partial \psi^*}{\partial n} = 0 \quad \text{on } \beta & \\ \psi^* \to 0 \quad \text{at infinity.} & \end{array} \right. \tag{3.7}$$

and

The reason of using (3.5) is that $\bar{\psi}$ is known in its exact form(Stimson & Jeffery 1926) and gets the following form for the boundary conditions(3.6)

$$\bar{\psi} = (\cosh\zeta - \mu)^{-3/2} \, \Sigma \, U_n V_n , \tag{3.8}$$

$$U_n(\zeta) = A_n \cosh(n - \tfrac{1}{2})\zeta + B_n \sinh(n - \tfrac{1}{2})\zeta + C_n \cosh(n + \tfrac{3}{2})\zeta + D_n \sinh(n + \tfrac{3}{2})\zeta,$$

$$V_n(\mu) = P_{n-1}(\mu) - P_{n+1}(\mu),$$

where P_n is the Legender Polynomial.

For equal spheres the constants B_n and D_n which can be obtained with the use of (3.6), vanishes and the analytic expressions of A_n and C_n are available. A similar expression like(3.8) can be imagined but with the contants A^*_n, ..., D^*_n which can be obtained with the use of (3.7) also in analytic form for equal spheres. In actual calculations these constants are computed and are checked (for the case of $\bar{\psi}$) with the available exact expressions' values.

The other part of the integrand appearing in I_4(3.3) is also available(Stimson & Jeffery) in bipolar co-ordinates and is given by

$$\frac{1}{r^2} E^2(\psi_0) = \frac{(\cosh\zeta - \mu)^{5/2}}{c^4(1-\mu^2)} \left[V_n\left\{(U_n)_{\zeta\zeta} - \frac{2\sinh\zeta}{\cosh\zeta - \mu}(U_n)_\zeta + \frac{3(\cosh\zeta - 3\mu)}{4(\cosh\zeta-\mu)}U_n\right\} + (1-\mu^2)U_n\left\{(V_n)_{\mu\mu} + \frac{2}{\cosh\zeta-\mu}(V_n)_\mu\right\}\right], \tag{3.9}$$

where

$c > 0$ and $(\ )_{\zeta/\mu}$ are respectively the derivatives with respect to ζ and μ. Note that $E^2(\bar{\psi}) = E^2(\psi_0)$.

Note that I_2(3.2) contains $(F_0)_z$ and $(F^*)_z$ which can be obtained with the informations of ψ_0 and ψ^* respectively. For the case of equal spheres

$(F_0)_z = 2(F^*)_z$ = Known(Stimson & Jeffery p. 116(equation(37))

Hence the first-order force on one of the spheres A ie $(F_1{}^A)_z$ in the uniform flow can be obtained from (3.1) numerically. For the computation of I_4(3.3), a library program AQ2DS of Nagoya University has been used. Formula (3.1) gives the force on sphere A in the leading position. To get force on the other sphere B, it is convenient to imagine the sphere A in the trailing position. It can be done if the direction of the uniform flow is reversed. Note that reversal of the uniform flow does not alter the value of I_4. With this procedure, drag forces on the sphere A are computed for the leading and for the trailing positions, for the arbitrary seperation between the spheres' centrs d. For the case of equal sized spheres difference between them gives the hydrodynamic interaction.

Appendix. Derivation of (3.3)

Consider the streaming flow past a body of revolutiion, parallel to its symmetry axis. Such motions which are termed as axially symmetric are characterized by the existence of a stream function. For such motioms it is possible to express the velocity field in terms of stream function. Further for streaming flow past spherical shaped objects, it is convenient to use cylindrical co-ordinates (r,z,ϕ). Let ψ_0 and ψ^* be the Stokes stream functions defined by the relations

$$\mathbf{u}_0 = \frac{1}{r}\,\mathbf{e}_\phi \times \nabla\psi_0,$$
$$\mathbf{u}^* = \frac{1}{r}\,\mathbf{e}_\phi \times \nabla\psi^*, \qquad \text{(I)}$$

where $\nabla = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\phi\,\frac{1}{r}\,\frac{\partial}{\partial\phi} + \mathbf{e}_z\,\frac{\partial}{\partial z}$,

and (e_r, e_z, e_ϕ) are the unit vectors in the directions of the cylindrical co-ordinates axis.

If the expressions

$$\nabla \times \mathbf{u}_0 = \frac{\mathbf{e}_\phi}{r}E^2\psi_0,$$

$$\mathbf{u}_0 \times(\nabla \times \mathbf{u}_0) = (\mathbf{u}_0.\mathbf{u}_0)\nabla - (\nabla \times \mathbf{u}_0)\mathbf{u}_0$$

are used then the integrarnd of I_4(2.8) may be written as

$$\mathbf{u}_0 \times (\nabla \times \mathbf{u}_0) = \frac{1}{r^2}\,E^2(\Psi_0)\,[\nabla\Psi_0 - (\mathbf{e}_\phi.\nabla\Psi_0)\mathbf{e}_\phi]. \qquad \text{(II)}$$

Taking the dot product in (II) both sides with respect to $\mathbf{u}^*$, the integral I_4 may be obtained in cylindrical co-ordinates as

$$I_4 = \int_{V_1} \frac{1}{r^3}\,E^2(\psi_0)\,\frac{\partial(\psi_0,\psi^*)}{\partial(r,z)}\,dV_1 \qquad \text{(III)}$$

Since in the cylindrical co-ordinates

$$dV_1 = r\, d\phi\, dr\, dz,$$

the integral in (III) after evaluating with respect to ϕ is given by

$$I_4 = 2\pi \int_{-\alpha}^{\infty} \int_0^{\alpha} \frac{1}{r^2} E^2 J(\psi_0, \psi^*)\, drdz, \qquad \text{(IV)}$$

where

$$J(\psi_0, \psi^*) = \frac{\partial \psi_0}{\partial r} \frac{\partial \psi^*}{\partial z} - \frac{\partial \psi_0}{\partial z} \frac{\partial \psi^*}{\partial r} \,.$$

For the problem of two spheres we take ζ, η curvillinear co-ordinates in the meridian plane(ϕ=constt), where

$$\zeta + i\eta = \log \frac{r + iz}{r - iz} , \qquad \text{(V)}$$

so that,

$$r = \frac{c \sin\eta}{\cosh\zeta - \cos\eta} , \qquad z = \frac{c \sinh\zeta}{\cosh\zeta - \cos\eta} . \qquad (c>0) \qquad \text{(VI)}$$

With these transformations, two spheres external to each other can be defined by $\zeta = \alpha$, $\zeta = \beta$ such that $\alpha>0$, $\beta<0$. For the case of equal spheres $\beta=-\alpha$ and the radius of each sphere and the sepera-tion d between their centers can respectively be obtained by the relations

$$a = c \operatorname{cosech}(\alpha) \quad , d = 2\, c \coth(\alpha) . \qquad \text{(VII)}$$

In these co-ordinates the expression of I_4(IV) may be given by

$$I_4 = 6\pi \int_{\beta}^{\alpha} \int_{-1}^{1} \frac{1}{r^2} E^2 \frac{\partial(\psi_0, \psi^*)}{\partial(\zeta , \mu)}\, d\mu\, d\zeta, \qquad \text{(VIII)}$$

$(\cos\eta = \mu, \ \beta=-\alpha)$

4. Summary of results

In order to understand the role played by the hydrodynamic interaction between two equal sized solid spheres in a uniform flow, the drag forces on one of the spheres in the leading and in the trailing positions are computed using (3.1) for the seperation d and radius a ratio(VII). For equal spheres, I_2(3.2) is available in exact form and the integral I_4(3.3) is evaluated (within an accuracy of 10^{-3}) numerically for $\beta=-\alpha$. These results are presented in Fig.1, which contain the values of the first-order force F_1(dimensionless) I(for leading position), III(for trailing position) and II=I-III(difference between the leading and trailing positions) verses seperation-radius ratio. d_1, d_2, d_3 and d_4 are the regions of seperation(in the increasing order), and the stage d/a=2 is the contact of spheres' surface. The behaviour of the forces in these regions, which give an idea of interaction between them are summerized as follows.

The leading sphere is always experienced a larger force than that of the trailing one even near the contact. Since the computation can not be done at the contact due to the singular behaviour of the integrand of I_4, the behaviour is still unknown there. But roughly, the leading sphere seems to attain a constant value of drag, whereas the drag on trailing sphere diminishes, giving a certain constant value of interaction at the contact.

In the region d_1, increase in the leading force is slower than that on the trailing one, which appears in the decrease of attraction. Later on in d_2, both the forces increase linearly such that the interaction remains constant. This region can be considered as an intermediate stage, since it distinguishes the behaviour between the small and comperatively large seperations. The region d_3 is characterized by the opposite behaviour as is found in the region d_1. Leading force increase faster than trailing and consequently the interaction gets weaker. In this region the coincedance of the present numerically determined values to those reported by Kaneda and Ishii(1982) start, which agreed well in the region d_4. Further, for the values greater than d_4, they seem to attain the constants(in the right hand side of the vertical axis Fig. 1) predicted by Kaneda & Ishii for higher values of seperation.

In the present study, though the attention has been paid for the equal sized spheres, the same analysis is also applicable to the case of unequal spheres. For such case, in addition to I_4(3.3), I_2(3.2) is required to compute numerically since the later is not available in exact form.

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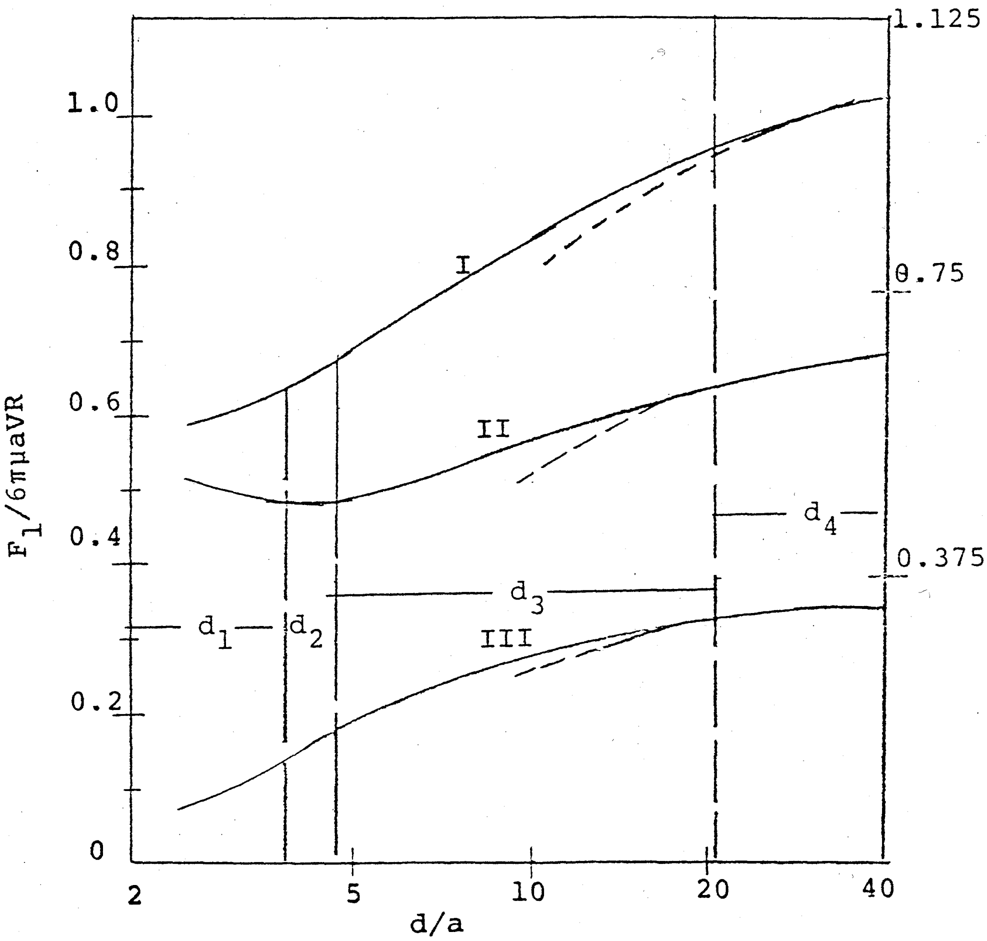

Fig.1 First-order force verses seperation

I: Leading position

III: Trailing position

II: Difference

(— — —): Asymptotic behaviour by Kaneda & Ishii