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<td>Author(s)</td>
<td>YASUDA, Yutaka</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 540: 8-59</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1984-10</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/98747">http://hdl.handle.net/2433/98747</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
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情報システム部（リポジトリ）
Structure theory for $\Pi^1_n$ sets in the plane with countable sections

By

Yutaka YASUDA

Abstract

We develop structure theory for $\Pi^1_n$ sets in $\omega \times \omega$ with countable sections under the assumption of the projective determinacy when it is needed. It is shown that several theorems of Luzin about Borel and analytic sets cannot be extended to $\Pi^1_n$ sets. We also generalize a Friedman's theorem to $\Pi^1_{2n+1}$ sets.
§0. Introduction.

Tanaka [39] extended several theorems of Lusin [24] about Borel and analytic sets in $\omega_\omega \times \omega_\omega$ with countable sections to $\Delta^1_{\omega_\omega \times \omega}$ and announced these also fail for $\Pi^1_{\omega_\omega \times \omega}$ sets. Our main theme is to complete structure theory for projective sets in $\omega_\omega \times \omega_\omega$ with countable sections under the assumptions of projective determinacy.

To this we prove that, among other things, every $\Sigma^1_{\Sigma^1_{\omega_\omega \times \omega}}$ set in $\omega_\omega \times \omega_\omega$ with countable sections can be uniformized by the difference of two $\Sigma^1_{\Sigma^1_{\omega_\omega \times \omega}}$ sets, and there is a $\Pi^1_{\Pi^1_{\omega_\omega \times \omega}}$ set in $\omega_\omega \times \omega_\omega$ with countable sections which cannot be covered by either countably many $\Sigma^1_{\Sigma^1_{\omega_\omega \times \omega}}$ or $\Pi^1_{\Pi^1_{\omega_\omega \times \omega}}$ curves. We also generalize a Friedman's theorem as follows: There is an infinitely countable $\Pi^1_{\Pi^1_{\omega_\omega \times \omega}}$ set of reals every member of which except one is $\Delta^1_{\Delta^1_{\omega_\omega \times \omega}}$ real. We present several applications of these results. Our proof methods are parametrization of $\Delta^1_{\Delta^1_{\omega_\omega \times \omega}}$ reals by integers, uniformization theorem and higher-level analogs of Gödel's $L$ which are all consequences of projective determinacy.

Acknowledgments. I wish to very thanks my teacher Professor M. Konô for his guidance and encouragement and most of all for creating our interest in descriptive set theory. I am also grateful to Professors J. W. Addison, J. Burgess, L. Harrington, D. A. Martin, Y. N. Moschovakis, Y. Sampei, J. R. Steel, H. Tanaka and T. Tugué for numerous helpful discussion on this subject. I especially wish to express my hearty thanks to Professor H. Tanaka for his constant
encouragement and very helpful suggestions in the preparation of this paper. Also I am grateful to Professor M. Davis and Courant Institute of Mathematical Sciences for hospitality during the months when parts of this paper was written.

§1. Preliminaries.

We use in this paper standard terminology and notation in descriptive set theory, following in most instances that of Moschovakis' monograph [27]. Our basic spaces will be $\omega$, $\omega_1$ (the reals, denoted by $\mathbb{R}$ in [27]) and $\omega_2$. (Product) spaces are of the form

$$X = X_1 \times \cdots \times X_k,$$

where $X_i$, $1 \leq i \leq k$, is a basic space, members of these spaces are called points and subsets of them pointsets or simply sets. Sometimes we think of them as predicates on the space $X$ and write interchangeably for each $x \in X$

$$x \in P \iff P(x).$$

A pointclass is a collection of pointsets, usually in all product spaces.

We will adhere to the following notational conventions throughout this paper. Letters $e, i, j, k, l, m, n$ denote always members of $\omega$, $d, \beta, \gamma, \delta$ members of $\omega_1$.

If $\Gamma$ is a pointclass, we put

$$\check{\Gamma} = \{ \chi - P : \text{for some } \chi \text{ and } P \in \Gamma, P \subseteq \chi \},$$

call it the dual of $\Gamma$,

$$(\Gamma)_P = \{ P \cap Q : \text{for some } \chi, P, Q, P \in \Gamma, Q \in \check{\Gamma}, P, Q \subseteq \chi \},$$
call it the \textit{difference} of two $\Gamma$ pointsets, for each $\alpha$

$$\Gamma(\alpha) = \{ P : \text{for some } \mathcal{X} \text{ and } Q \in \Gamma, \ P(x) \leftrightarrow Q(\alpha, x) \}$$

call it the \textit{relativization} of $\Gamma$ to $\alpha$, and for each product space $\mathcal{X}$

$$\Gamma \upharpoonright \mathcal{X} = \{ P \subseteq \mathcal{X} : P \in \Gamma \}$$

After Kondō [20] for each $x \in \mathcal{X}$ and $P \subseteq \mathcal{X} \times \mathcal{Y}$

$$P^{\langle x \rangle} = \{ y \in \mathcal{Y} : P(x, y) \}$$

and call it the \textit{section} of $P$ at $x$. We also call a partial function $\Phi : \mathcal{X} \rightarrow \mathcal{Y}$ a \textit{curve} and identify $\Phi$ with its graph $\{ (x, y) : \Phi(x) = y \}$.

We shall be concerned in this paper the \textit{projective} pointclasses:

$$\Sigma^0_1 = \text{all open pointsets},$$

$$\Pi^0_1 = \text{all closed pointsets},$$

$$\Sigma^1_1 = \{ P \subseteq \mathcal{X} : \text{for some product space } \mathcal{X} \text{ and some }$$

$$\Pi^0_1 \text{ pointset } Q \subseteq \omega_1 \times \mathcal{X}, \ P(x) \leftrightarrow \exists \alpha Q(\alpha, x) \}$$

In classical terminology these are the \textit{analytic} sets or $\Delta$-sets. Then we let $\Pi^1_1$ be the pointclass of all complements of $\Sigma^1_1$ sets, i.e.

$$\Pi^1_1 = \Sigma^1_1$$

Classicaly again these are known as the \textit{coanalytic} sets or $\text{CA}$-sets.

Then we let

$$\Sigma^1_2 = \{ P : \text{for some product space } \mathcal{X} \text{ and some } \Pi^1_1$$

$$\text{pointset } Q \subseteq \omega_1 \times \mathcal{X}, \ P(x) \leftrightarrow \exists \alpha Q(\alpha, x) \}$$

(the classical $\text{PCA}$-set), and

$$\Pi^1_2 = \Sigma^1_2$$

(the classical $\text{CPCA}$-set),
and in general inductively

\[ \Sigma_{n+1}^1 = \{ P : \text{for some product space } \mathcal{X} \text{ and some } \Pi_n^1 \text{ pointset } Q \subseteq \omega_\omega \times \mathcal{X}, P(x) \iff \exists \alpha Q(\alpha, x) \} , \]

\[ \Pi_{n+1}^1 = \Sigma_{n+1}^1. \]

We also define the ambiguous pointclass

\[ \Delta_n^1 = \Sigma_n^1 \cap \Pi_n^1. \]

The class of projective sets is the union

\[ \bigcup_{n=0}^{\infty} \Sigma_n^1. \]

It only remain to clarify the relationship between the Borel set and \( \Delta_1^1 \). According to a excellent theorem of Souslin [34]

The Borel sets = \( \Delta_1^1 \)

i.e. the Borel sets coincide with the analytic-coanalytic ones.

We shall also be concerned the analytical pointclasses:

\[ \Sigma_1^0 = \text{all recursively enumerable pointsets}, \]

\[ \Pi_1^0 = \Sigma_1^0, \]

\[ \Sigma_1^1 = \{ P : \text{for some } \mathcal{X} \text{ and } \Sigma_1^0 \text{ set } Q \subseteq \omega_\omega \times \mathcal{X}, P(x) \iff \exists \alpha Q(\alpha, x) \}, \]

\[ \Pi_1^1 = \Sigma_1^1, \]

\[ \Sigma_{n+1}^1 = \{ P : \text{for some } \mathcal{X} \text{ and } \Pi_n^1 \text{ set } Q \subseteq \omega_\omega \times \mathcal{X}, P(x) \iff \exists \alpha Q(\alpha, x) \}, \]

\[ \Pi_{n+1}^1 = \Sigma_{n+1}^1, \]

\[ \Delta_n^1 = \Sigma_n^1 \cap \Pi_n^1. \]

The class of analytical sets is the union
\[ \bigcup_{n=0}^{\infty} \Sigma_n^1. \]

One can also define the effective analogs of the Borel sets, which are known as hyperarithmetic (HYP) sets. Kleene's theorem [16] (the effective analog of Souslin's theorem) asserts that

\[ \text{HYP} = \Delta^1_1. \]

What is the interrelationship between the classical and the effective notions? The key lies in the concept of relativization, introduced in Kleene [16].

We proceed exactly as before to define, for each real \( \alpha \), \( \Sigma^0_1(\alpha) \), \( \Pi^0_1(\alpha) \), \( \Sigma^1_1(\alpha) \), etc.

Here is now the precise relationship between the classical and the (relativized) effective notions:

\[
\begin{align*}
\Sigma^0_1 &= \bigcup_{\alpha \in \omega} \Sigma^0_1(\alpha), \\
\Pi^0_1 &= \bigcup_{\alpha \in \omega} \Pi^0_1(\alpha), \\
\Sigma^1_1 &= \bigcup_{\alpha \in \omega} \Sigma^1_1(\alpha), \\
\Pi^1_1 &= \bigcup_{\alpha \in \omega} \Pi^1_1(\alpha), \text{ etc.}
\end{align*}
\]

Also for a function \( f : \mathcal{X} \to \mathcal{Y} \), \( f \) is continuous if and only if \( f \) is recursive in some real \( \alpha \). Thus the effective notions are refinements of the classical ones. Note also the following facts. For each \( n \geq 1 \) there is a \( \Sigma^1_n \) set \( G \) in \( \omega_\omega \times \mathcal{X} \) which is universal for the \( \Sigma^1_n \) sets in \( \mathcal{X} \), i.e. a set \( P \subseteq \mathcal{X} \) is \( \Sigma^1_n \) if and only if for some real \( \alpha \),

\[ P = g^{\langle \alpha \rangle}. \]

Similarly for \( \Sigma^0_1, \Pi^0_1, \Sigma^1_n \). Thus e.g. the \( \Sigma^1_1 \) sets are just the
sections of the $\Sigma^1_1$ set (Addison [1] and Tugó).

Almost any proof in effective descriptive set theory involving the absolute notions: recursive, $\Sigma^0_1$, $\Pi^0_1$, etc, relativizes immediately to an arbitrary parameter $\alpha$, by just "plugging-in" the parameter at all appropriate places in the proof and so yields its relativized version. In view of the previous explained relationship between the relativized concepts and the classical ones, this makes clear that the effective result immediately implies its classical version. In that sense effective descriptive set theory is a strengthening of the classical one. And usually the methods of the effective theory allow for much simpler and elegant proofs.

Many times now the formulation and proof of an effective result involve concepts which are only meaningful in the effective theory, but nevertheless throw a lot of light to a related classical result.

For example, the classical perfect set theorem for $\Sigma^1_1$ sets asserts that a $\Sigma^1_2$ set which contains no nonempty perfect subset must be countable. Let us say that a real $\beta$ is $\Delta^1_1(\alpha)$ if and only if its graph $\{(i, j) : \beta(i) = j\}$ is in $\Delta^1_1(\alpha)$. Denote by $\mathcal{J}^1_1(\alpha)$ the set of all $\Delta^1_1(\alpha)$ reals. Clearly $\mathcal{J}^1_1(\alpha)$ is countable. We have now the following basic result of effective descriptive set theory.

**Theorem** (Harrison [11]). Let $A \subseteq \omega^\omega$ be $\Sigma^1_1(\alpha)$. If $A$ contains no nonempty perfect set, then $A \subseteq \mathcal{J}^1_1(\alpha)$.

Thus not only we know that every $\Sigma^1_1(\alpha)$ set with no nonempty perfect subset is countable, but we know what kind of members it contains, namely only $\mathcal{J}^1_1(\alpha)$ ones. Put in another way, we have a fixed countable
set \( \Sigma^1_1(\alpha) \) such that a necessary and sufficient condition for \( A \in \Sigma^1_1(\alpha) \) to contain a nonempty perfect subset is to contain at least one element not in it. So the fact that effective descriptive set theory can develop a concept of classification of "complexity" of individual reals (as for example being \( \Delta^0_1, \Delta^1_1, \Delta^1_2, \ldots \)) serves to clarify considerably a classical situation. (Note that the concept of a real being \( \Delta^0_1, \Delta^1_1, \Delta^1_2, \ldots \) is trivial; every real is such.)

Finally, and very importantly, effective set theory provides powerful methods for the solution of problems of undoubtedly classical character and contents, for which no classical type proofs are known at present. A such example is Steel and Martin negative solution of one of Luzin's uniformization problems [24]: That is to say there is a \( \Sigma^1_1 \) set in \( \omega \times \omega \) which cannot be uniformized by the difference of two \( \Sigma^1_1 \) sets (see Moschovakis [27; 4F.21]). Also this is an example of strongest negative solution for classical type problems, since counter example is a light face \( \Sigma^1_1 \) set.

As Luzin [25] predicted, the classical methods of descriptive set theory are not successful in solving non-trivial problems concerning projective sets for levels beginning with the third, and sometimes for the second and even the first level. Powerful as they are, the methods from logic and recursion theory cannot solve this "difficulties of the theory of projective sets", since they too are restricted by the limitations of Zermelo-Fraenkel set theory. (see Cohen [4], Gödel [7], Harrington [9, 10], Lévy [23]).
Since properties of definable sets can usually be established effectively, without use of the full axiom of choice AC, we shall work in set theory without the full axiom of choice. However, we shall assume a weak form of the axiom of choice. The reason is that in descriptive set theory one frequently considers unions and intersections of "countably many countable sets is countable". Thus we shall work, throughout this paper, in set theory ZF + DC, where DC is the axiom of dependent choices:

Axiom of dependent choices (DC). For every set of pairs \( P \subseteq A \times A \) from nonempty set \( A \),

\[ \forall x \in A \ \exists y \in A \ P(x, y) \Rightarrow \exists f : \omega \to A \forall n \ P(f(n), f(n+1)). \]

Recall some consequences of DC:

(1) The countable axiom of choice.

(2) Every infinite set has a countable subset.

(3) The union of countably many countable sets is countable.

(4) A binary relation without infinite descending chains is wellfounded.

The full axiom of choice implies DC easily and Kechris [16] has shown that DC is consistent with PD.
For any pointclass $\Gamma$, 
\[ \text{Det}(\Gamma) \]
is an abbreviation for the assertion that all games in $\Gamma$ are determined. In particular,
\[ \text{PD} \iff \bigvee_{n=0}^{\infty} \text{Det}(\Sigma^1_n) \]
is the projective determinacy.

As stated in introduction, this paper is a sequel to well-known classic book Luzin [25]. Its main purpose is to show how game theoretic hypotheses, definable determinacy, can be used to natural extension of Luzin’s theory of structural properties about Borel and analytic sets in $\omega_1 \times \omega_0$ with countable sections to higher level projective sets.

We assume from now on for the rest of this section $\text{Det}(\Delta^1_{2n})$.

(For $n = 0$, $\Sigma^1_0 = \Sigma^0_1$, $\Pi^1_0 = \Pi^0_1$, $\Delta^1_0 = \Delta^0_1$ so $\text{Det}(\Delta^1_0)$ is just clopen determinacy which is provable in ZF + DC, thus no strong hypothesis is being made in this case.)

Let $\Gamma$ be a pointclass, $A$ a pointset. A norm on $A$ is a map $\varphi : A \to \kappa$ from $A$ onto an ordinal $\kappa$. We call $\varphi$ a $\Gamma$-norm if the two relations below
\[ x \leq_{\varphi} y \iff x \in A \quad \& \quad (\varphi(x) \leq \varphi(y)), \]
\[ x <_{\varphi} y \iff x \in A \quad \& \quad (\varphi(x) < \varphi(y)) \]
are in $\Gamma$ where we put $\varphi(y) = \text{an ordinal bigger than } \sup \{ \varphi(x) : x \in A \}$, for all $y \notin A$. Finally we say that $\Gamma$ is normed if and only if every pointsets in $\Gamma$ admits a $\Gamma$-norm.

The pointclasses $\Pi^1_{2n+1}$, $\Sigma^1_{2n+2}$ are normed (First Periodicity Theorem; Moschovakis [27; 6B. 1]). Some corollaries of this fact are the following:
(1) $\Pi^1_{2n+1}$ ($\Sigma^1_{2n+2}$) satisfy reduction and $\Sigma^1_{2n+1}$ ($\Pi^1_{2n+2}$) satisfy separation (Moschovakis [29; 43.10-11]).

(2) The number uniformization theorem for $\Pi^1_{2n+1}$, i.e. if $P(x, n)$ is $\Pi^1_{2n+1}$ there is $P^*(x, n)$ in $\Pi^1_{2n+1}$ such that

$$P^* \subseteq P$$

and

$$\exists n P(x, n) \iff \exists! n P^*(x, n)$$

(Moschovakis [29; 43.4]).

(3) Let $\mathcal{D}^1_{2n+1}$ denote the set of $\Delta^1_{2n+1}$ reals and $\mathcal{D}^1_{2n+1}(\alpha)$ the relativized notion, i.e.

$$\beta \in \mathcal{D}^1_{2n+1}(\alpha) \iff \beta \in \Delta^1_{2n+1}(\alpha).$$

Then there are partial functions $d: \omega \times \omega \times \omega \rightarrow \omega$ and $\zeta: \omega \times \omega \rightarrow \omega$ with $\Pi^1_{2n+1}$ graphs such that

$$\beta \in \mathcal{D}^1_{2n+1}(\alpha) \iff \exists e \forall i (\beta(i) = d(e, \alpha, i)),$$

$$\beta, \alpha \in \text{dom}(\zeta) \iff \beta \in \mathcal{D}^1_{2n+1}(\alpha),$$

for $\beta \in \mathcal{D}^1_{2n+1}(\alpha)$

$$\forall i (\zeta(\beta(\alpha), \alpha, i) = \beta(i)),$$

and the relations "$\beta(i) = d(e, \alpha, i)$" and "$e = \zeta(\beta, \alpha)$" are $\Delta^1_{2n+1}$, uniformly on $(e, \alpha, i) \in \text{dom}(d)$ and $(\beta, \alpha) \in \text{dom}(\zeta)$ respectively, i.e., for example, there are $Q, R$ in $\Sigma^1_{2n+1}$, $\Pi^1_{2n+1}$ respectively, such that for $(e, \alpha, i) \in \text{dom}(d)$

$$\beta(i) = d(e, \alpha, i) \iff Q(e, \alpha, i, \beta) \iff R(e, \alpha, i, \beta)$$

-1-
The bounded quantification theorem for \( \prod_{2n+1}^{1} \), i.e. for each \( P(\alpha, \beta, x) \) in \( \prod_{2n+1}^{1} \) the pointset

\[ R(\beta, x) \iff \exists \alpha \in \prod_{2n+1}^{1}(\beta) \ P(\alpha, \beta, x) \]

is also in \( \prod_{2n+1}^{1} \) (Moschovakis [29; 4D.3 and 6B.2]). In particular the relation

\[ \beta \in \Delta_{2n+1}^{1}(\alpha) \]

is \( \prod_{2n+1}^{1} \).

Harrington [10] has shown that "First Periodicity Theorem" is consistent with \( ZF + DC_{\kappa}^{+} \) so its all consequences, e.g. (1)-(4), are consistent with \( ZF + DC + I \), where \( I \) is the statement which say there is an inaccessible cardinal.

Again let \( \Gamma \) be a pointclass and \( A \) a pointset. A scale on \( A \) is a sequence \( \vec{\phi} = \{\phi_{n}\} \) of norms on \( A \) such that

(i) If \( x_{i} \in A, \ i = 0, 1, ... \) and \( x_{i} \rightarrow x \)

and

(ii) For each \( n, \) and for all large enough \( i \)

\[ \phi_{n}(x_{i}) = \text{constant} = \lambda_{n} \]

then \( x \in A \) and \( \phi_{n}(x) \leq \lambda_{n} \). We call \( \{\phi_{n}\} \) a \( \Gamma \)-scale if the pointsets

\[ R(n, x, y) \iff x \leq^{*} y, \]

\[ S(n, x, y) \iff x <^{*} y \]

are in \( \Gamma \). We say that \( \Gamma \) is scaled if every \( A \in \Gamma \) admits a \( \Gamma \)-scale.

-/-2-
The pointclasses $\Pi^1_{2n+1}$, $\Sigma^1_{2n+2}$ are scaled (Second Periodicity Theorem; Moschovakis [29; 6C.3]). Some corollaries of this fact are the following:

1. The uniformization theorem for $\Pi^1_{2n+1}$, i.e. if $P(x, y)$ is $\Pi^1_{2n+1}$ there is $P^*(x, y)$ in $\Pi^1_{2n+1}$ such that

$$P^* \subseteq P$$

and

$$\exists y P(x, y) \iff \exists! y P^*(x, y)$$

(Moschovakis [29; 6C.5]).

2. The basis theorem for $\Sigma^1_{2n+2}$, i.e. every nonempty $\Sigma^1_{2n+2}$ set contains a $\Delta^1_{2n+2}$ real (Moschovakis [29; 6C.6]).

We turn now to definability estimates for winning strategies. The basic theorem here is the Third Periodicity Theorem (Moschovakis [29; 6E.1]), which asserts that in every $\Sigma^1_{2n}$ game in which Player I has a winning strategy, he actually has a $\Delta^1_{2n+1}$ winning strategy. We shall also use the following consequences of this result:

1. The Spector-Gandy Theorem for $\Pi^1_{2n+1}$, which asserts that every $\Pi^1_{2n+1}$ pointset $P(x)$ can be written as

$$P(x) \iff \exists \alpha \in \mathcal{B}^1_{2n+1}(x) R(\alpha, x),$$

for some $R$ in $\Pi^1_{2n}$ (Moschovakis [29; 6E.7]).

2. Every thin (i.e. containing no nonempty perfect subset) $\Sigma^1_{2n+1}$ set contains only $\Delta^1_{2n+1}$ reals (so in particular is countable). Also every nonempty $\Delta^1_{2n+1}$ thin set $A$ can be written as

$$\{(\xi)_n : n \in \omega\}$$

for some $\Delta^1_{2n+1}$ real $\xi$ (Moschovakis [29; 6E.5]).
§2. The uniformization of $\Sigma^{1}_{\omega_{2n+1}}$ sets with countable sections.

We assume in this section Det($\Delta^{1}_{\omega_{2n}}$).

Theorem 2.1. (Yasuda [43, 45]). Every $\Sigma^{1}_{\omega_{2n+1}}$ set in $\omega_{\omega} \times \omega_{\omega}$ with countable section $\Sigma^{1}_{\omega_{2n+1}}$ can be uniformized by a $\Sigma^{1}_{\omega_{2n+1}}$ set.

Proof. Let $P$ be a $\Sigma^{1}_{\omega_{2n+1}}$ set in $\omega_{\omega} \times \omega_{\omega}$ with countable sections. Since for each $\alpha$, $P^{(\alpha)} \subseteq \Sigma^{1}_{\omega_{2n+1}}(\alpha)$, let

$$P^{(\alpha)} \subseteq \bigcup_{\omega_{2n+1}}(\alpha).$$

Let $P^{*}$ define by

$$P^{*}(\alpha, \beta) \iff P(\alpha, \beta) \& \forall \gamma (P(\alpha, \gamma) \Rightarrow g(\beta, \alpha) \leq g(\gamma, \alpha)).$$

where $\leq$ is the usual wellordering on $\omega$. Then we have

$$P^{*} \subseteq P,$$

and

$$\exists \beta P^{*}(\alpha, \beta) \iff \exists \beta P^{*}(\alpha, \beta).$$

Thus $P^{*}$ uniformizes $P$, and from our definition of $P^{*}$ it is clearly a $(\Sigma^{1}_{\omega_{2n+1}})$ set.

From this theorem, using the remarks in preliminaries, we have

Corollary 2.2. Every $\Sigma^{1}_{\omega_{2n+1}}$ set in $\omega_{\omega} \times \omega_{\omega}$ with countable sections can be uniformized by a $(\Sigma^{1}_{\omega_{2n+1}})$ set. \(\square\)
**Theorem 2.3.** (Yasuda [45]) There is a $\Sigma^1_{2n+1}$ set $P$ in $\omega^\omega \times \omega^\omega$ with the properties:

(i) For each $\alpha$, $P^{<\alpha}$ is nonempty and has at most two elements,

(ii) $P$ cannot be uniformized by either a $\Sigma^1_{2n+1}$ or a $\Pi^1_{2n+1}$ set.

**Proof.** Let $G$ be a $\Sigma^1_{2n+1}$ set in $\omega^\omega \times \omega^\omega \times \omega^\omega$ which is universal for all $\Sigma^1_{2n+1}$ sets in $\omega^\omega \times \omega$ and $Q$ define by

$$Q(\alpha, e) \iff \forall \beta (G(\alpha, \alpha, \beta) \Rightarrow \forall i (\beta(i) = q(e, \alpha, i))).$$

Since $Q$ is a $\Pi^1_{2n+1}$ set in $\omega^\omega \times \omega$, using the number uniformization theorem we can find a $\Pi^1_{2n+1}$ set $Q^*$ which uniformizes $Q$. Now let $R$ define by

$$R(\alpha, \beta) \iff \exists e (Q^*(\alpha, e) \& \forall i (\beta(i) = q(e, \alpha, i))).$$

Then $R$ is a $\Pi^1_{2n+1}$ set whose each section has at most one $\Delta^1_{2n+1}(\alpha)$ real as an elements. The following is clear from the properties of $R$.

**Fact.** For each $\alpha$, if $G^{<\alpha}$ contains just one element then $G^{<\alpha} = R^{<\alpha}$.

Finally, put

$$P(\alpha, \beta) \iff (\forall i (\beta(i) = 0) \vee \forall i (\beta(i) = 1)) \& \neg R(\alpha, \beta).$$

Then $P$ is $\Sigma^1_{2n+1}$ and for each $\alpha$, $P^{<\alpha}$ is nonempty and at most two elements. We will show that $P$ satisfies (ii). Let $A$ be a $\Sigma^1_{2n+1}$ subset of $P$ which is the graph of a partial function from $\omega^\omega$ into $\omega^\omega$. Then there is a real $\alpha_0$ such that

$$A = G^{<\alpha_0}.$$
Since $A^{\langle \alpha \rangle}$ has at most one element, if $A^{\langle \alpha \rangle}$ is nonempty then by the fact

$$A^{\langle \alpha \rangle} = R^{\langle \alpha \rangle}.$$ 

But

$$A^{\langle \alpha \rangle} \subseteq P^{\langle \alpha \rangle} \subseteq \omega_2 = R^{\langle \alpha \rangle}.$$ 

We must have $A^{\langle \alpha \rangle}$ is empty. Thus $A^{\langle \alpha \rangle}$ is the empty set. This means $P$ cannot be uniformized by a $\Sigma^1_{2n+1}$ set. Now suppose that $P$ can be uniformized by a $\Pi^1_{2n+1}$ set $C$. Then for some real $\delta$, $C$ is in $\Pi^1_{2n+1}(\delta)$ and we have

$$\neg C(\alpha, \beta) \iff \exists \gamma \in \bigcup_{2n+1}^1(\alpha, \gamma)(C(\alpha, \gamma) \land \gamma \neq \beta).$$

By the bounded quantification theorem, this equivalence shows that the complement of $C$ is also $\Pi^1_{2n+1}(\delta)$, i.e., $C$ is $\Delta^1_{2n+1}$. Thus it is a $\Sigma^1_{2n+1}$ uniformizer for $P$, so we have a contradiction. Therefore $P$ also cannot be uniformized by a $\Pi^1_{2n+1}$ set. □

**Corollary 2.4.** There is a $\Sigma^1_{2n+1}$ set in $\omega_\omega \times \omega_\omega$ with countable sections which cannot be uniformized by either a $\Sigma^1_{2n+1}$ or a $\Pi^1_{2n+1}$ set. □

**Corollary 2.5.** There is a $\Sigma^1_{2n+1}$ set in $\omega_\omega \times \omega_\omega$ which cannot uniformized by either a $\Sigma^1_{2n+1}$ or a $\Pi^1_{2n+1}$ set. □

Let $D$ define by

$$D = \{ \alpha : R(\alpha, \lambda n \in [0]) \lor R(\alpha, \lambda n \in [1]) \},$$

and $C^*$ by
\[ C^*(\alpha, \beta) \iff \alpha \in D \land ((R(\alpha, \lambda_{\mathbb{N}[0]})) \land \beta = \lambda_{\mathbb{N}[1]} \lor (R(\alpha, \lambda_{\mathbb{N}[1]})) \land \beta = \lambda_{\mathbb{N}[0]})) \]

Then \( D \) and \( C^* \) are \( \prod_{2n+1}^1 \) and \( C^* \) is a partial function which is contained in \( P \). Now put

\[ P^*(\alpha, \beta) \iff (\alpha \in D \land C^*(\alpha, \beta)) \lor (\alpha \notin D \land \beta = \lambda_{\mathbb{N}[0]}) \]

Then \( P^* \) uniformizes \( P \), and it is the sum of a \( \Sigma_{2n+1}^1 \) and a \( \Pi_{2n+1}^1 \) sets.

**Problem.** Is there a \( \Sigma_{2n+1}^1 \) set in \( \omega_\omega \times \omega_\omega \) with countable sections which cannot be uniformized by the sum of a \( \Sigma_{2n+1}^1 \) and a \( \Pi_{2n+1}^1 \) sets?

For this problem we have no answer at present, but a related result for \( n = 0 \).

**Theorem 2.6.** (Tanaka [39] for \( n = 0 \)). There is an uncountable \( \Sigma_{2n+1}^1 \) set in \( \omega_2 \) with no nonempty \( \Pi_{2n+1}^1 \) subsets.

**Proof.** Let \( G \) be a \( \Pi_{2n+1}^1 \) set in \( \omega \times \omega_2 \) which is universal for all \( \Pi_{2n+1}^1 \) sets in \( \omega_2 \). By the uniformisation theorem for \( \Pi_{2n+1}^1 \), we can find a \( \Pi_{2n+1}^1 \) set \( G^* \) in \( \omega \times \omega_2 \) which uniformizes \( G \). Put

\[ Q(\alpha) \iff \exists e \ G^*(e, \alpha) \]

Then \( Q \) is a \( \Pi_{2n+1}^1 \) set and intersects with every nonempty \( \Pi_{2n+1}^1 \) set in \( \omega_2 \). Therefore the complement \( A = \omega_2 - Q \) is a \( \Sigma_{2n+1}^1 \) set.
and it contains no nonempty $\prod_{2n+1}^1$ subsets. □

(Theorem 2.7) For each uncountable $\sum_1^1$ set $A$ in $\omega_2$, we can find a nonempty perfect subset $A^*$ of $A$ which is also $\sum_1^1$. □
**Corollary 2.10.** There is a perfect Polish space \( D \) which is a \( \Sigma^1_1 \) set in \( \omega^2 \) with no nonempty \( \Pi^1_1 \) subsets.

**Proof.** Let \( A \) be an uncountable \( \Sigma^1_1 \) set with the properties as in theorem 2.6. Then, using theorem 2.7, we can find a nonempty perfect \( \Sigma^1_1 \) subset \( D \) of \( A \). Clearly \( D \) is a perfect Polish space, since \( D \) has a countable base

\[
\left\{ N_s \cap D : s \in 2^{<\omega_1} \right\},
\]

where

\[
N_s = \{ d \in \omega^2 : d \in \text{the length of } s = s \}.
\]

\[\square\]

**Theorem 2.11.** There is a \( \Sigma^1_2 \) set \( P \) in \( \omega^2 \times \omega^2 \) with the following properties:

(i) For each real \( \alpha \), \( P <^< \alpha \) has at most two reals as elements.

(ii) \( P \) cannot be uniformized by a \( \Sigma^1_2 \) set.

(iii) \( P \) contains no nonempty \( \Pi^1_1 \) curves.

**Proof.** Let \( D \) be a perfect Polish space as in corollary 2.10, and \( f : D \to \omega^2 \) a Borel isomorphism (we cannot find such \( f \) which is \( \Delta^1_1 \) isomorphism, since if such \( f \) exists then \( D \) must contain nonempty \( \Pi^1_1 \) sets), and \( G \) a \( \Sigma^1_1 \) set in \( \omega^2 \times \omega^2 \times \omega^2 \) which is universal for all \( \Sigma^1_1 \) sets in \( \omega^2 \times \omega^2 \). We can define a new universal

\[\ldots\]
set $G^+$ for all $\Sigma^1_1$ sets in $\omega_2 \times \omega_2$ by

$$G^+(\alpha, \beta, \gamma) \iff \alpha \in D \land G(f(\alpha), \beta, \gamma).$$

Since $D$ is $\Sigma^1_1$ and $f$ is a Borel function, $G^+$ is a $\Sigma^1_1$ in $\omega_2 \times \omega_2 \times \omega_2$, then we use this $G^+$ as $G$ in the proof of theorem 2.3. \[\Box\]

From this theorem, using a $\Delta^1_1$ isomorphism between two spaces $\omega_2$ and $\omega$, we have

**Corollary 2.12.** There is a $\Sigma^1_1$ set in $\omega_\omega \times \omega_\omega$ with countable sections which cannot be uniformized by the sum of a $\Sigma^1_1$ and a $\Pi^1_1$ sets. \[\Box\]

The same reason as before the set $P$ in theorem 2.11 can be uniformized by the sum of a $\Sigma^1_1$ and a $\Pi^1_1$ sets. We do not know at present whether corollary 2.12 can be extended to $\Sigma^1_{2n+1}$ set, where $n > 0$.

Note in the proof of theorem 2.7 we really used the fact that $C$ is $\Pi^0_1$ only to prove that the game $G^*$ is determined. Thus we have

**Theorem 2.13.** If Det($\Pi^1_{2n+1}$) then every uncountable $\Sigma^1_{2n+2}$ set in $\omega_2$ has a nonempty $\Sigma^1_{2n+3}$ perfect subset. \[\Box\]
§3. On the partial $\prod_n^1$ functions from $\omega_\omega$ into $\omega_\omega$.

Luzin [26] proved that every analytic curve can be extended to a Borel curve. We assume that Det($\Delta^1_{2n}$) holds in this section. Under this assumption, Tanaka [42] extend this Luzin's result as follows.

Theorem 3.1. (Tanaka [26]). Every $\Sigma^1_{2n+1}$ partial function from $\omega_\omega$ into $\omega_\omega$ can be extended to a $\Delta^1_{2n+1}$ function. \hfill \Box

From this, using the notion of relativization, we have

Corollary 3.2. Every $\Sigma^1_{2n+1}$ partial function from $\omega_\omega$ into $\omega_\omega$ can be extended to a $\Delta^1_{2n+1}$ function. \hfill \Box

Now we show that these results do not hold for $\prod_{2n+1}^1$ and $\Sigma_{2n+2}^1$ partial functions in the strongest form.

Lemma 3.3. Let $\Gamma$ be an analytical pointclass, and $G$ a $\Gamma$ pointset in $\omega_\omega \times \omega_\omega \times \omega_\omega$ which is universal for all $\gamma$ sets in $\omega_\omega \times \omega_\omega$, and put

$$D = \{ (\alpha', \beta') : G(\alpha', \alpha', \beta') \}.$$  

Then $D$ cannot be embedded in a $\gamma$ set $P$ in $\omega_\omega \times \omega_\omega$ with the property: for each $\alpha'$

$$P \not< \alpha' \neq \omega_\omega.$$  

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Proof. Assume, in order to obtain a contradiction, that there is a set $P$ in $\omega_\omega \times \omega_\omega$ such that
\[ D \subseteq P \]
and for each $\alpha$
\[ P^{<\alpha>} \not\subseteq \omega_* \]
Now consider the set $Q = (\omega_\omega \times \omega_\omega) - P$. Since $Q$ is $\omega_\omega$, there is a real $\alpha_0$ such that
\[ Q = \mathcal{G}^{<\alpha_0>} \]
Then
\[ D^{<\alpha_0>} = \{ \beta ; \mathcal{G}(\alpha_0, \alpha_0, \beta) \} = \mathcal{Q}^{<\alpha_0>} \]
so we have
\[ Q^{<\alpha_0>} = \emptyset \]
But $Q^{<\alpha_0>}$ is not empty since $P^{<\alpha_0>}$ is not equal to $\omega_\omega$. Thus we have a contradiction, so the lemma is proved. $\square$

Theorem 3.4. (i) There is a partial $\prod^{1}_{2n+1}$ function from $\omega_\omega$ into $\omega_\omega$ which cannot be extended to: a $\Delta^{1}_{2n+1}$ function.

(ii) There is a partial $\Sigma^{1}_{2n+1}$ function from $\omega_\omega$ into $\omega_\omega$ which cannot be extended to a $\Delta^{1}_{2n+2}$ function.

Proof. Let $\Gamma$ be $\prod^{1}_{2n+1}(\Sigma^{1}_{2n+2})$ and $\bar{\Gamma}$ a partial $\prod^{1}_{2n+1}(\Sigma^{1}_{2n+2})$ function from $\omega_\omega$ into $\omega_\omega$ which uniformizes the $\prod^{1}_{2n+1}(\Sigma^{1}_{2n+2})$ set $D$ in the lemma 3.3, we can find such function using the uniformization theorem for $\prod^{1}_{2n+1}(\Sigma^{1}_{2n+2})$. By lemma 3.3, $\bar{\Gamma}$ cannot be extended to $\Delta^{1}_{2n+1}(\Delta^{1}_{2n+2})$ function. $\square$
We have also

**Theorem 3.5.** There is a total $\prod_{2n+1}^1$ function from $\omega_\omega$ into $\omega_\omega$ which is not $\Delta_{2n+1}^1$.

To prove this we need first

**Lemma 3.6.** There is a $\prod_{2n}^1$ set $P$ in $\omega_\omega \times \omega_\omega$ such that

(i) $\forall \alpha \exists \beta P(\alpha, \beta)$,

(ii) $\forall \alpha \exists \beta \in \bigcup_{2n+1}^1(\alpha) P(\alpha, \beta)$.

**Proof.** Since $\bigcup_{2n+1}^1(\alpha)$ is $\prod_{2n+1}^1$, there is a $\prod_{2n}^1$ set $R$ in $\omega_\omega \times \omega_\omega \times \omega_\omega$ such that

$\beta \notin \bigcup_{2n+1}^1(\alpha) \iff \exists \gamma R(\alpha, \beta, \gamma)$.

Put

$P(\alpha, \beta) \iff R(\alpha, (\beta)_0, (\beta)_1)$.

Then $P$ is $\prod_{2n}^1$ and since

$\forall \alpha \exists \beta \exists \gamma R(\alpha, \beta, \gamma)$,

$\forall \alpha \exists \beta P(\alpha, \beta)$.

Since if

$\forall \alpha \exists \beta \in \bigcup_{2n+1}^1(\alpha) P(\alpha, \beta)$

then for some real $\alpha$ and $\Delta_{2n+1}^1(\alpha)$ real $\beta$

$R(\alpha, (\beta)_0, (\beta)_1)$,

so $(\beta)_0$ is not $\Delta_{2n+1}^1(\alpha)$ real. This contradiction shows

$\forall \alpha \exists \beta \in \bigcup_{2n+1}^1(\alpha) P(\alpha, \beta)$.

$\square$
Proof of theorem 3.5. Let P be as in lemma 3.6. By the uniformization theorem for $\prod_{2n+1}^1$ we can find a $\prod_{2n+1}^1$ set $P^*$ which uniformizes $P$. By lemma 3.6 (i), $P^*$ is a total function.

Assume, in order to obtain a contradiction, that $P^*$ is $\Delta_{2n+1}^1$. Then for some real $\alpha_0$ $P^*$ is in $\Delta_{2n+1}^1(\alpha_0)$. Thus there is just one $\Delta_{2n+1}^1(\alpha_0)$ real $\beta_0$ such that

$$P^*(\alpha_0, \beta_0).$$

This implies

$$\neg \forall \alpha \neg \exists \beta \in \bigcup_{2n+1}^1(\alpha) P(\alpha, \beta).$$

But this formula contradicts with the lemma 3.6 (ii). \qed

Theorem 3.5 can be extended to even levels using Moschovakis' models $\mathcal{N}_{2n+2}(\alpha)$ which is the smallest $\Sigma_{2n+2}^1$-correct standard model of ZFC containing all ordinals and real $\alpha$. We also need the sharp operation for these models (for detailed theory of these models and its sharps, the reader consults Becker [3]).

Lemma 3.7 (Becker [3]). Assuming $\sum_{\omega \cap \mathcal{N}_{2n+2}(\alpha)}^1$ is countable, the real $\alpha_{2n+2}^#$ exists, the relation

$$\forall \alpha, \beta \exists \beta \in \mathcal{M}_{2n+2}(\alpha) P(\alpha, \beta) \iff \beta = \alpha_{2n+2}^#$$

is $\prod_{2n+2}^1$, and

$$\forall \alpha \exists ! \beta P(\alpha, \beta).$$

\qed
Lemma 3.7. If $\beta$ is $\Delta^1_{2n+2}(\alpha)$ real, then $\beta$ is in $\mathcal{M}^{2n+2}(\alpha)$.

Proof. Let $\beta$ be a $\Delta^1_{2n+2}(\alpha)$ real. Then

$$\beta(i) = j \iff P(i, j),$$

where $P$ is $\Sigma^1_{2n+2}(\alpha)$

$$\iff \mathcal{M}^{2n+2}(\alpha) \models P(i, j).$$

Since

$$ZF + V = \mathcal{M}^{2n+2}(\alpha) \models \exists X (X \subseteq \omega \times \omega \land \forall i \forall j ((i, j) \in X \iff P(i, j))),$$

$$\mathcal{M}^{2n+2}(\alpha) \models \exists X (X \subseteq \omega \times \omega \land \forall i \forall j ((i, j) \in X \iff P(i, j))).$$

This implies that there is a set $X$ in $\mathcal{M}^{2n+2}(\alpha)$ such that

$$X \subseteq \omega \times \omega,$$

and for each $i, j$

$$\mathcal{M}^{2n+2}(\alpha) \models (i, j) \in X \iff P(i, j),$$

so by the $\Sigma^1_{2n+2}$ correctness of $\mathcal{M}^{2n+2}(\alpha)$

$$(i, j) \in X \iff \mathcal{M}^{2n+2}(\alpha) \models P(i, j)$$

$$\iff P(i, j)$$

$$\iff \beta(i) = j.$$

Thus $X = \beta$ and hence $\beta \in \mathcal{M}^{2n+2}(\alpha)$. □

Corollary 3.8. Assuming $\omega \omega \cap \mathcal{M}^{2n+2}(\alpha)$ is countable, $\alpha_{2n+2}^\#$ is not in $\Delta^1_{2n+2}(\alpha)$. □

Theorem 3.9. Assume that for all $\alpha$, $\mathcal{M}^{2n+2}(\alpha) \cap \omega \omega$ is countable. Then there is a total $\Pi^1_{2n+2}$ function form $\omega \omega$ into $\omega \omega$ which is not in $\Delta^1_{2n+2}$. □
Proof. Consider the $P$ in the lemma 3.7 which is a total
$$\prod_{2n+2}^1$$ function from $\omega_\omega$ into $\omega_\omega$. Suppose that $P$ is in $\Delta_{2n+2}^1$,
so for some real $\alpha_0$ $P$ is in $\Delta_{2n+2}^1(\alpha_0)$. Then $\alpha_0 \#^+$ is in
$\Delta_{2n+2}^1(\alpha_0)$. This contradicts with corollary 3.8. \qed

Remark. $\text{Det}(\sum_{2n+2}^1)$ implies $\omega_\omega \cap \mathbb{2}^{2n+2}(\alpha)$ is countable
(see Becker [3]).

Luzin [26] proved the so-called "Théorème sur la projection
d'ensemble d'unicité". This says that let $E$ be a Borel set in
$\omega_\omega \times \omega_\omega$ and $E_1$ denote the set of all points $(\alpha, \beta)$ of $E$
such that the section $E^{<\alpha>$} consists of a single point . Further,
let $E_1$ be the projection of $E_1$ on the first axis :

$$E_1 = \text{Proj } E_1.$$

Luzin called $E_1$ "l'ensemble d'unicité" of $E$, and showed
that both $E_1$ and $E_1$ are $\prod_{1}^1$ sets. Tugué and Tanaka [43]
obtained the effective version of this classical fact and also proved
this classical fact from its effective version.

We will extend these facts to higher levels of projective sets.

Theorem 3.10. (Tugué and Tanaka [43] for $n = 0$). For each
$\Delta_{2n+1}^1$ set $B$ in $\omega_\omega \times \omega_\omega$ the pointset

$$P(\alpha) \iff \exists! \beta B(\alpha', \beta)$$
is also in $\prod_{2n+1}^1$. 

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Proof. As is well known

\[ (*) \quad \exists! \; \beta \; B(\alpha, \beta) \Rightarrow B^{<\alpha>} \text{ is a } \Sigma^1_{2n+1}(\alpha) \text{ singleton} \]

\[ \Rightarrow \exists \beta \in \bigotimes_{2n+1}^1(\alpha) \; B(\alpha, \beta). \]

Since, by the bounded quantification theorem, \( \exists \beta \in \bigotimes_{2n+1}^1(\alpha) \; B(\alpha, \beta) \) is \( \Pi^1_{2n+1} \), from (*) we have

\[ \exists! \; \beta \; B(\alpha, \beta) \iff \exists \beta \in \bigotimes_{2n+1}^1(\alpha) \; B(\alpha, \beta) \land \forall \beta', \beta \; (B(\alpha, \beta) \land B(\alpha, \beta') \Rightarrow \beta = \beta'). \]

This shows that \( P \) is \( \Pi^1_{2n+1} \). \( \Box \)

**Corollary 3.11.** (Tugue and Tanaka [43] for \( n = 0 \)). For each \( \alpha \in \Delta^1_{2n+1} \) set \( B \) in \( \omega \times \omega \) the pointset

\[ R(\alpha) \iff \exists! \; \beta \; B(\alpha, \beta) \]

is also in \( \Pi^1_{2n+1} \). \( \Box \)

Let \( E \) and \( E_1 \) be the sets in \( \omega \times \omega \) defined as follows.

\[ (\alpha, \beta) \in E \iff B(\alpha, \beta) \]

and

\[ (\alpha, \beta) \in E_1 \iff (\alpha, \beta) \in E \land \forall \beta' \; ((\alpha, \beta') \in E \Rightarrow \beta = \beta'). \]

Since \( E \) is \( \Delta^1_{2n+1} \), \( E_1 \) is \( \Pi^1_{2n+1} \). Since

\[ \alpha \in E_1 \iff \exists! \beta\; ((\alpha, \beta) \in E), \]

we have

\[ \alpha \in E_1 \iff R(\alpha), \]

so \( E_1 \) is also \( \Pi^1_{2n+1} \).
\((\alpha, \beta) \in \Phi \iff \alpha \in D \land \beta = \emptyset\).

Then \(\Phi\) is a \(\sum_{2n+1}^1\) partial function from \(\omega\omega\) into \(\omega\omega\), but whose domain \(D\) is not in \(\Delta_{2n+1}^1\).

Finally, we notice that every partial \(\Delta_{2n+1}^1\) function from \(\omega\omega\) into \(\omega\omega\) can be extended to a total \(\Delta_{2n+1}^1\) function.
Corollary 3.12. (Luzin [26], Tugué and Tanaka [43]). Let $B$ be a $\Delta^1_{2n+1}$ set in $\omega_\omega \times \omega_\omega$ such that
\[ \exists \beta B(\alpha, \beta) \Rightarrow \exists! \beta B(\alpha, \beta). \]
Then $\exists \beta B(\alpha, \beta)$ is also in $\Delta^1_{2n+1}$. \qed

Corollary 3.13. (Tanaka [42]). The domain of a $\Delta^1_{2n+1}$ partial function $\Phi : \omega_\omega \to \omega_\omega$ is also $\Delta^1_{2n+1}$.

**Proof.** In theorem 3.11 or corollary 3.12 set $B = \Phi$. Then $\text{Dom}(\Phi) = R$ is in $\Delta^1_{2n+1}$. \qed

This corollary 3.13 does not hold for $\Delta^1_{2n+2}$ partial functions.

Theorem 3.14. There is a partial $\Pi^1_{2n+1}$ function $\Phi : \omega_\omega \to \omega_\omega$ whose domain is not in $\Delta^1_{2n+2}$.

**Proof.** Let $D$ be a set in $\omega_\omega$ such that $D \in \Sigma^1_{2n+2} = \Delta^1_{2n+2}$.

By the uniformization theorem, we can find a $\Pi^1_{2n+1}$ partial function $\Phi : \omega_\omega \to \omega_\omega$ such that
\[ \text{dom}(\Phi) = D. \] \qed

Clearly this theorem 3.14 holds for $\Sigma^1_{2n+1}$ partial functions.

Let $D$ be a set in $\omega_\omega$ as follows
\[ D \in \Sigma^1_{2n+1} = \Delta^1_{2n+1}. \]

Let $\beta_0$ be a fixed recursive real, and put
§4. On the $\bigcup_{n}^{1}$ pointsets in $\omega \times \omega$ with countable sections.

One of well known theorems of Luzin is

**Theorem 4.1. (Luzin [26]).** Every $\Delta_{1}^{1}$ set in $\omega \times \omega$ with countable sections is the union of countably many $\Delta_{1}^{1}$ curves. □

From this he also obtained

**Corollary 4.2. (Luzin [26]).** Every $\Sigma_{1}^{1}$ set in $\omega \times \omega$ with countable sections is the union of countably many $\Sigma_{1}^{1}$ curves. □

Effective versions and extensions of these results are obtained by Kondō [23] and Tanaka [42]. We assume from now on $\text{Det}(\Delta_{2n}^{1})$ holds.

**Theorem 4.3. (Kondō [23] for $n = 0$, Tanaka [42]).** For every $\Delta_{2n+1}^{1}$ set $P$ in $\omega \times \omega$ with countable sections, we can find a $\Delta_{2n+1}^{1}$ set $P^{*}$ in $\omega \times \omega \times \omega$ such that for each $n$ and $\alpha$

\[ p^{* \langle n, \alpha \rangle} \text{ has at most one element and} \]

\[ P(\alpha, \beta) \iff \exists n P^{*}(n, \alpha, \beta). \] □

**Corollary 4.4. (Kondō [23] for $n = 0$, Tanaka [42]).** For every $\Sigma_{2n+1}^{1}$ set $P$ in $\omega \times \omega$ with countable sections, we can find a $\Sigma_{2n+1}^{1}$ set $P^{*}$ in $\omega \times \omega \times \omega$ such that for each $n$ and $\alpha$

\[ p^{* \langle n, \alpha \rangle} \text{ has at most one element and} \]

\[ P(\alpha, \beta) \iff \exists n P^{*}(n, \alpha, \beta). \] □
Then Luzin [26] proposed the question:

Does every $\prod_{n=1}^{1}^{\omega}$ set in $\omega \times \omega$ can be covered by countably many $\prod_{n=1}^{1}^{\omega}$ curves?

A negative answer of this and its effective analogue are obtained by Tanaka [42]. Here we shall show

**Theorem 4.5.** Assume that $\det(\sum_{2n+1}^{1})$. Then there is a $\prod_{n=1}^{1}^{\omega}$ set in $\omega \times \omega$ with countable sections which cannot be covered by either countably many $\sum_{2n+2}^{1}$ or $\prod_{2n+2}^{1}$ curves.

**Proof.** Let $G$ be a $\sum_{2n+1}^{1}$ set in $\omega \times \omega \times \omega$ which is universal for all $\sum_{2n+1}^{1}(\alpha)$ sets in $\omega$ for each real $\alpha$, and for each $e$ and $\alpha$ $\beta_{e,\alpha}$ the characteristic function of the set $g^{<e,\alpha}_{1}$ in $\omega$.

Since for each $e$ and $\alpha$ $g^{<e,\alpha}_{1}$ is $\sum_{2n+2}^{1}(\alpha)$, the real $\beta_{e,\alpha}$ is in $(\sum_{2n+2}^{1}(\alpha))_{\rho}$. Put

$$P = \{ (\alpha, \beta) : \exists e \ (\beta = \beta_{e,\alpha}) \}.$$  

We want to prove that the set $P$ is really $\sum_{2n+2}^{1}$. To this we need the following results of Kechris and Moschovakis (for details see Kechris [12]).

**Theorem 4.6.** (Solovay [36] for $n = 0$, Kechris and Moschovakis [13]). Assume that $\det(\sum_{2n+1}^{1})$ holds. Then for each real $\alpha$

1. There is a largest thin (not containing nonempty perfect subsets) $\prod_{2n+1}^{1}(\alpha)$ set of reals $C_{2n+1}(\alpha)$. (For $n = 0$, this is also due independently to Saks [30].)
(ii) If
\[ c_{2n+2}(\alpha) = \{ \beta : \exists \gamma \in c_{2n+1}(\alpha) (\beta \text{ is recursive in } \gamma) \} , \]
then \( c_{2n+2}(\alpha) \) is a \( \sum_1^{2n+2}(\alpha) \) set containing all thin \( \sum_1^{2n+2}(\alpha) \) sets.
(iii) \( c_{2n+2}(\alpha) \) is the largest countable \( \sum_1^{2n+2}(\alpha) \) set.

We say that for each real \( \alpha \) a wellordering \( \prec_\alpha \) on a pointset \( A \) is \( \Delta(\alpha) \)-good, where \( \Gamma(\alpha) \) is a pointclass and
\[ \Delta(\alpha) = \Gamma(\alpha) \cap \bar{\Gamma}(\alpha), \]
if and only if for each real \( \beta \in A \)
\[ \{ \gamma : \gamma \prec_\alpha \beta \} \]
is countable
and the relation
\[ \text{InSeq} \prec_\alpha (\gamma, \beta) = \{(\gamma_n : n \in \omega) \} = \{ \delta : \delta \prec_\alpha \beta \} \]
is in \( \Delta(\alpha) \) for \( \beta \in A \), i.e. for \( Q, R \) in \( \Gamma(\alpha) \) \( \bar{\Gamma}(\alpha) \) respectively.
\[ \beta \in A \Rightarrow (\text{InSeq} \prec_\alpha (\gamma, \beta) \iff Q(\gamma, \beta) \iff R(\gamma, \beta)). \]

**Theorem 4.7.** (Kechris [12, 15]). Assume that \( \text{Det}(\sum_1^{2n+1}) \) for
\( n \geq 1 \) \( c_{2n+2}(\alpha) \) admits a \( \sum_1^{2n+2}(\alpha) \)-good wellordering \( \prec_\alpha \) which has
the property:
\[ \beta \prec_\alpha \gamma \Rightarrow \beta \in \Delta_1^{2n+2}(\alpha, \gamma). \]
Lemma 4.9. The set \( \{ (\alpha, \gamma) : \gamma \in d^{2n+2}(\alpha) \} \) is in \( \Sigma_{2n+2}^1 \).

Proof. This is clear from \( \Sigma_{2n+2}^1 \)-goodness of wellordering \( <_{\alpha} \) and the definition of \( d^{2n+2}(\alpha) \).

Lemma 4.10. If \( \gamma \) is in \( d^{2n+2}(\alpha) \) and for each \( \Delta_{2n+2}^1(\alpha) \) real \( \beta \)

\[ \beta <_{\alpha} \gamma, \]

then for each \( e \)

\[ \beta_{e, \alpha} = \beta_{e, \alpha}^\gamma. \]

Proof. Since \( \gamma \) is in \( c_{2n+2}(\alpha) \), it is sufficient to show that

\[(*) \quad \exists \beta \phi(\beta, e, \alpha, i) \iff \exists \beta <_{\alpha}^{2n+2} \gamma \phi <_{\alpha}^{2n+2} (\beta, e, \alpha, i).\]

Suppose that \( \exists \beta \phi(\beta, e, \alpha, i) \). Then \( \Sigma_{2n+1}^1(\alpha) \) set \( \{ \beta : \phi(\beta, e, \alpha, i) \} \) is nonempty. By the basis theorem for \( \Sigma_{2n+2}^1 \), there is a \( \Delta_{2n+2}^1(\alpha) \) real \( \beta_0 \) such that

\[ (**) \quad \phi(\beta_0, e, \alpha, i). \]

Since \( \gamma \) is in \( d^{2n+2}(\alpha) \) and \( \beta_0 <_{\alpha}^{2n+2} \gamma \), by \((**)

\[ \beta_0 <_{\alpha}^{2n+2} \gamma \phi <_{\alpha}^{2n+2} (\beta_0, e, \alpha, i). \]

Thus we have

\[ \exists \beta <_{\alpha}^{2n+2} \gamma \phi <_{\alpha}^{2n+2} (\beta, e, \alpha, i). \]

To prove right to left implication of \((*)\), suppose that \( \exists \beta <_{\alpha}^{2n+2} \gamma \phi(\beta, e, \alpha, i) \). Since \( \gamma \) is in \( d^{2n+2}(\alpha) \), we have

\[ \exists \beta \phi(\beta, e, \alpha, i). \]

So we have proved the formula \((*)\).
Lemma 4.11. There is a real \( \gamma \) in \( C_{2n+2}(\alpha) \) such that

(i) \( L^{2n+2}(\alpha) \models " \text{For each } (\Sigma_{2n+2}(\alpha))_p \text{ real } \beta, \beta <^\alpha \gamma " \).

(ii) \( L^{2n+2}(\alpha) \models " \gamma \in D^{2n+2}(\alpha)". \)

Proof. Now we work in the model \( L^{2n+2}(\alpha) \). Let \( \gamma \) be the \( <^\alpha \gamma \) -least real such that for all real \( \beta \)

\[ \beta <^\alpha \gamma \Rightarrow \beta \in D_{2n+2}^1(\alpha). \]

(i) Suppose that there is a \( (\Sigma_{2n+2}(\alpha))_p \) real \( \beta \) such that

\[ \neg \beta <^\alpha \gamma. \]

Then

\[ \gamma <^\alpha \beta \lor \gamma = \beta. \]

So \( \gamma \) is a \( \Delta^1_{2n+3}(\alpha) \) real. This contradicts with our choice of \( \gamma \).

(ii) This can be proved by the induction on the construction of the \( \Pi^1_{2n+1} \) formula \( \varphi \). Let \( \psi \) be a \( \Sigma^1_{2n} \) formula such that

\[ \forall \delta \psi(\delta, \beta, e, \alpha, i) \iff \varphi(\delta, \beta, e, \alpha, i). \]

Suppose that

\[ \forall \beta <^\alpha \gamma (\psi <^\alpha \gamma \Rightarrow \varphi(\delta, \beta, e, \alpha, i)). \]

(*)

We must show that for all \( \beta <^\gamma \gamma \),

\[ \forall \delta <^\alpha \gamma (\psi <^\alpha \gamma (\delta, \beta, e, \alpha, i) \iff \forall \delta \psi(\delta, \beta, e, \alpha, i)). \]

(**)

To prove from left to right of formula (**), suppose that

\[ \neg \forall \delta \psi(\delta, \beta, e, \alpha, i). \]

Then \( \exists \delta \neg \psi(\delta, \beta, e, \alpha, i). \) Since the \( \Sigma^1_{2n+2} \) set \( \{ \delta : \neg \psi(\delta, \beta, e, \alpha, i) \} \) is nonempty, by the basis theorem for \( \Sigma^1_{2n+2} \) there's a \( \Delta^1_{2n+2}(\alpha) \) real \( \delta_0 \) such that

\[ \neg \psi(\delta_0, \beta, e, \alpha, i), \]

I.e.
\[ \exists \gamma \in 2^{n+2} \gamma \quad \neg \psi(\sigma, \beta, e, \alpha, i). \]

Thus we have a contradiction. This prove that the left to right implication of (**) using (*) from the right to the left of (**) is clear. Therefore (ii) is proved. \( \square \)

**Lemma 4.12.** For each \( \alpha \) there is a \( \gamma \) such that

(i) \( \gamma \in 2^{2n+2}(\alpha) \).

(ii) If \( \beta \) is a \( (\Sigma^1_{2n+2}(\alpha))^\rho \) real, then \( \beta <_{\alpha} 2^{2n+2} \gamma \).

**Proof.** Since the model \( L^{2n+2}(\alpha) \) is \( \Sigma^1_{2n+2} \)-absolute, by lemmas 4.9 and 4.11 (i) and (ii) are clear. \( \square \)

**Lemma 4.13.** The following equality holds.

\[ P = \{ (\alpha, \beta) : \exists e \exists \gamma (\gamma \in 2^{2n+2}(\alpha) \land \beta <_{\alpha} 2^{2n+2} \gamma \land \beta = \beta_{e, \alpha}^\gamma) \}. \]

**Proof.** To prove the inclusion from the left to the right, let \( P(\alpha, \beta) \). Then there is an \( e \) such that \[ \beta = \beta_{e, \alpha}^\gamma. \]

By lemma 4.12 there is a \( \gamma \) such that \[ \gamma \in 2^{2n+2}(\alpha) \]

and \[ \beta <_{\alpha} 2^{2n+2} \gamma. \]

By lemma 4.10 \[ \beta = \beta_{e, \alpha}^\gamma. \]

---
Conversely let \((\alpha, \beta)\) be such that
\[
\exists \gamma \exists \beta \left( \gamma \in B^{2n+2}(\alpha) \land \beta <_{\alpha}^{2n+2} \gamma \land \beta = \beta_{e, \alpha} \right).
\]
for each \(\Delta_{2n+2}(\alpha)\) real \(\gamma\),
\[
\gamma <_{\alpha}^{2n+2} \beta.
\]
Then by lemma 4.10
\[
P(\alpha, \beta).
\]
If there is a \(\Delta_{2n+2}(\alpha)\) real \(\gamma\) such that
\[
\neg \gamma <_{\alpha}^{2n+2} \beta,
\]
then, since the absoluteness of the \(\Delta_{2n+2}(\alpha)\) reals \(\gamma \in C_{2n+2}(\alpha)\),
\[
\gamma <_{\alpha}^{2n+2} \gamma \lor \gamma = \gamma.
\]
Thus
\[
\beta <_{\alpha}^{2n+2} \gamma.
\]
Therefore \(\beta\) is a \(\Delta_{2n+2}(\alpha)\) real. Since \(\beta = \beta_{e, \alpha}\) is the characteristic function of the \(\Sigma_{2n+2}(\alpha)\) set \(\{ i : \beta(i) = 1 \}\),
we have
\[
P(\alpha, \beta).
\]

Lemma 4.14. The set \(P\) has the following properties:

(i) For each \(\alpha\) \(P \alpha\) is countable.

(ii) There is no countable family of \(\Sigma_{2n+2}^{1}\) curves whose union contains \(P\).

(iii) There is no countable family of \(\Pi_{2n+2}^{1}\) curves whose union contains \(P\).
Proof. (i) Since for each $\alpha$ there are countably many $\Sigma^1_{2n+2}(\alpha)$ sets in $\omega$, $P^{<\alpha}$ is countable.

(ii) Assume, in order to obtain a contradiction, that there is a countable family $\{P_m\}$ of $\Sigma^1_{2n+2}$ curves such that

$$P \subseteq \bigcup_{m=0}^{\infty} P_m.$$ 

Take a real $\alpha_0$ and a set $S$ in $\omega$ as follows:

$$\forall m (P_m \in \Sigma^1_{2n+2}(\alpha_0)),$$

and

$$S \in \Sigma^1_{2n+2}(\alpha_0) \setminus \Delta^1_{2n+2}(\alpha_0).$$

Let $\beta_0$ be the characteristic function of the set $S$. Then

$$P(\alpha_0, \beta_0),$$

so there is a $m_0$ such that

$$P_{m_0}(\alpha_0, \beta_0).$$

Since $P_{m_0}$ is a curve, $\beta_0$ is in $\Sigma^1_{2n+2}(\alpha_0)$, so $\beta_0$ is a $\Delta^1_{2n+2}(\alpha_0)$ real. The set $S$ is written as

$$S = \{ i : \beta_0(i) = 1 \},$$

$S$ is in $\Delta^1_{2n+2}(\alpha_0)$. This contradicts with our choice of $S$.

To prove lemma 4.14, (iii) we need the following lemma.

Lemma 4.16. There is a $\beta$ in $P^{<\alpha}$ which is not $\prod^1_{2n+2}(\alpha)$ singleton, i.e. $\{ \beta \}$ is in $\prod^1_{2n+2}(\alpha)$.

Proof. Since $P^{<\alpha}$ is $\Sigma^1_{2n+2}(\alpha)$ and $\{ \beta : \{ \beta \} \in \prod^1_{2n+2}(\alpha) \}$ is in $\prod^1_{2n+2}(\alpha) \setminus \Sigma^1_{2n+3}(\alpha),$
Proof of lemma 4.14, (iii). Suppose that there is a countable family \( \{ P_m \} \) of \( \prod_{2n+2}^1 \) curves such that
\[
P \subseteq \bigcup_{m \neq 0} P_m.
\]
Then there is a \( \alpha'_0 \) such that
\[
\forall m \ (P_m \in \prod_{2n+2}^1 (\alpha'_0)).
\]
By lemma 4.16, there is a \( \beta_0 \) in \( P^{\alpha'} \) which is not \( \prod_{2n+2}^1 (\alpha'_0) \) singleton. Since
\[
P(\alpha'_0, \beta_0),
\]
there is a \( m_0 \) such that
\[
P_{m_0}(\alpha'_0, \beta_0).
\]
Since \( P_{m_0} \) is a \( \prod_{2n+2}^1 (\alpha'_0) \) curve, \( \beta_0 \) is a \( \prod_{2n+2}^1 (\alpha'_0) \) singleton. Thus we have a contradiction. □

Let \( h : \omega_\omega \times \omega_\omega \to \omega_\omega \) be a recursive homeomorphism and for \( i=0,1 \)
\( h_i : \omega_\omega \to \omega_\omega \) recursive functions such that for each \( \alpha', \)
\[
h(h_0(\alpha'), h_1(\alpha')) = \alpha'.
\]
By the uniformization theorem there is a \( \prod_{2n+1}^1 \) set \( P^* \) in
\( \omega_\omega \times \omega_\omega \times \omega_\omega \) such that
\[
\text{dom}(P^*) = P
\]
and
\[
\forall \gamma, \delta' (P^*(\alpha', \beta, \gamma) \& P^*(\alpha, \beta, \gamma') \Rightarrow \gamma = \gamma').
\]
Put

\[ p^{**} = \{(\alpha_0, h(\beta, \gamma)) : P^*(\alpha, \beta, \gamma)\} \cdot \]

We will show that the \( \prod_{2n+1}^{1} \) set \( p^{**} \) cannot covered by either countably many \( \sum_{2n+2}^{1} \) or \( \prod_{2n+2}^{1} \) curves.

Suppose that there is countably many \( \sum_{2n+2}^{1} \) curves \( \{p_m\} \) such that

\[ p^{**} \subseteq \bigcup_{m = 0}^{\infty} p_m. \]

Take \( \alpha_0 \) such that

\[ \forall m \ (p_m \in \sum_{2n+2}^{1}(\alpha_0)) \]

and, take \( \beta_0 \) in \( \omega_2 \) such that

\[ \beta_0 \in (\sum_{2n+2}^{1}(\alpha_0)) \rho - \Delta_{2n+2}^{1}(\alpha_0) \]

and

\[ P(\alpha_0, \beta_0). \]

Let \( \gamma_0 \) be a real such that

\[ p^{**}(\alpha_0, h(\beta_0, \gamma_0)). \]

Then there is a \( \alpha_0 \) such that

\[ p_{m_0}^{*}(\alpha_0, h(\beta_0, \gamma_0)), \]

so \( h(\beta_0, \gamma_0) \) is a \( \Delta_{2n+2}^{1}(\alpha_0) \) real. By the substitution property (see Moschovakis [29]) \( \beta_0 = h_0(h(\beta_0, \gamma_0)) \) is a \( \Delta_{2n+2}^{1}(\alpha_0) \) real. This contradicts with our choice of \( \beta_0 \).

Now suppose that there is countably many \( \prod_{2n+2}^{1} \) curves \( \{p_m\} \) such that

\[ -??- \]
\[ P** \subseteq \bigcup_{m = 0}^{\infty} P_m \]

Take \( \alpha_0 \) such that

\[ \forall m \ (P_m \in \prod_{2n+2}^1 (\alpha_0)) \]

and take \( \beta_0 \) such that

\[ P(\alpha_0, \beta_0) \]

but \( \beta_0 \) is not \( \prod_{2n+2}^1 (\alpha_0) \) singleton (by lemma 4.16 such a \( \beta_0 \) exists). Let \( \gamma_0 \) be a real such that

\[ P**(\alpha_0, h(\beta_0, \gamma_0)) \]

Then there is a \( m_0 \) such that

\[ P_{m_0}(\alpha_0, h(\beta_0, \gamma_0)) \]

so \( h(\beta_0, \gamma_0) \) is a \( \prod_{2n+2}^1 (\alpha_0) \) singleton. By the substitution property \( \beta_0 = h_1(h(\beta_0, \gamma_0)) \) is also \( \prod_{2n+2}^1 (\alpha_0) \) singleton. This contradicts with our choice of \( \beta_0 \). Therefore the proof of theorem 4.5 is completed. \( \square \)

Since by theorem 4.1 every \( \prod_{2n+1}^1, \Sigma_{2n+2}^1 \) and \( \prod_{2n+2}^1 \) set in \( \omega_\omega \times \omega_\omega \) with countable sections can be covered by countably many \( \Delta_{2n+3}^1 \) curves, theorem 4.5 is the best possible extension of theorem II.7 of Tanaka [42].
§5. A generalization of a Friedman’s theorem.

Friedman proved the following theorem.

**Theorem 5.1.** (See Mathias [28 ; T3210]). There is an infinitely countable $\Pi^1_1$ set of reals every member of which except one is $\Delta^1_2$ real.

Using the method developed in §4, we shall prove the following generalization of theorem 5.1.

**Theorem 5.2.** Assume that $\text{Det}(\Sigma^1_{2n+1})$ for $n > 0$. There is an infinitely countable $\Pi^1_{2n+1}$ set of reals every member of which except one is $\Delta^1_{2n+2}$ real.

**Proof.** Let $G$ be a $\Sigma^1_{2n+2}$ set in $\omega \times \omega$ which is universal for all $\Sigma^1_{2n+2}$ sets in $\omega$, $\emptyset$ a $\Pi^1_{2n+1}$ formula such that

$$\exists \beta \emptyset(\beta, i) \iff G(1, i).$$

Let $\emptyset <^{2n+2} \gamma(\beta, i)$ be the $\Sigma^1_{2n+2}$ formula which is obtained from the formula $\emptyset(\beta, i)$ by replacing quantifiers $\forall \delta$, $\exists \delta$ in $\emptyset$ by $\forall \delta <^{2n+2} \gamma$, $\exists \delta <^{2n+2} \gamma$ respectively, where $<^{2n+2}$ is a $\Sigma^1_{2n+2}$-good wellordering on $\mathcal{C}_{2n+2}$ such that

$$\beta <^{2n+2} \alpha \Rightarrow \beta \in \Delta^1_{2n+2}(\alpha).$$

**Lemma 5.3.** There is a real $\gamma$ such that if $\beta$ is a $\Delta^1_{2n+2}$ real then

$$\beta <^{2n+2} \gamma.$$
Proof is similar one of lemma 4.12. □

**Lemma 5.4.** Let $\gamma$ be such that if $\beta$ is a $\Delta^1_{2n+2}$ real then $\beta < \gamma$. Then following formula holds.

$$\exists \beta \in (\beta, i) \iff \exists \beta < 2^{n+2} \gamma \beta < 2^{n+2} \gamma (\beta, i).$$

Proof is similar one of lemma 4.11 of lemma 4.10. □

Put

$$A = \{ \xi \in \omega^2 : \forall i (\xi(i) = 1 \iff \exists \beta < 2^{n+2} \gamma \beta < 2^{n+2} \gamma (\beta, i)) \}. $$

Then $A$ is $\Sigma^1_{2n+2}$. Since the set $\{ i : G(i, i) \}$ is in $\Sigma^1_{2n+2} - \Delta^1_{2n+2}$, its characteristic function $\xi^*$ is in $(\Sigma^1_{2n+2}) - \Delta^1_{2n+2}$.

By lemmas 5.3, 5.4, there is a real $\gamma$ such that

$$\xi^*(i) = 1 \iff G(i, i) \iff \exists \beta \in (\beta, i) \iff \exists \beta < 2^{n+2} \gamma \beta < 2^{n+2} \gamma (\beta, i).$$

Thus $\xi^*$ is in $A$. By the basis theorem for $\Sigma^1_{2n+2}$, $A$ must have infinitely many $\Delta^1_{2n+2}$ reals. Let $\gamma_0$ be the smallest real such that

$$\beta \in \Omega^1_{2n+2} \Rightarrow \beta < 2^{n+2} \gamma_0$$

(such $\gamma_0$ exists by lemma 5.3). Then, by lemma 5.4, if $\gamma_0 < 2^{n+2} \gamma$

$$\exists \beta < 2^{n+2} \gamma \beta < 2^{n+2} \gamma (\beta, i) \iff \exists \beta \in (\beta, i).$$

Since $\xi^*$ is in $\Delta^1_{2n+3}$, $\gamma_0 < 2^{n+2} \xi^*$. Clearly between $\gamma_0$ and $\xi^*$ there is no elements of $A$.

\[\gamma_0 \quad \quad \quad \xi^*\]

In this interval, there is no element of $A$, i.e. the real $\xi^*$
Let $A^*$ be a $\prod_{2n+1}^{1}$ set in $\omega_\omega \times \omega_\omega$ such that
\[ \text{dom}(A^*) = A \]
and
\[ \forall \beta, \beta' (A^*(\alpha, \beta) \land A^*(\alpha, \beta') \Rightarrow \beta = \beta') \]
by the uniformization theorem such $A^*$ can be found. Now put
\[ A^{**} = h(A^*) \]
where $h$ is a recursive homeomorphism from $\omega_\omega \times \omega_\omega$ onto $\omega_\omega$. Let $\beta^*$ be the unique real such that
\[ A^*(\xi^*, \beta^*) \]
Since $\xi^*$ is $(\Sigma^1_{2n+2})^c = \Delta^1_{2n+2}$, $\xi^*$ is a $\Delta^1_{2n+3} = \Delta^1_{2n+2}$ real, so is $h(\xi^*, \beta^*)$ which is in $A^{**}$.
\[ A^{**}(\alpha) \land \alpha \neq h(\xi^*, \beta^*) \Rightarrow \exists \xi \neq \xi^* \exists \beta (\alpha = h(\xi, \beta) \land A^*(\xi, \beta) \land A(\xi)) \]
\[ \Rightarrow \alpha \in \bigcap_{2n+2}^{1} \]
Thus the $\prod_{2n+1}^{1}$ set $A^{**}$ has just one non-$\Delta^1_{2n+2}$ real $h(\xi^*, \beta^*)$
and other members of $A^{**}$ are all $\Delta^1_{2n+2}$ reals. Therefore theorem 5.2 is proved. □

From this theorem we have

**Corollary 5.5.** Effective perfect set theorem for $\Sigma^1_{2n+2}$ fail. □
Finally, we state one more theorem which is essentially included in the theorem 4.5, but it is interested itself.

**Theorem 5.6.** Assume that $\text{Det}(\Sigma_{2n+1}^1)$. Then there is a $\Pi_{2n+1}^1$ set in $\omega \times \omega$ which contains at least one non $\Pi_{2n+2}^1$ singleton real.

**Proof.** Let $G$ be a $\Sigma_{2n+2}^1$ set in $\omega \times \omega$ which is universal for all $\Sigma_{2n+2}^1$ sets in $\omega$. Now put

$$A = \left\{ \xi \in \omega^2 : \exists e \exists 1 \ (\xi(1) = 1 \iff G(e, 1)) \right\}.$$

Then applying the proof method of theorem 4.5 to the set $A$ using the fact

$$\emptyset \not\subseteq \left\{ \xi \in \omega^2 : \{\xi\} \in \Pi_{2n+2}^1 \land \xi \in A \right\} \not\subset A.$$

$\square$
§6. **Enumerability.**

We begin an application of theorem 5.2 to the problem of effective enumerability of countable projective sets of reals.

Since there is non-$\Delta^1_{2n+1}$ infinitely countable $\Sigma^1_{2n+1}$ set in $\omega$, the elements of a $\Sigma^1_{2n+1}$ set of reals are not necessarily enumerated by a $\Delta^1_{2n+1}$ function, but

**Theorem 6.1.** Assume that $\text{Det}(\Delta^1_{2n})$. Then the elements of a countable $\Delta^1_{2n+1}$ set in $\omega$ can be enumerated by a $\Delta^1_{2n+1}$ function. \(\Box\)

Assume that $\text{Det}(\Delta^1_{2n})$

**Theorem 6.2.** (Tanaka [40] for $n = 0$). An infinitely countable $\Sigma^1_{2n+1}$ set $P$ in $\omega$ cannot be contain $\Delta^1_{2n+1}$ reals of arbitrarily high degrees: that is, there is a $\Delta^1_{2n+1}$ real $\mathcal{E}$ such that

$\forall \alpha (P(\alpha) \Rightarrow \alpha$ is recursive in $\mathcal{E})$.

**Proof.** By Moschovakis [29; 4P.5], there is $\Delta^1_{2n+1}$ real $\mathcal{E}$ such that

$P \subseteq \{(\mathcal{E})_0, (\mathcal{E})_1, (\mathcal{E})_2, \cdots \}.$

Using this real $\mathcal{E}$ we have $(*).$ \(\Box\)

Is was a difficult work the one performs any enumeration of a countable $\Pi^1_1$ set in $\omega$. In fact it is undecidable in ZFC. But under the projective determinacy, we can prove, using theorem 5.2,

**Theorem 6.3.** Assume that $\text{Det}(\Sigma^1_{2n+1})$. Then there is a $\Pi^1_{2n+1}$ set in $\omega$ which cannot be enumerated by a $\Delta^1_{2n+2}$ function. \(\Box\)

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Therefor the following theorem is the best possible one.

**Theorem 6.4.** Assume that \( \text{Det}(\sum_{2n+1}^1) \). Then every infinitely countable \( \prod_{2n+1}^1 \) set in \( \omega_\omega \) can be enumerated without repetition by a \( \Delta_{2n+3}^1 \) function.

**Proof.** Let \( P \) be a \( \prod_{2n+1}^1 \) set in \( \omega_\omega \). Put
\[
P_0(\beta) \iff \forall i, j \ (i \neq j \Rightarrow (\beta)_i \neq (\beta)_j) \land \forall i \ (P((\beta)_i) \land \forall \alpha (P(\alpha) \Rightarrow \exists i \ (\alpha = (\beta)_i))).
\]

Then \( P_0 \) is a countable \( \Delta_{2n+3}^1 \) set, by the \( \Delta \) -uniformization criterion (Moschovakis [29; 4D.4]), we can find a \( \Delta_{2n+3}^1 \) set \( P_0^* \) such that
\[
\exists \beta \ P_0^*(\beta)
\]
and
\[
\forall \beta, \beta' \ (P_0^*(\beta) \land P_0^*(\beta') \Rightarrow \beta = \beta').
\]

Now we can define the function \( \Phi : \omega \rightarrow \omega_\omega \) by
\[
\Phi(i) = \alpha \iff \exists \beta (P_0^*(\beta) \land (\beta)_1 = \alpha ) \land \forall \beta (P_0^*(\beta) \Rightarrow (\beta)_1 = \alpha ) .
\]

Thus the function \( \Phi \) is in \( \Delta_{2n+3}^1 \) and enumerates without repetition the elements of \( P^* \). \( \square \)

Closing this section, we state an extension of Sampei [33] and Tanaka [39] theorem.
Theorem 6.5. (sampsol [33] and Tanaka [39] for n = 0). Assume that \( \text{Det}(\Delta_{2n}^1) \). Then every \( \Sigma_{2n+1}^1 \) set in \( \omega \omega \) can be enumerated by a \( \Delta_{2n+2}^1 \) function without repetition.

Proof is similar one of theorem 6.4, using the uniformization theorem for \( \Delta_{2n+2}^1 \) (see Kondō [21]) instead of the \( \Delta \)-uniformization criterion. \( \square \)
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