THE CAUCHY PROBLEM FOR THE
COUPLED MAXWELL-SCHRÖDINGER EQUATIONS

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Abstract: The Cauchy problem for the coupled Maxwell-Schrödinger equations in $\mathbb{R}^d$ in the Lorentz gauge is considered. The viscosity method is used to establish local existence. In one and two space dimensions, global solutions are obtained.

1. Introduction

Due in part to the developments of lasers, there has been a revived interest in the theory of the interaction of the radiation and non-relativistic charged particles in recent years [3]. In this paper we shall study the Cauchy problem for the closely related minimally coupled Maxwell-Schrödinger equations, by specializing to the Lorentz gauge. These equations are the classical approximation to the quantum field equations for an electrodynamical non-relativistic many body system [7], and may be written as

\begin{align}
(1.1) & \quad \partial_\mu F_{\mu \nu} = J_\nu, \quad F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \\
(1.2) & \quad (iD_0 + D_j D_j)\psi = V\psi, \quad D_\mu = \partial_\mu - iA_\mu,
\end{align}

with the components $A_\mu(t,x)$'s of the electromagnetic real vector poten-
tial and the complex scalar field $\psi(t,x)$ of non-relativistic charged particles. Here $x \in \mathbb{R}^d$, $\mu, \nu$ run over $0, 1, \ldots, d$, $j$ ranges over $1, \ldots, d$ (repeated indices always imply summation), $\partial^0 = \partial_t = \partial/\partial t$, $(-\partial^1, \ldots, -\partial^d) = (\partial_1, \ldots, \partial_d) = \nabla$, $V = V(x)$ is a given real external potential, and the $J_\nu$ are the charge-current densities given by

$$J_0 = - \bar{\psi} \psi, \quad J_j = - i (\bar{\psi} D_j \psi - \psi D_j \bar{\psi}), \quad j = 1, \ldots, d.$$ 

The Lorentz gauge condition is expressed as

$$(1.3) \quad \partial^\mu A_\mu = 0.$$ 

In Sect. 2, we shall show that the Cauchy problem for Eqs. (1.1)–(1.3) with $d \geq 1$ has a local solution, provided the initial data and the external potential $V$ are sufficiently regular. Our local existence argument uses the viscosity method (see e.g. [4][8][9]) to deal with the difficulty arising from the presence of highly singular derivative coupling terms in the Schrodinger equation, which is an unwelcome feature of the equations.

In Sect. 3, we prove the global existence of solutions in the cases of one and two space dimensions. The needed a priori estimates for the solutions will be obtained by using the energy method in the form developed in [5], together with the covariant Sobolev inequalities.

Throughout this paper we shall use Greek indices $\mu, \nu, \ldots$ to run from 0 to $d$ and Latin indices $j, k, \ldots$ to run from 1 to $d$, and the
summation convention for both types of indices. We use the standard notation $H^s$ for the Sobolev space of order $s$ and exponent 2. If $X$ is a normed space, we write $\| \cdot \|_X$ for its norm, and if $X$ is also an inner product space, $(\cdot, \cdot)_X$ for its inner product. The $L^p$-norm will be denoted simply by $\| \cdot \|_p$.

2. Local Existence

Introducing the momenta $P_{\mu} = \partial_0 A_{\mu}$, and the vector notation $f = (f_1, \ldots, f_d)$, we write Eq. (1.1)-(1.3), with initial data $A_0^0, P_0^0, \Lambda^0, \Gamma^0, \psi^0$, in the form

\begin{align}
(2.1) & \quad \frac{du}{dt} + 2u = K(u), \quad u = (A_0, P_0, \Lambda, \Gamma, \psi) \\
(2.2) & \quad P_0 - \partial_j A_j = 0
\end{align}

\begin{align*}
u(0) = u_0 = (A_0^0, P_0^0, \Lambda^0, \Gamma^0, \psi^0),
\end{align*}

where the components $A_{\mu}, P_{\mu}$, and $\psi$ of the unknown $u$ take values in $L^2$, and the operator $Z$ and the function $K(u)$ are respectively defined by

\begin{align*}
Zu = (-P_0, -\Delta A_0, -\Gamma, -\Delta \Lambda, -i\psi),
\end{align*}

\begin{align*}
K(u) = (0, J_0, 0, \hat{\mathcal{J}}, -iV\psi + iA_0\psi + P_0\psi + 2A_j\partial_j\psi - iA_jA_j\psi),
\end{align*}

the indicated differential operators being defined by Fourier trans-
formation. Use has been made of the side condition Eq.(1.3) to convert Eqs.(1.1)-(1.2) into Eq.(2.1). Eqs.(2.1)-(2.2) impose the following initial value constraints on the components of the data \( u_0 \):

\[
(2.3) \quad p_0^0 - \partial_j A_j^0 = 0, \quad \partial_j p_j^0 - \Delta A_0^0 + \overline{\psi_0} \psi_0 = 0.
\]

Next we form for \( s \geq 1 \) the direct sum Hilbert spaces

\[
X^s = H^{s+1}(\mathbb{R}^d; \mathbb{R}) \oplus H^{s}(\mathbb{R}^d; \mathbb{R}) \oplus H^{s}(\mathbb{R}^d; \mathbb{R}^d) \oplus H^{s-1}(\mathbb{R}^d; \mathbb{R}^d) \oplus H^{s}(\mathbb{R}^d; \mathbb{C}),
\]

\[
Y^s = H^{s+1}(\mathbb{R}^d; \mathbb{R}) \oplus H^{s}(\mathbb{R}^d; \mathbb{R}) \oplus H^{s}(\mathbb{R}^d; \mathbb{R}^d) \oplus H^{s-1}(\mathbb{R}^d; \mathbb{R}^d) \oplus H^{s-1}(\mathbb{R}^d; \mathbb{C}).
\]

We now state the main result of this section. We will denote by \([p]\) the integer part of \( p \in \mathbb{R} \). Let \( d \geq 1 \).

**Theorem 2.1.** Let \( m \) be an integer satisfying \( m \geq \left[ \frac{d}{2} + 2 \right] \), and let \( V \in H^m(\mathbb{R}^d; \mathbb{R}) \). Let \( u_0 \) be any initial data lying in \( X^m \), not necessarily satisfying (2.3). Then Eq.(2.1) has a unique solution \( u \) on \([0, T)\) for some \( T, 0 < T \leq \infty \), such that \( u \in C([0, T); X^m) \cap C^1([0, T); Y^{m-1}) \) and \( u(0) = u_0 \), where we may assume that either \( T = \infty \) or \( \lim_{t \to T} \|u(t)\|_{X^m} = \infty \); the solution \( u \) depends continuously on the initial data \( u_0 \), in the sense that if \( \|u\|_{X^m} \leq C \) on \([0, T')\) for some fixed \( C \), \( T' > 0 \) when \( u_0 \) converges to some \( u_0^\prime \in X^m \) in \( X^m \), then the solution \( u' \) of Eq.(2.1) corresponding to the data \( u_0^\prime \) also exists on \([0, T')\), and \( u \) converges to \( u' \) weakly in \( X^m \) uniformly on \([0, T')\). Furthermore, if in addition the data \( u_0 \) satisfies the initial value constraints (2.3), the solution \( u \) satisfies Eq.(2.2).
The proof of Theorem 2.1 depends on

**Lemma 2.2.** Let \( m \) and \( V \) be as in Theorem 2.1, and let \( 0 \leq \alpha \leq 1 \). Then

\[
(2.4) \quad \|K(u) - K(u')\|_{X^{m-\alpha}} \leq \omega(\|u\|_{X^{m-\alpha}}, \|u'\|_{X^{m-\alpha}})\|u-u'\|_{X^{m-\alpha}}, \quad u, u' \in X^{m-\alpha},
\]

\[
(2.5) \quad \text{Re}(K(u), u)_{X^m} \leq c(1 + \|u\|_{X^{[\frac{d}{2}+2]}} + \|u'\|_{X^{[\frac{d}{2}+2]}}^{2})\|u\|_{X^m}, \quad u \in X^{m+1},
\]

\[
(2.6) \quad \text{Re}(K(u) - K(u'), u - u')_{X^{m-1}} \leq \omega(\|u\|_{X^m}, \|u'\|_{X^m})\|u-u'\|_{X^{m-1}}^{2}, \quad u, u' \in X^m,
\]

where \( \omega(a, b) = c(1 + a + b + a^2 + b^2) \).

**Proof.** The inequality (2.4) immediately follows from the multiplication lemma:

\[
(2.7) \quad \|fg\|_{H^{s}} \leq c \|f\|_{H^{s_1}} \|g\|_{H^{s_2}} \quad \text{if } s_1, s_2 \geq s \geq 0, \text{ and } s_1 + s_2 - \frac{d}{2} > s,
\]

which implies that \( \|fg\|_{H^{s-1}} \leq c \|f\|_{H^{s}} \|g\|_{H^{s-1}} \) for \( s > \frac{d}{2} \).

In proving (2.5) and (2.6), we may assume that \( u \) and \( u' \) are \( C_0^\infty \)-vectors in view of (2.4). Let \( m \) be an integer satisfying \( m \geq [\frac{d}{2} + 2] \).

To show (2.5), we make use of the following inequality:

\[
(2.8) \quad \|\partial^\alpha(fg) - f\partial^\alpha g\|_2 \leq c(\|f\|_{H^{m-1}} \|g\|_{H^{[\frac{d}{2}+1]}} + \|f\|_{H^{[\frac{d}{2}+2]}} \|g\|_{H^{m-1}}), \quad f, g \in C_0,
\]

where \( |\alpha| \leq m \). For \( |\alpha| \leq \frac{d}{2} + 1 \), (2.8) (the right side becomes simpler in this case) results from (2.7) after an application of Leibnitz rule. For
the case \( \frac{d}{2} + 1 < |\alpha| \leq m \), see e.g. [6]. Now let \( k \) be any integer satisfying \( 0 \leq k \leq m \). Then (2.8) in particular implies that

\[
(2.9) \| f \|_{H_k^m} \leq c \left( \| f \|_{H_{m-1}^{d+1}} + \| f \|_{H_{m+1}^{d+2}} \| g \|_{H^k} + \| f \|_{H_{m+1}^{d+1}} \| g \|_{H^k} \right),
\]

since for \( |\alpha| = k \), \( \| f^{\alpha} \|_{L^2} \leq \| f \|_{H^k} \| g \|_{H^k} \leq c \| f \|_{H_{m+1}^{d+1}} \| g \|_{H^k} \). A repeated application of (2.9) yields

\[
(2.10) \| f_1 f_2 f_3 \|_{H_k^m} \leq c \sum_{j_1,j_2,j_3} \| f_{j_1} \|_{H_{m+1}^{d+2}} \| f_{j_2} \|_{H_{m+1}^{d+2}} \| f_{j_3} \|_{H_{j_3}^m} \| f_{j_1} f_2 f_3 \|_{C_0},
\]

where the sum is taken over all cyclic permutations \((j_1,j_2,j_3)\)'s of \((1,2,3)\). Except for the term \( \text{Re}(A_{\alpha} \psi, \psi)_{H^m} \), the inequality (2.5) can be proved by using (2.9) and (2.10) in the left hand side, after an application of Schwartz inequalities. To estimate the term \( \text{Re}(A_{\alpha} \psi, \psi)_{H^m} \), we shall use (2.8) directly. We write

\[
(2.11) \text{Re}(A_{\alpha} \psi, \psi)_{H^m} = \text{Re}(\alpha A_{\alpha} \psi, \psi)_{L^2} + \text{Re}(A_{\alpha} \psi, \psi)_{L^2},
\]

Applying (2.8) to the first term on the right side in the obvious manner, and noting in the second term that, by an integration by parts,

\[
\text{Re}(A_{\alpha} \psi, \psi)_{L^2} = -\frac{1}{2} (A_{\alpha} A_{\alpha} \psi, \psi)_{L^2},
\]

and that \( \| A_{\alpha} \|_\infty \leq c \| A_{\alpha} \|_{H_{m+1}^{d+1}} \), we find that the right side of (2.11) is bounded by a constant times \( \| u \|_{X_{d+2}^m}^2 \). Thus (2.5) follows.

The proof of the inequality (2.6) is, except for the term
Re(\(A_j \partial_j \psi - A'_j \partial_j \psi', \psi - \psi'\))_{H^{m-1}}, \text{ straightforward, and in fact reduces to that of (2.4)}. \text{ We now write}

\[
\text{Re}(A_j \partial_j \psi - A'_j \partial_j \psi', \psi - \psi')_{H^{m-1}} = \text{Re}((A_j - A'_j) \partial_j \psi', \psi - \psi')_{H^{m-1}} + \sum_{\alpha, \beta, \gamma} \frac{c_{\alpha \beta}}{\| \alpha \|_{X^{m-1}} \| \beta \|_{X^m}} \langle \partial_j \psi - \partial_j \psi', \partial_j \psi' \rangle_{L^2}^2
\]

Using (2.7) in the first and the second term on the right side after the application of Schwartz inequalities, and estimating the last term in the same way as in the last part of the proof of (2.5), one finds that the right side of the above equality is bounded by a constant times

\[
\|u\|_{X^m} + \|u'\|_{X^m}\|u - u'\|_{X^{m-1}}^2, \text{ and obtains (2.6)}.
\]

Proof of Theorem 2.1. \text{ Let } m \text{ and } V \text{ be as in the statements of Theorem 2.1, and assume first that the initial data } u_0 \text{ is an arbitrary element of } X^m. \text{ In order to construct the solution of Eq. (2.1) corresponding to the data } u_0, \text{ we shall introduce the following approximate Cauchy problem:}

\[
(2.12) \quad \frac{du}{dt} + B_\varepsilon u = F(u),
\]

\[ u(0) = u_0, \]

where \( \varepsilon > 0, \text{ and } B_\varepsilon = \varepsilon T + S, F(u) = -Mu + K(u), \text{ with } T = I - \Delta \) and

\[ Mu = (-P_0, -\Delta A_0, -\hat{P}, -\Delta \hat{A}, 0), \quad Su = (0, 0, 0, 0, -i\Delta \psi), \]
(so that $M + S = 2$). Note that the norms $\| \cdot \|_{X^{1+2\alpha}}^{1+2\alpha}$ and $\| B_\epsilon^\alpha(\cdot) \|_{X^1}$ are equivalent for each $\alpha \geq 0$, $B_\epsilon^\alpha$'s being the fractional powers of $B_\epsilon$. The operator $B_\epsilon^\alpha$ generates a holomorphic semigroup on $X^{[\frac{d}{2}+1]}$. On the other hand, from (2.4) we have

$$\| F(u) - F(u') \|_{X^s} \leq \omega(\| u \|_{X^{s+1}}, \| u' \|_{X^{s+1}}) \| u - u' \|_{X^{s+1}}$$

for any $s \geq \left[ \frac{d}{2} + 1 \right]$, where $\omega(a, b) = c(1 + a + b + a^2 + b^2)$. Thus, by the well established theory of semilinear parabolic equations (see e.g. [1]), Eq. (2.12) has a unique solution $u_\epsilon$ on $[0, T_\epsilon)$, for some $0 < T_\epsilon = \infty$, such that $u_\epsilon \in C([0, T_\epsilon); X^m) \cap C^1([0, T_\epsilon); X^{m-1})$ and $u_\epsilon(0) = u_0$, and here we may assume that either $T_\epsilon = \infty$ or $\lim_{t \to T_\epsilon} \| u_\epsilon(t) \|_{X^{[\frac{d}{2}+2]}} = \infty$.

We shall now consider the convergence of $u_\epsilon$, $\epsilon > 0$, in the limit $\epsilon \to 0$. Taking the $X^m$-inner product of Eq. (2.12) for $u_\epsilon$ with $u_\epsilon$ and adding the complex conjugate of the result, we have

$$(2.13) \quad \frac{1}{2} \frac{d}{dt} \| u_\epsilon \|_{X^m}^2 + \epsilon \langle Tu_\epsilon, u_\epsilon \rangle_{X^m} = \text{Re}(J(u_\epsilon), u_\epsilon)_{X^m}.$$ 

Using (2.5) and noting that the second term on the left side is non-negative, we obtain

$$\frac{d}{dt} \| u_\epsilon \|_{X^m} \leq P(\| u_\epsilon \|_{X^{[\frac{d}{2}+2]}}) \| u_\epsilon \|_{X^m},$$

where $P(a) = c(1 + a + a^2)$. It follows that

$$(2.14) \quad \| u_\epsilon(t) \|_{X^m} \leq \| u_\epsilon(0) \|_{X^m} \exp \left[ \int_0^t P(\| u_\epsilon(s) \|_{X^{[\frac{d}{2}+2]}}) ds \right] \quad \text{on } [0, T_\epsilon).$$
With the solution \( b \) of the scalar Cauchy problem

\[
\frac{db}{dt} = p(b), \quad b(0) = L \geq \|u_0\|_{X^{[2,2]}}^2,
\]

which exists and is bounded on a time interval \([0, T_0]\), \( T_0 = T_0(L) > 0 \), it also follows that \( \|u_\varepsilon(t)\|_{X^{[2,2]}} \leq b(t) \) on \([0, T_\varepsilon) \cap [0, T_0] \), from which we may assume that \( T_\varepsilon > T_0 \). Then by (2.14),

\[
(2.15) \quad \|u_\varepsilon(t)\|_{X^m} \leq \|u_0\|_{X^m} e^{Ct} \quad \text{on} \quad [0, T_0],
\]

where \( C \) is a positive constant independent of \( \varepsilon \).

Next let \( 0 < \varepsilon_1 < \varepsilon_2 \), and put \( w = u_\varepsilon_1 - u_\varepsilon_2 \). From (2.12) we have

\[
(2.16) \quad \frac{dw}{dt} + B_{\varepsilon_1} w = (\varepsilon_2 - \varepsilon_1) T_{\varepsilon_2} - Mw + J(u_\varepsilon_1) - J(u_\varepsilon_2).
\]

Taking the \( X^{m-1}\)-inner product of this equation with \( w \) and adding the complex conjugate of the result, we get

\[
\frac{1}{2} \frac{d}{dt} \|w\|_{X^{m-1}}^2 + \varepsilon_1 (T_{\varepsilon_1} w, w)_{X^{m-1}}
\]

\[
= (\varepsilon_2 - \varepsilon_1) \text{Re}(T_{\varepsilon_1} w, w)_{X^{m-1}} + \text{Re}(J(u_\varepsilon_1) - J(u_\varepsilon_2), w)_{X^{m-1}}.
\]

Noting that the second term on the left side is non-negative and using (2.6) and (2.15), we obtain

\[
(2.17) \quad \frac{1}{2} \frac{d}{dt} \|w\|_{X^m}^2 \leq (\varepsilon_2 - \varepsilon_1) \|u_\varepsilon_2\|_{X^m} \|w\|_{X^m} + \omega(\|u_\varepsilon_1\|_{X^m}, \|u_\varepsilon_2\|_{X^m}) \|w\|_{X^{m-1}}^2
\]
\[ \leq c\varepsilon_2 + c\|w\|_{X^{m-1}}^2. \]

Application of Gronwall's inequality then gives \[ \|w(t)\|_{X^{m-1}}^2 \leq c\varepsilon_2 \] on \([0,T_0]\) since \(w(0) = 0\). Thus by letting \(\varepsilon_2 \to 0\), we find a function \(u \in C([0,T_0]; X^{m-1})\) such that \(u_\varepsilon \to u\) in \(X^{m-1}\) uniformly on \([0,T_0]\) as \(\varepsilon \to 0\). By (2.15), it also follows that \(u \in L^\infty([0,T_0]; X^m)\) with \[ \|u(t)\|_{X^m} \leq \|u_0\|_{X^m} e^{Ct}, \] that \(u_\varepsilon \to u\) weakly in \(X^m\) uniformly on \([0,T_0]\) as \(\varepsilon \to 0\), and that \(u\) is weakly continuous from \([0,T_0]\) to \(X^m\).

Now let \(\phi\) be a smooth element of \(Y^{m-1}\). Then we have

\[-B_\varepsilon u_\varepsilon + F(u_\varepsilon), \phi\]_{Y^{m-1}} = -\varepsilon(u_\varepsilon, T\phi)_{Y^{m-1}} + (u_\varepsilon, Z\phi)_{Y^{m-1}} + (J(u_\varepsilon), \phi)_{Y^{m-1}}.\]

This together with (2.4) implies that \((-B_\varepsilon u_\varepsilon + F(u_\varepsilon), \phi)_{Y^{m-1}}\) converges to \((-Zu + J(u), \phi)_{Y^{m-1}}\) uniformly on \([0,T_0]\). Thus, integrating the equality \((-\frac{d}{dt} - B_\varepsilon u_\varepsilon + F(u_\varepsilon), \phi)_{Y^{m-1}} = 0\) on a time interval in \([0,T_0]\), and changing the order of the inner product and the time integral after taking the limit \(\varepsilon \to 0\), we find that \(u\) is a solution of Eq. (2.1) lying in the class \(C([0,T_0]; X^{m-1}) \cap L^\infty([0,T_0]; X^m)\). By taking the \(X^{m-1}\)-inner product of Eq. (2.1) with \(u\) and using (2.6) and Gronwall's inequality, we also find that the solution \(u\) is unique in this class. Note that \(u\) is strongly continuous in \(X^m\) at \(t = 0\) since \(u\) is weakly continuous in \(X^m\) at \(t = 0\) and \(\limsup_{t \to 0} \|u(t)\|_{X^m} \leq \|u_0\|_{X^m}\). The fact that Eq. (2.1) is time translational then implies that the solution corresponding to the initial data \(u(t_0)\), given at \(t = t_0 > 0\), is also right continuous in \(X^m\) at \(t = t_0\). By the above uniqueness result, it follows that \(u\) is right continuous in \(X^m\) at any \(t\) in \([0,T_0]\). Since Eq. (2.1) is also time reversion.
sible (in a suitable sense), we deduce that \( u \in C([0, T_0]; X^m) \). Note that
the choice of \( T_0 \) was uniform for all initial data \( u_0 \) satisfying
\[
\|u_0\|_{X^{d+2}} \leq L,
\]
for each fixed \( L > 0 \). Thus the above solution \( u \) extends
to some larger interval \([0, T)\) in such a way that \( u \in C([0, T); X^m) \) with
either \( T = \infty \) or \( \lim_{t \to T} \|u(t)\|_{X^{d+2}} = \infty \). From the equation, it also
follows that \( u \in C^1([0, T); Y^{m-1}) \).

To prove the continuous dependence of the solution \( u \) of Eq.(2.1) on
the initial data \( u_0 \), let \( \{u_{n0}\}_{n=1}^{\infty} \) be a sequence in \( X^m \) that converges to
\( u_0 \) in \( X^m \), and let \( u_n \) be the solution of Eq.(2.1) corresponding to the
initial data \( u_{n0} \), for each \( n \). Suppose that \( \{u_n\} \) satisfies
\[
\|u_n\|_{X^{d+2}} \leq C
\]
on \([0, T']\) for some constants \( C, T' > 0 \) independently of \( n \). Now an
argument similar to that which led from (2.13) to (2.15) (but now setting
\( \varepsilon = 0 \) and replacing \( u_0 \) by \( u_{n0} \)) shows that
\[
\|u_n\|_{X^m} \leq \|u_{n0}\|_{X^m} e^{Ct}
\]
on \([0, T']\) for some constant \( C > 0 \) independent of \( n \). An analysis similar to that
which led from (2.17) to (2.18) then shows that \( \|u_n(t) - u_{n',}(t)\|_{X^{m-1}} \leq C \|u_{n0} - u_{n'0}\|_{X^{m-1}} \)
on \([0, T']\) again for some constant \( C > 0 \) independent of
\( n \) and \( n' \). Then by the same argument as above, we deduce that \( u_n \to u \) in
\( X^m \) uniformly on \([0, T']\).

It remains to show the last part of the theorem. To see this,
assume further that \( u_0 \) satisfies (2.3), and let \( u = (A_0, P_0, \Phi, \bar{\psi}) \) be
the corresponding solution of Eq.(2.1). Put
\[
f = P_0 - \partial_j A_j, \quad g = \partial_j P_j - \Delta A_0 + \bar{\psi} \psi.
\]

From Eq.(2.1) we have

\[
/ / \]
\[ \frac{df}{dt} = -g, \quad \frac{dg}{dt} = -\Delta f + 2\overline{\psi}\psi f. \]

Using these equation, we obtain

\[ \frac{1}{2} \frac{d}{dt} \left( \|f\|_2^2 + \|\nabla f\|_2^2 + \|g\|_2^2 \right) = \int_{\mathbb{R}^d} (2\overline{\psi}\psi - 1)fg \, dx \]

\[ \leq \left\{ \frac{1}{2} + \|\psi\|_\infty^2 \right\} \left( \|f\|_2^2 + \|g\|_2^2 \right), \]

on the interval of existence \([0,T)\) of \(u\). Since \(\|\psi\|_\infty \leq \|\psi\|_{H^{d+1}_2} \leq c(T')\) on each \([0,T']\), \(T' < T\), and \(f(0) = g(0) = 0\) by assumption, if follows that \(\|f\|_2^2 + \|\nabla f\|_2^2 + \|g\|_2^2 = 0\) on \([0,T)\). Thus \(f = 0\), which is the desired result.

3. Global Existence in One and Two Space Dimensions

In this section we shall prove

Theorem 3.1. Let \(d \in \{1,2\}\), \(m\) an integer satisfying \(m \geq \left[ \frac{d}{2} + 4 \right]\), and \(V \in H^m(\mathbb{R}^d; \mathbb{R})\). Let \(u_0\) be any initial data lying in \(X^m\) and satisfying the constraints (2.3). Then Eqs. (2.1)-(2.2) has a unique solution \(u\) on \([0,\infty)\) such that \(u \in C([0,\infty); X^m) \cap C^1([0,\infty); Y^{m-1})\) and \(u(0) = u_0\).

The proof of Theorem 3.1 depends on the energy and the charge conservation laws, which we shall state here as a lemma. Note that we have the identities
\[ \partial_{\mu}(fg) = D_{\mu}fg + fD_{\mu}g, \quad D_{\mu}(fg) = \partial_{\mu}fg + fD_{\mu}g, \]

\[ D_{\mu}D_{\nu}f = D_{\nu}D_{\mu}f + iF_{\nu\mu}f, \]

\[ \partial_{\mu}F_{\nu\lambda} + \partial_{\nu}F_{\lambda\mu} + \partial_{\lambda}F_{\mu\nu} = 0, \]

whenever \( f, g \) and the \( A_{\mu} \) are smooth in \((t,x)\) and the \( A_{\mu} \) are real valued.

**Lemma 3.2.** Let \((A_0, A_1, \ldots, A_d, \psi)\) be a smooth solution of Eqs. (1.1)–(1.2) with \( V \) smooth, and assume that \( A_0, A_1, \ldots, A_d, \psi, V \) and their derivatives (of suitable order) are square integrable on \( \mathbb{R}^d \). Then the energy \( E_1 \) and the charge \( Q \) of the solution, that is,

\[ E_1 = \int_{\mathbb{R}^d} \left\{ D_j \psi \bar{D}_j \psi + \nabla \psi \bar{\nabla} \psi + \frac{1}{2} F_{\mu0} F_{\mu0} + \frac{1}{4} F_{jk} F_{jk} \right\} dx, \]

\[ Q = \int_{\mathbb{R}^d} \psi \bar{\psi} dx, \]

are finite constant functions of time.

**Proof.** In fact, we have from the equations that \( \frac{d}{dt} E_1 = \frac{d}{dt} Q = 0 \). The proof is facilitated by using the above identities for the \( D_{\mu} \) and the \( F_{\mu\nu} \).

**Proof of Theorem 3.1.** We shall prove the theorem under slight weaker assumptions. We replace the condition \( m \geq \left[ \frac{d}{2} + 4 \right] \) by \( m \geq \left[ \frac{d}{2} + 2 \right] \), and assume the existence of a sequence \( \{u_{n0}\} \) of initial data in \( X^k \), with \( k \geq \max(m, \left[ \frac{d}{2} + 4 \right] \) \), such that \( u_{n0} \xrightarrow{\ast} u_0 \) in \( X^m \), and that each \( u_{n0} \)
satisfies the constraints (2.3). Let \( \{ V_n \} \) be a sequence of external potentials in \( H^k \otimes \mathbb{R}^d \), \( k \) being as above, that converges to \( V \) in \( H^m \), and for each \( n \), \( u_n \in C([0,T_n]; X^k) \) the solution of Eqs. (2.1)-(2.2) corresponding to the initial data \( u_n0 \) and the external potential \( V_n \), given by Theorem 2.1. We shall show that for such \( \{ u_n \} \), there is a locally bounded function \( C(\cdot) \) on \([0,\infty)\) which can be chosen independently of \( n \), such that \( \| u_n(t) \|_{X^{d/2} + 2} \leq C(t) \) on \([0,T_n]\). One can then show, by an obvious change of the proof of the previous result on continuous dependence of solutions on initial data (to include the dependence on the external potential) that, for every \( T > 0 \), the solution \( u \) of Eqs. (2.1)-(2.2) corresponding to the initial data \( u_0 \) and the external potential \( V \) exists in \( C([0,T]; X^m) \) as the uniform limit of \( \{ u_n \} \) on \([0,T] \) in the weak topology of \( X^m \), and thus conclude the desired global existence result.

To derive the above estimate, we will use the covariant Sobolev inequality (see e.g. [2] Appendix):

\[
\| f \|_p \leq K \left( \sum_{1 \leq j \leq d} \| D_j f \|_q^a \right)^{1-a} \| f \|_r^{d-a}, \quad D_j = \partial_j - iA_j,
\]

where \( \frac{1}{p} = a \left( \frac{1}{q} - \frac{1}{d} \right) - (1 - a) \frac{1}{r} \), with \( d \geq 1 \), \( 1 \leq p \leq \infty \), \( 1 \leq q \leq \infty \), \( 1 \leq r < \infty \) and \( 0 \leq a \leq 1 \) (if \( p = \infty \), only \( a < 1 \) is allowed), \( K = K(d,p,q,r) \), and the \( A_j \) are real and \( f \) is a complex valued, with \( f \in L^r \), \( \partial_j f \in L^q \) and \( A_j f \in L^q \). We will need the following particular estimates:

\[
(3.1) \quad \| f \|_q \leq K \| Df \|_2^{d/a} \| f \|_2^{1-d/4}, \quad d = 1,2,
\]
(3.2) \[ \| \varphi \|_\infty \leq K \| D\varphi \|_{2}^{1/2} \| \varphi \|_{2}^{1/2}, \quad d = 1, \]

(3.3) \[ \| \varphi \|_\infty \leq K \| D^2\varphi \|_{2}^{\varepsilon/2} \| D\varphi \|_{2}^{1-\varepsilon} \| \varphi \|_{2}^{\varepsilon/2}, \quad 0 < \varepsilon < 1, \quad d = 2, \]

and also use the usual estimates obtained by setting \( A_j = 0 \) for all \( j \) in (3.1)-(3.3). Here as in the following, we write \( \| D^S f \|_2 \) for \[ \| \sum_{j_1 \geq 1} \cdots \sum_{j_S \geq 1} \| D_{j_1} \cdots D_{j_S} f \|_2^{1/2} \]. We will also use the notation \( \| \partial^S f \|_2 \) to designate \[ \| \sum_{j_1 \geq 1} \cdots \sum_{j_S \geq 1} \| \partial_{j_1} \cdots \partial_{j_S} f \|_2^{1/2} \].

We shall denote \( u^n \) simply as \( u \), and any positive locally bounded function of \( t \in [0,\infty) \) (including any positive constant) which can be chosen independently of \( n \) by the same letter \( C \). Note that \( u = (A_0, \psi) \in C([0,T_n]; \mathbb{R}^k) \) implies that \( A_0 \in C^\lambda([0,T_n]; H^{k+1+\lambda}), \quad A_j \in C^\lambda([0,T_n]; H^{k+\lambda}) \) \( (j = 1, \ldots, d) \) and \( \psi \in C^\lambda([0,T_n]; H^{k-2\lambda}) \), for \( \lambda = 0, 1, \ldots, \lfloor \frac{k}{2} \rfloor \). The identities for the \( D_{\mu} \) and the \( F_{\mu\nu} \) will be freely used in the following arguments.

Lemma 3.2 and the fact that the sequence of initial data with which we are concerned is bounded in \( X^m \) first give

(3.4) \[ E_1 < C, \quad \| \psi \|_2 < C. \]

Consider now the second order pseudo-energy \( E_2 \) defined by

\[ E_2 = \int_{\mathbb{R}^d} \left( \sum_{j} \partial_j \psi D_{j} \partial_j \psi + \frac{1}{2} \partial_j F_{j0} \partial_j F_{j0} + \frac{1}{2} \partial_j F_{j\ell} \partial_j F_{j\ell} \right) \ dx. \]

We note that (3.1) and (3.4) imply
\[
\|D^2\psi\|_2^2 = \int_{\mathbb{R}^d} D_j D_k \psi D_j D_k \psi \, dx
\]

\[
= \int_{\mathbb{R}^d} \{D_j D_k \psi D_j D_k \psi + 2iF_{jk} D_j \psi D_k \psi + i\partial_{jk} F \psi D_k \psi\} \, dx
\]

\[
\leq \|D_j D_k \psi\|_2^2 + \|F_{jk}\|_2 \|D_j \psi\|_4 \|D_k \psi\|_4 + \|\partial_{jk} F\|_2 \|\psi\|_4 \|D_k \psi\|_4
\]

\[
\leq E_2 + C\|D^2\psi\|_2^{d/2} + C\|D^2\psi\|_2^{d/4},
\]

so that

(3.5) \quad \|D^2\psi\|_2^2 \leq CE_2 + C.

We compute the time derivative of \(E_2\) using Eqs.(1.1)-(1.2), and then estimate the result by means of (3.1)-(3.5). It results that

\[
\frac{1}{2} \frac{d}{dt} E_2 = \text{Re} \int_{\mathbb{R}^d} \{2iF_{jk} D_k \psi D_j D_k \psi - i\partial_{jk} \psi D_k D_j \psi
\]

\[- 2i\partial_{jk} V D_j \psi D_k \psi - \partial_{jk} F_{jk} \psi \psi\} \, dx
\]

\[
\leq 2\|F_{jk}\|_4 \|D_k \psi\|_4 \|D_j \psi\|_2 + \|\partial_{jk} V\|_2 \|\psi\|_4 \|D_k D_j \psi\|_2
\]

\[
+ 2\|\partial_{jk} V\|_4 \|D_j \psi\|_4 \|D_k D_j \psi\|_2 + \|\partial_{jk} F_{jk}\|_2 \|F_{jk}\|_4 \|\psi\|_4 \|\psi\|_6
\]

\[
\leq CE_2 + C.
\]

(\(\|\partial_j V\|_4\) should be estimated by the standard Sobolev inequality). Recall—
ing that the sequence of initial data is bounded in $X^m$, we obtain

\[(3.6) \quad E_2 < C.\]

In a completely analogous way, we further estimate for $d = 2$ the third order pseudo-energy

\[E_3 = \int_{\mathbb{R}^2} \left\{ D_\kappa D_j D_\lambda \overline{\psi} D_k D_\kappa \psi + \frac{1}{2} \kappa \overline{\alpha_k} \overline{F}_j \alpha_k \alpha_j \overline{F}_j + \frac{1}{4} \kappa \overline{\alpha_k} \overline{F}_j \alpha_k \alpha_j \overline{F}_j \right\} dx,
\]

by using the covariant Sobolev inequalities and the above estimates. One first finds that

\[\|D^3 \psi\|_2^2 = \int_{\mathbb{R}^2} D_\kappa D_j D_\lambda \overline{\psi} D_k D_\kappa \psi dx\]

\[= \int_{\mathbb{R}^2} \left\{ D_\kappa D_j D_\lambda \overline{\psi} D_k D_\kappa \psi + i F_{\xi j} D_\kappa \overline{\psi} D_k D_\kappa \psi - i F_{\xi j} D_\kappa \overline{\psi} D_k D_\kappa \psi \right\}
\]

\[+ i \overline{\psi} D_\kappa D_j D_\lambda D_\kappa \psi + 2 i F_{\xi j} D_\kappa \overline{\psi} D_k D_\kappa \psi - i F_{\xi j} D_\kappa \overline{\psi} D_k D_\kappa \psi \]

\[- 2 i F_{\xi k} D_\kappa D_j D_\lambda \overline{\psi} D_\kappa \psi \right\} dx\]

\[\leq CE_3 + C\|D^3 \psi\|_2 + C\]

and thus obtains

\[(3.7) \quad \|D^3 \psi\|_2^2 \leq CE_3 + C.\]
Using Eqs. (1.1)-(1.2) and noting this estimate, one has

\[
\frac{1}{2} \frac{d}{dt} E_3 = \int_{\mathbb{R}^d} \left\{ (2iF_{j 0} D_{k 0} \psi + iF_{0} D_{j 0} \psi + 2i \partial_{x_j} F_{j 0} D_{0} \psi + i \partial_{x_j} F_{j 0} D_{0} \psi \\
+ i \partial_{x_j} F_{j 0} \psi - i \partial_{x_j} F_{j 3} \psi - i \partial_{x_j} F_{j 3} \psi - 2i \partial_{x_j} F_{3 0} \psi \\
- 2i \partial_{x_j} F_{3 0} \psi - i \partial_{x_j} F_{3 0} \psi ) \frac{D_{x_k} D_{x_k} \psi}{D_{x_k} D_{x_k} \psi} + (iD_{j 3} \psi) \frac{D_{x_k} D_{x_k} \psi}{D_{x_k} D_{x_k} \psi} + iD_{j 3} \psi \frac{D_{x_k} D_{x_k} \psi}{D_{x_k} D_{x_k} \psi} \right\} dx
\]

\[\leq C E_3 + C,\]

which gives

(3.8) \quad E_3 < C.

The results (3.4)-(3.8) show that

(3.9) \quad \|
\psi\|_2 < C, \quad \|D^s \psi\|_2 < C, \quad s = 1, \ldots, [\frac{d}{2} + 2],

(3.10) \quad \|F_{\mu \nu}\|_{H^{[\frac{d}{2}+1]}} < C, \quad \mu, \nu = 0, 1, \ldots, d,

for d = 1, 2. To derive the needed bound on \((A_0, \partial_0 A_0, \vec{A}, \partial_0 \vec{A}, \psi)\), we shall consider the following quantities:

\[E_{A_0} = \|\vec{A}\|^2_2 + \|A_0\|^2_{H^{[\frac{d}{2}+3]}} + \|\partial_0 A_0\|^2_{H^{[\frac{d}{2}+2]}}.\]
\[ = \int_{\mathbb{R}^d} A_0 A_\mu \, dx + \sum_{|\alpha| \leq \left[ \frac{d+2}{2} \right]} \int_{\mathbb{R}^d} \delta^{\alpha_0} A_0 \delta^{\alpha_\mu} A_\mu \, dx \]

\[ E_A = \| \delta_0 \|_{H^{\left[ \frac{d}{2} + 1 \right]}}^2 + \| \delta_0 A_0 \|_{H^{\left[ \frac{d}{2} + 1 \right]}}^2 \]

\[ = \sum_{|\alpha| \leq \left[ \frac{d+2}{2} \right]} \int_{\mathbb{R}^d} \delta^{\alpha_0} A_0 \delta^{\alpha_\mu} A_\mu \, dx. \]

Here we use the notation \( \delta^{\alpha} \) in the usual sense; thus \( \delta^{\alpha} = \delta_1 \cdots \delta_s \) with \( |\alpha| = s \). Eq. (1.1), Eq. (1.3) and (3.10) give

\[ \frac{1}{2} \frac{d}{dt} A_0 = - \int_{\mathbb{R}^d} F_{\mu} A_\mu \, dx + \sum_{|\alpha| \leq \left[ \frac{d+2}{2} \right]} \int_{\mathbb{R}^d} \delta^{\alpha_0} A_0 \delta^{\alpha_\mu} F_{\mu} \, dx \]

\[ \leq \| F_{\mu} \|_{L^2} \| A_\mu \|_{L^2} + \| \delta_0 A_0 \|_{H^{\left[ \frac{d}{2} + 2 \right]}} \| \psi \|_{H^{\left[ \frac{d}{2} + 2 \right]}}^2 \]

\[ \leq \left( C + \| \psi \|_{H^{\left[ \frac{d}{2} + 2 \right]}} \right)^{1/2} A_0. \]

But from (3.1)–(3.3) and (3.9) one obtains

\[ \| \psi \|_{L^2} \leq \| \psi \|_{H^4} < C, \]

\[ \| \delta \psi \|_{L^2} \leq 2 \| \psi \|_{H^4} < C, \]

\[ \| \delta^2 \psi \|_{L^2} \leq 2 \| \psi \|_{H^4} \| D^2 \psi \|_{L^2} + 2 \| D \psi \|_{L^4}^2 < C, \]

\[ \| \delta^3 \psi \|_{L^2} \leq 2 \| \psi \|_{H^4} \| D^3 \psi \|_{L^2} + 6 \| D \psi \|_{L^4} \| D^2 \psi \|_{L^4} < C, \]

the last estimate being needed only for \( d = 2 \). Thus it follows that
\( E_{A_0} < C \). Eq. (1.3), (3.10) and this result yield

\[
E_A^+ = \sum_{|\alpha| \leq \left\lfloor \frac{d}{2} \right\rfloor + 1} \int_{\mathbb{R}^d} \{ \partial_{\alpha} \mu_j \partial_{\alpha} \varphi A_j + \partial_{\alpha} \varphi A_0 \partial_{\alpha} \varphi A_j + \partial_{\alpha} \varphi A_0 \partial_{\alpha} \varphi A_0 \} \, dx
\]

\[
\leq \| F_{\mu j} \|_{H^\left\lfloor \frac{d}{2} \right\rfloor + 1} \| \varphi A_j \|_{H^\left\lfloor \frac{d}{2} \right\rfloor + 1} + \| \varphi A_0 \|_{H^\left\lfloor \frac{d}{2} \right\rfloor + 1} \| \varphi A_0 \|_{H^\left\lfloor \frac{d}{2} \right\rfloor + 1}
\]

\[
+ \| \varphi A_0 \|_{H^\left\lfloor \frac{d}{2} \right\rfloor + 1}^2
\]

\[
\leq C E_A^{1/2} + C,
\]

giving \( E_A^+ < C \). Therefore,

\[
\| A_0 \|_{H^\left\lfloor \frac{d}{2} + 3 \right\rfloor} < C, \quad \| \varphi A_0 \|_{H^\left\lfloor \frac{d}{2} + 2 \right\rfloor} < C, \quad \| \varphi \Lambda \|_{H^\left\lfloor \frac{d}{2} + 2 \right\rfloor} < C, \quad \| \varphi A_0 \|_{H^\left\lfloor \frac{d}{2} + 1 \right\rfloor} < C
\]

Finally, from the definition of the \( D_\mu \), one has

\[
\partial_j \psi = D_j \psi + i A_j \psi,
\]

\[
\partial_j \partial_k \psi = D_j D_k \psi + i \partial_j A_k \psi + i A_k \partial_j \psi + i A_j D_k \psi,
\]

\[
\partial_j \partial_k \partial_\lambda \psi = D_j D_k D_\lambda \psi + i \partial_j \partial_k A_\lambda \psi + i \partial_\lambda A_k \partial_j \psi + i \partial_j A_\lambda \partial_k \psi + i A_\lambda \partial_j \partial_k \psi
\]

\[
+ i \partial_j A_k \partial_\lambda \psi + i A_k \partial_j \partial_\lambda \psi + A_k \partial_j A_\lambda \psi - A_\lambda \partial_k \partial_j \psi - i A_j D_k D_\lambda \psi.
\]

Using these expressions, one finds, with the help of (3.1)-(3.3) and the usual Sobolev inequalities, that (3.9) and the above estimate on \( \Lambda \) imply
that $\|\psi\|_{H^{\frac{d}{2}+2}} < C$, which completes the proof of the desired global result.

References


