A decision method for a set of first order classical formulas and its application to decision problems for non-classical propositional logics

MOTOHASHI, Nobuyoshi

数理解析研究所講究録 (1984), 540: 80-84

1984-10

http://hdl.handle.net/2433/98745

Departmental Bulletin Paper

Kyoto University
A decision method for a set of first order classical formulas and its application to decision problems for non-classical propositional logics.

Nobuyoshi MOTOHASHI

I. Main Theorem

Let LN be the first order classical predicate logic without equality which has a fixed binary predicate symbol R, unary predicate symbols P1, ..., PN and no other non-logical constant symbols. Suppose that X is a set of sentences in LN. Then a decision method for X is a method by which, given a sentence A in X, we can decide in a finite number of steps whether or not it has a model. X is said to be decidable if there is a decision method for X. It is well-known that the set of all the R-free sentences (sentences in LN which have no occurrences of R) is decidable, but the set of all the sentences in LN is not. R-formulas are formulas belonging to the least set such that: (i) R-free formulas belong to X, (ii) X is closed under ¬, ∧, ∨, ⊃, (iii) If A(x) belongs to X, then ∃vA(v), ∃v(R(x,v)∧A(v)), ∃v(R(v,x)∧A(v)) belong to X. R-positive formulas are formulas which have no negative occurrences of R. Also, Tr is the sentence ∀u∀v∀w(R(u,v)∧R(v,w), ⊃ R(u,w)) and Sym is the sentence ∀u∀v(R(u,v)⊃R(v,u)). Let FN be the set of finite conjunctions of sentences: R-sentences, R-positive
sentences, Tr and Sym. Then, our main theorem is:

**MAIN THEOREM.** FN is decidable.

In fact, we show that for each sentence $A$ in FN, we can calculate a natural number $n(A)$. This fact clearly implies our main theorem, such that if $A$ has a model, then $A$ has a model whose cardinality is at most $n(A)$.

II. Applications.

Suppose that $L$ is a formal logic. Then a decision method for $L$ is a method by which, given a formula of $L$, we can decide in a finite number of steps whether or not it is provable in $L$.

1) Intuitionistic propositional logic.

Let IPL be the intuitionistic propositional logic whose propositional variables are $pl, \ldots, pN$. For each formula $A$ in IPL, and each free variable $x$ in LN, let $(A, x)$ be the formula in LN defined by:

$(pi, x)$ is $Pi(x)$, $(-A, x)$ is $\forall v(R(x, v) \supset \neg (A, v))$, $(A \land B, x)$ is $(A, x) \land (B, x)$, $(A \lor B, x)$ is $(A, x) \lor (B, x)$, and, $(A \supset B, x)$ is $\forall v(R(x, v) \supset ((A, v) \supset (B, v)))$.

Then, Kripke’s completeness theorem for IPL, we have:

Completeness Theorem for IPL. For each formula $A$ in IPL, $A$ is provable in IPL iff the sentence $Tr_{i_n} \land \land_{i_n} Tr(Pi) \land \exists v \neg (A, v)$ has no models, where $Tr(Pi)$ is the $R$-sentence $\forall u(Pi(u) \supset \forall v(R(u, v) \supset P(v)))$. 
Since $\forall u R(u,u) \land \forall v \neg A, v$ belongs to FN, our main theorem clearly implies that the logic IPL is decidable.

2) Modal propositional logics.

Let MPL be the modal propositional language whose logical constants are $\neg, \land, \lor, \supset$ and $\Box$, and whose propositional variables are $p_1, \ldots, p_n$. For each formula $A$ in MPL and each free variable $x$ in LN, let $<A, x>$ be the formula in LN defined by: $<p_i, x>$ is $p_i(x)$, $<\neg A, x>$ is $\neg <A, x>$, $<A \land B, x>$ is $<A, x> \land <B, x>$, $<A \lor B, x>$ is $<A, x> \lor <B, x>$, $<A \supset B, x>$ is $<A, x> \supset <B, x>$ and $<\Box A, x>$ is $\forall v (R(x,v) \supset <A, v>)$.

Let $M$, $S_4$, $B$, $S_5$ be modal propositional logics in Kripke [ ], whose language is MPL. Then, by Kripke’s completeness theorem for modal logics, we have:

**Completeness Theorem for modal logics.** For any formula $A$ in MPL,

(i) $A$ is provable in $M$ iff $\forall u R(u,u) \land \exists v \neg A, v$ has no models,

(ii) $A$ is provable in $S_4$ iff $\forall u R(u,u) \land \exists v \neg A, v$ has no models,

(iii) $A$ is provable in $B$ iff $\forall u R(u,u) \land \forall v \neg A, v$ has no models,

(iv) $A$ is provable in $S_5$ iff $\forall u R(u,u) \land \exists v \neg A, v$ has no models.

Since finite conjunctions of sentences $\forall u R(u,u)$, $\forall v \neg A, v$ belong to FN, our main theorem clearly implies that four logics $M$, $S_4$, $B$, $S_5$ are all decidable.
III. A proof.

1) R-degree. For each R-formula $A$, let $R$-deg($A$) be the non-negative integer, called the R-degree of $A$, defined by: $R$-deg($A$) = 0 if $A$ is R-free, $R$-deg($\neg A$) = $R$-deg($A$), $R$-deg($A \land B$) = $R$-deg($A \lor B$) = $R$-deg($A \rightarrow B$) = max ($R$-deg($A$), $R$-deg($B$)), $R$-deg($\exists vA(v)$) = $R$-deg($A(x)$), and $R$-deg($\exists v(R(x,v) \land A(v))$) = $R$-deg($\exists v(R(v,x) \land A(v))$) = $R$-deg($A(x)$) + 1.

2) R-basic sentences.

Define $\Sigma n(n = 0,1,2,...)$ and $\Sigma$ by: $\Sigma 0 = \text{Pow}(\{1,...,N\})$

$\Sigma(n+1) = \Sigma n \times \text{Pow}(\Sigma n) \times \text{Pow}(\Sigma n), (n = 0,1,2,...)$ and $\Sigma = \bigcup_{n<\omega} \Sigma n$,

where $\text{Pow}(Z)$ is the power set of $Z$.

For each $\sigma$ in $\Sigma$, let $A(\sigma,x)$ be the unary formula defined by:

$A(\sigma,x)$ is $\bigwedge \Pi(\sigma) \land \bigwedge \Pi(x)$ if $\sigma \in \Sigma 0$ and $i \in \sigma$ $i \notin \sigma$.

$A(\sigma,x)$ is $A(\nu,x) \land \exists v(R(v,x) \land A(\alpha,v)) \land \forall \nu \in \alpha \land \exists v(R(v,x) \land A(\alpha,v)) \land \exists v(R(v,x) \land A(\alpha,v))$ if $\sigma =< \nu,1,1> \in \Sigma(n+1)$.

Then, $A(\sigma,x)$ is an R-formula whose R-degree is $n$ if $\sigma \in \Sigma n$.

For each subset $X$ of $\Sigma n$, let $AX$ be the sentence:

$\bigwedge \exists vA(\sigma,v) \land \bigwedge \neg \exists vA(\sigma,v)$ if $\sigma \in X$ $\sigma \notin X$.

$AX (X \subseteq \Sigma n)$ are called R-basic sentences of R-degree $n$. 
3) Representation theorem.

(1) For each $R$-formula $A(x,\ldots,y)$ of $R$-degree $n$, whose free variables are among $x,\ldots,y$, we can concretely construct a Boolean combination $B(x,\ldots,y)$ of formulas of the forms: $\exists v A(\sigma,v), A(\sigma,x), A(\sigma,y)$, where $\sigma \in \Sigma_n$ such that $A$ and $B$ are equivalent in LN.

(2) For each $R$-sentence $A$ of $R$-degree $n$, we can concretely obtain finite subsets $X_1,\ldots,X_n$ of $\Sigma_n$ such that $A X_1 \lor \cdots \lor A X_n$ and $A$ are equivalent in LN.

4) Reduction lemmas.

Let $GN$ ($HN$) be the set of sentences in $FN$ which are finite conjunctions of the sentences: $R$-basic sentence ($R$-basic sentences of $R$-degree 1), $R$-positive sentences, $Tr$ and $Sym$. Then $HN \subseteq GN \subseteq FN$.

Reduction Lemma 1. If $GN$ is decidable, then $FN$ is decidable.

Reduction Lemma 2. If $HN$ ($N = 1,2,\ldots$) are all decidable, then $GN$ ($N = 1,2,\ldots$) are all decidable.

5) Main Lemma. For each sentence $A$ in $HN$, if $A$ has a model, then $A$ has a model of cardinality no more than $2^\kappa \times 2^{2^\kappa} \times 2^{2^\kappa}$.

Clearly, Reduction Lemma 1, Reduction lemma 2 and Main Lemma imply our main theorem.