数理解析研究所講究録
540巻 1984年 1-7

A naive example of a proper Π_4^0 set of reals

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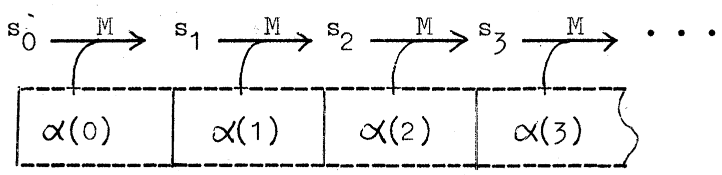
We show that a subset A of the product space 2^ω which is defined by

$A = \{\alpha \in 2^\omega :$ there are infinitely many integers $m \in \omega$ such that α contains infinitely many segments of the form $10^m1\}$

is proper Π_4^0.

§1. This is concerned with Landweber's work on finite automaton theory. A finite automaton $\mathfrak{M}$ consists of four elements S, $2 = \{0, 1\}$, s_0, and M. S is a finite set, whose elements are called states. s_0 is an element of S and is called the starting states. M is a function from $S \times 2$ into S, which is called a next state function. When a sequence $\alpha \in 2^\omega$ was given (which is identified with a data tape), the machine $\mathfrak{M}$ runs as illustrated below.



The sequence $\langle s_i \rangle_{i \in \omega}$ in the above is determined by α, we denote it by $\langle s_i(\alpha) \rangle_{i \in \omega}$.

§2. Let $D \subseteqq S$. Then $\mathcal{M}$ accepts α with respect to D if some member of D occurs in the sequence $\langle s_i(\alpha) \rangle_{i \in \omega}$, and rejects α otherwise. This condition of acceptance of a sequence is of the most standard type. Some other conditions have been studied by Hartmanis, Stearns, Büchi, McNaughton, etc. ([1], [2], [3], [4], [5], [6])

Let $D \subseteqq S$.

$\mathcal{M}$ 1-accepts α w.r.t. D if $\exists i\ s_i(\alpha) \in D$,

$\mathcal{M}$ 1'-accepts α w.r.t. D if $\forall i\ s_i(\alpha) \in D$,

$\mathcal{M}$ 2-accepts α w.r.t. D if $\exists^\infty i\ \ s_i(\alpha) \in D$,

where $\exists^\infty i$ means that there are infinitely many i's such that... .

Put $In(\alpha) = \{s \in S : \exists^\infty i\ \ s = s_i(\alpha)\}$. Let $\mathcal{D} \subseteq \mathcal{P}(S)$.

$\mathcal{M}$ 2'-accepts α w.r.t. $\mathcal{D}$ if $\exists D \in \mathcal{D}\ \ In(\alpha) \subseteqq D$,

$\mathcal{M}$ 3 -accepts α w.r.t. $\mathcal{D}$ if $\exists D \in \mathcal{D}\ \ In(\alpha) = D$.

Let i stand for 1, 1', 2, 2', or 3. $A \subseteqq 2^\omega$ is i-definable if there is $\mathcal{M}$ and D (or $\mathcal{D}$) such that $A = \{\alpha \in 2^\omega : \mathcal{M}$ i-accepts α w.r.t. D (or $\mathcal{D}$)$\}$. The following are easily observed:

<u>Proposition</u> (Landweber [6]). (1) Every 1-definable set is Σ^0_1.

(2) Every 1'-definable set is Π^0_1.

(3) Every 2 -definable set is Π^0_2.

(4) Every 2'-definable set is Σ^0_2.

(5) Every 3- definable set is Δ^0_3.

These estimations are known to be proper by the following examples:

<u>Examples</u> (Landweber [6]). Put $A^+ = \{\alpha \in 2^\omega$: in the sequence α, only a finite number of 1's occur$\}$. Then:

(1) A^+ is 2'-definable and in $\Sigma^0_2 - \underset{\sim}{\Pi}^0_2$.

(2) A^{+c} is 2-definable and in $\Pi^0_2 - \underset{\sim}{\Sigma}^0_2$.

(3) $A^\# = \{\alpha : \alpha(0) = 0\ \ \&\ \ \alpha \in A^+\} \cup \{\alpha : \alpha(0) = 1\ \ \&\ \ \alpha \in A^{+c}\}$ is 3-definable and in $\Delta^0_3 - (\underset{\sim}{\Sigma}^0_2 \cup \underset{\sim}{\Pi}^0_2)$.

<u>Problem</u> (Landweber [6]). Find a N A T U R A L condition for acceptance which enables finite machines to define sets above Δ^0_3.

To answer this question, it needs to find a set of reals in a higher hierarchy which is however easily handled by finite automaton. Unfortunately we could not find satisfactory examples of such sets in the literatures. So we give here such an example. Our answer to Landweber's problem is under improvemant, which will appear elsewhere in a satisfactory form.

§3. When a finite sequence $x \in 2^{\underset{\smile}{\omega}}$ is an initial segment of a sequence $y \in 2^{\underset{\smile}{\omega}} \cup 2^{\omega}$, then we write $x \prec y$. The concatenation of two sequences x and y is denoted by xy. Our results are as follows:

Put $A_0 = \{\alpha \in 2^{\omega}$: There is an $m \in \omega$ such that 0-blocks of length m (0^m-blocks) occur infinitely often in the sequence $\alpha\}$, i.e.:

$$\alpha \in A_0 \quad \text{iff} \quad (\exists m)(\exists^{\infty} x \in 2^{\underset{\smile}{\omega}}) \quad x10^m1 \prec \alpha .$$

<u>Theorem 1</u>. A_0 is in $\Sigma^0_3 - \underset{\sim}{\Pi}^0_3$.

Put $A_1 = \{\alpha \in 2^{\omega}$: There are infinitely many integers m such that 0^m-blocks occur infinitely often in the sequence $\alpha\}$, i.e. :

$$\alpha \in A_1 \quad \text{iff} \quad (\exists^{\infty} m) \text{ s. t. } (\exists^{\infty} x \in 2^{\underset{\smile}{\omega}}) \quad x10^m1 \prec \alpha .$$

<u>Theorem 2</u>. A_1 is in $\Pi^0_4 - \underset{\sim}{\Sigma}^0_4$.

Theorem 1 is essentially a part of Theorem 2. The rest of the paper is entirely devoted to the proof of Theorem 2.

§ 4. <u>Proof of Theorem 2</u>. Identifying the finite sequence 0^m1 with m for each integer m, we can translate the subset A_1 of 2^{ω} into the subset A of the space ω^{ω} as follows:

$$A = \{\alpha \in \omega^{\omega} : \exists^{\infty} m \text{ s.t. } \exists^{\infty} i \quad \alpha(i) = m\}.$$

Thus we show that A is in $\Pi_4^0 - \underset{\sim}{\Sigma}_4^0$. Since A is clearly Π_4^0 by the definition, it needs only to show $A \notin \underset{\sim}{\Sigma}_4^0$. For the purpose we use the following lemma:

<u>Lemma 1</u> (Landweber [6], in modified form). For every subset X of ω^ω, X is $\underset{\sim}{\Pi}_2^0$ iff $\exists B \subseteq \omega^{\underset{\smile}{\omega}}$ s.t. $X = \widehat{B}$, where $\widehat{B}$ is the set of all $\alpha \in \omega^\omega$ such that infinitely many segments of α belongs to B, i.e. : $\widehat{B} = \{\alpha \in \omega^{\underset{\smile}{\omega}} : (\exists^\infty y \in B)\ y \prec \alpha\}$.

To the end of contradiction we suppose $A \in \underset{\sim}{\Sigma}_4^0$, equivalently $A^c \in \underset{\sim}{\Pi}_4^0$. Then by the lemma there is an indexed family of sets of finite sequences $\{B_{l,i}\}_{l,i \in \omega}$ such that $A^c = \bigcap_{l \in \omega} \bigcup_{i \in \omega} \widehat{B_{l,i}}$.

<u>Definition</u>. Let $x \prec y$ and $J \subseteq \omega$, and z be the sequence satisfying $xz = y$. If in the sequence z, only the elements of J occur, we say that y is a J-restricted extension of x, or shortly y J-extends x, and write $x \xrightarrow{J} y$.

Notice that by the definition of A, A^c is the set of all α such that the set $\{m : m$ occurs infinitely often in $\alpha\}$ is finite.

<u>Lemma 2.</u> If a family of sets of finite sequences $\{C_i\}_{i \in \omega}$ satisfies that
$$A^c \subseteq \bigcup_{i \in \omega} \widehat{C_i}\,,$$
then

(*) for every x in $\omega^{\underset{\smile}{\omega}}$, each <u>infinite</u> set $I \subseteq \omega$, and each <u>finite</u> set $F \subseteq \omega$, we can take $x_0 \in \omega^{\underset{\smile}{\omega}}$ which $F \cup I$-extends x, a set C_i in $\{C_i\}_{i \in \omega}$, and an infinite set $I_1 \subseteq I$ such that for every x_1 which $F \cup I_1$-extends x_0, for every infinite set $J \subseteq I_1$, there is x^* which $F \cup J$-extends x_1 such that $x^* \in C_i$: in symbols,

$$(\forall x, F, I)\ \exists x_0 \xleftarrow[FI]{} x\ \ \exists I_1 \subseteq I\ \ \exists C_i \in \{C_i\}_{i \in \omega} \text{ such that}$$
$$\forall x_1 \xleftarrow[FI_1]{} x_0\ \ \forall J \subseteq I_1\ \ \exists x^* \xleftarrow[FJ]{} x_1\ \ x^* \in C_i\,.$$

<u>Proof of Lemma 2</u>. If not, we can obtain x, F, and I such that:

(#) $(\forall x_0 \xleftarrow[FI]{} x\ \ \forall I_1 \subseteq I\ \ \forall C_i$ in $\{C_i\}_{i \in \omega})\ \exists x_1 \xleftarrow[FI_1]{} x_0\ \ \exists J \subseteq I_1$ such that $\forall x^* \xleftarrow[FJ]{} x_1\ \ x^*$ is not in C_i.

We extends this x to $\alpha \in \omega^\omega$ as follows:

$$x \xrightarrow[FI]{} x_0 \xrightarrow[FJ_0]{} x_1 \xrightarrow[FJ_1]{} \ldots \quad x_i \xrightarrow[FJ_i]{} \ldots,$$

where infinite sets J_i, $i \in \omega$, are chosen so that:

($) $I \supsetneqq J_0 \supseteqq J_1 \supseteqq J_2 \supseteqq \cdots \supseteqq J_i \supseteqq \cdots$, and $J_n \cap \{0,1, \ldots, n\} = \emptyset$ for each n.

Moreover in addition x_i and J_i, $i \in \omega$, are chosen so as to satisfy that: $(\%)_0$ Every $F \cup J_0$-extension of x_0 is not in C_0,

$(\%)_1$ Every $F \cup J_1$-extension of x_1 is not in C_1,

...

$(\%)_i$ Every $F \cup J_i$-extension of x_i is not in C_i,

... .

This is clearly possible by the condition (#). Now put $\alpha = \bigcup_{i \in \omega} x_i$. Then by the condition $(\%)_i$, any initial segment of α which belongs to C_i must be shorter than x_i for each i, so $\alpha \notin \bigcup_{i \in \omega} \widehat{C_i}$. But by the condition ($), for every m in the co-finite set $\omega - F$, all occurrences of m in the sequence α are in the initial segment x_m. This means that α is in A^c; a contradiction. Lemma 2 is thus proved.

In the present case, $A^c \subseteqq \bigcup_{i \in \omega} B_{l,i}$ for every l. Hence by this lemma, the following holds for every l:

$(@)_l$ $(\forall x, F, I)\ \exists x_0 \xleftarrow[FI]{} x\ \exists B_{l,i} \in \{B_{l,i}\}_{i \in \omega}\ \exists I' \subseteqq I$ such that $(\forall x_1 \xleftarrow[FI']{} x_0)(\forall J \subseteqq I')\ \exists x^* \xleftarrow[FJ]{} x_1$ s.t. $x^* \in B_{l,i}$.

Using this, we construct α which is in $\bigcap_{l \in \omega} \bigcup_{i \in \omega} B_{l,i}$ but not in A^c. Starting with null sequence λ, we repeat extensions of finite sequences in the following manner:

$$\lambda\ (= y_{-1}) \xrightarrow[\omega(= F_{-1} \cup I_{-1})]{} x_0 \xrightarrow[F_0 I_0]{} y_0 \xrightarrow[F_0 I_0]{} x_1 \xrightarrow[F_1 I_1]{} y_1 \xrightarrow[F_1 I_1]{} x_2 \cdots,$$

where (1) $F_0 \cup I_0 \supseteqq F_1 \cup I_1 \supseteqq F_2 \cup I_2 \supseteqq \cdots$ and

(2) $F_0 \subsetneqq F_1 \subsetneqq F_2 \subsetneqq \cdots$.

I. Stage to define $x_\ell \xleftarrow[F_{\ell-1}\cup I_{\ell-1}]{} y_{\ell-1}$.

The extension x_ℓ is taken together with a set $I_\ell \subsetneqq I_{\ell-1}$ and a set $B_{\ell,i_\ell} \in \{B_{\ell,i}\}_{i\in\omega}$ so that :

$(¥)_\ell$ $(\forall x \xleftarrow[F_{\ell-1}I_\ell]{} x_\ell)(\forall J \subsetneqq I_\ell)\ \exists y \in B_{\ell,i_\ell}$ s.t. $y \xleftarrow[F_{\ell-1}J]{} x$.

This is possible by the condition $(@)_\ell$. We put :

$F_\ell = F_{\ell-1} \cup \{\min(I_\ell - F_{\ell-1})\}$. Hence:

(3) $F_{\ell-1} \cup I_\ell = F_\ell \cup I_\ell \supsetneqq \cdots \supsetneqq F_m \cup I_m \supsetneqq \cdots$.

Notice by the above conditions $(¥)_\ell$, (1), and (3) that:

$(\ddagger)_\ell^m$ Every x_m with $m > l$ or its any $F_m \cup I_m$-extension can be $F_m \cup I_m$-extended to a member of B_{ℓ,i_ℓ}.

II. Stage to define $y_\ell \xleftarrow[F_\ell I_\ell]{} x_\ell$.

By the help of the conditions $(\ddagger)_0^\ell, (\ddagger)_1^\ell, \ldots, (\ddagger)_\ell^\ell$, we repeat extensions ℓ times starting from x_ℓ as follows:

$$x_\ell \xrightarrow[F_\ell I_\ell]{} y_{\ell,0} \xrightarrow[F_\ell I_\ell]{} y_{\ell,1} \xrightarrow[F_\ell I_\ell]{} \cdots \xrightarrow[F_\ell I_\ell]{} y_{\ell,\ell},$$

where $y_{\ell,j}$ is a member of B_{j,i_j} for each $j \leqq \ell$. Now we $F_\ell \cup I_\ell$-extend $y_{\ell,\ell}$ to y_ℓ so that all members of F_ℓ occur in the newly added part; for example, put $y_\ell = y_{\ell,\ell}k_0k_1\ldots k_l$, the concatenation of $y_{\ell,\ell}$ and a list of all members of $F_\ell = \{k_0, k_1, \ldots, k_\ell\}$.

The construction is completed. Put $\alpha = \bigcup_{\ell\in\omega} x_\ell$. Let $j \in \omega$. Then for every ℓ larger than j, the sequence $y_{\ell,j}$ belongs to B_{j,i_j}; thisshows $\alpha \in \widehat{B}_{j,i_j}$. Thus $\alpha \in \bigcap_{\ell\in\omega}\bigcup_{i\in\omega}\widehat{B}_{\ell,i}$.

On the other hand, for each $\ell \in \omega$, the segment z_ℓ that satisfies $x_\ell z_\ell = y_\ell$ has an occurrence of each element of F_ℓ. Since $\{F_\ell\}_{\ell\in\omega}$ is strictly increasing, every element of infinite set $\bigcup_{\ell\in\omega}F_\ell$ occurs infinitely often in the sequence α, that means $\alpha \notin A^c$; a contradiction. Theorem is thus proved.

R e f e r e n c e s

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