数理解析研究所講究録
542巻 1985年 97-101

A Heegaard-Diagram of the 3-Sphere

阪大 理学部 落合豊行 (Mitsuyuki Ochiai)

1. Introduction.

As an algorithm for recognizing S^3 in 3-manifolds, there is the Whitehead algorithm [10], [11]. Partly this algorithm is really true by Homma-Ochiai-Takahashi [4], but in general not true by Viro [9], Ochiai [7], and Morikawa [6]. That is, Whitehead conjecture, which asserts that all Heegaard diagrams of S^3 other than the canonical one have always waves, is true in the case of genus two but not true in the case of genus greater than two. All already known counterexamples to the conjecture was constructed as Heegaard diagrams of 2-fold branched coverings branched along knot diagrams of the trivial knot. In this paper, we construct such a counterexample through the different method using presentations of the trivial group, and later we will set up a new conjecture which permit us to deform the example and other examples to the canonical one (see the conjecture in Chapter 3).

2. A new counterexample to Whitehead conjecture.

For all definitions of Heegaard diagrams, complete systems of meridians, waves, band moves, and others we refer to [4].

At first, let's choose a trivial group

$$G = \{X,Y,Z;\ XY^2X^{-1} = Y^3,\ YX^2Y^{-1} = X^3,\ Z = 1\}$$

Partially supported by Grand-in-Aid Scientific Research.

It will be noticed that the group G is trivial by Crowell-Fox [3] and by Birman-Hilden [2] there are no Heegaard diagrams which have relators of G as that of the fundamental group induced by them. Hence we may change the relators of G and to get new relators of the trivial group;

$$H = \{ X,Y,Z \ ; \ \ XY^2X^{-1}(Y^{-1}Z)^3 = 1, \\ YX^2Y^{-1}(X^{-1}Z^{-1})^3 = 1, \\ ZXYX^{-1}Y^{-1} = 1 \ \}$$

It will be noticed that H is obtained from G by the trial and error method and that the relators of H is induced by some Heegaard diagram of S^3. Next we may also change the diagram through a band move, and the resulting diagram Ψ is obtained which is illustrated in Figure 1. Let Ψ denote $(\Gamma \ ; \ v,w)$. Then one complete system of meridians of Γ, v is illustrated as Figure 1. Moreover the Heegaard surface Γ is obtained from Figure 1 by identifying meridians ∂A, ∂C, ∂E with ∂B, ∂D, ∂F, respectively, where A,B,C,D,E,F are meridian-disks in Figure 1. At the same time, another complete system of meridians, w is obtained as follows; identify points $A_1, A_2, \ldots, A_9$ with $B_1, B_2, \ldots, B_9$, points $C_1, C_2, \ldots, C_{14}$ with points $D_1, D_2, \ldots, D_{14}$, and points $E_1, E_2, \ldots, E_{15}$ with $F_1, F_2, \ldots, F_{15}$, respectively. The last Heegaard diagram has as the fundamental group the following group H';

$$H' = \{ X,Y,Z \ ; \ \ XZY^2Z^{-1}X^{-1}(Y^{-1}Z^{-1})^3 = 1, \\ ZY(XZ)^2Y^{-1}Z^{-1}X^{-3} = 1, \\ XZY^3Z^{-1}X^{-1}(Y^{-1}Z^{-1})^4 = 1 \ \}$$

Figure 1.

By the way, the fundamental group of the dual diagram $(\Gamma \ ; \ w,v)$ of Ψ is the following;

$H'' = \{ A,B,C \; ; \; ABC^2 = C^3BA, \; (AB)^2AC = CA(BA)^3, \; CABC^2 = (AB)^3ACBA\}$

It will be noticed that both of H' and H" are not simply trivial different from Kaneto's example [5]. The Heegaard diagram Ψ has no waves and so is an counterexample to Whitehead conjecture and moreover, different from already known examples, it does not permit us to directly reduce Heegaard genus of it, but permit us to make another Heegaard diagrams which have arbitrary many intersections of one complete system of meridians and another one. Such a Heegaard diagram has as the fundamental group the following;

$$\begin{aligned} H(n,m) = \{X,Y,Z \; ; \; & XZY^{n-1}Z^{-1}X^{-1}(Y^{-1}Z^{-1})^n = 1, \\ & ZY(XZ)^{m-1}Y^{-1}Z^{-1}X^{-m} = 1, \\ & XZY^nZ^{-1}X^{-1}(Y^{-1}Z^{-1})^{n+1} = 1\} \end{aligned}$$

3. A new algorithm for recognizing S^3 in 3-manifolds.

Let M be a closed connected 3-manifold and $(\Gamma \; ; \; v,w)$ be a Heegaard diagram of genus g of M, where $v = v_1 \cup v_2 \cup \ldots \cup v_g$ and $w = w_1 \cup w_2 \cup \ldots \cup w_g$. Then a simple closed curve γ on Γ is called a bridge associated with v (resp. w) if it is disjoint from v (resp. w) and there exists some meridian v_i in v (resp. w_j in w) such that v' (resp. w') is a complete system of meridians of F and that the number of the intersections of $v' \cap w$ (resp. $v \cap w'$) is less than that $v \cap w$, where $v' = v_1 \cup \ldots \cup v_{i-1} \cup \gamma \cup v_{i+1} \cup \ldots \cup v_g$ (resp. $w' = w_1 \cup \ldots \cup w_{j-1} \cup \gamma \cup w_{j+1} \cup \ldots \cup w_g$). Given a Heegaard diagram, to observe the two Whitehead graph of it, it is easy to determine whether the diagram has a bridge or not. For example, the diagram given by Figure 1 has a bridge γ as illustrated in the one.

Then our assrtion is as follows; All Heegaard diagrams of S^3, other than the canonical one, have always bridges.

If Heegaard diagrams have waves, then they have also bridges. And so this assertion is true in the case of genus two. Bridges is a generalization of waves and the method in the proof of Lemma 6 in [8].

References

[1] J. S. Birman, Heegaard Splittings, Diagrams and Sewings for Closed, Orientable 3-Manifolds, Lecture notes for CBMS conference at Blacksburg, Va., Oct. 8-12, 1977

[2] J. S. Birman and H. M. Hilden, Heegaard splittings of branched coverings of S^3, TRANS. AMER. MATH. SOC.,213(1975),315-352.

[3] R. Crowell and R. Fox, INTRODUCTION TO KNOT THEORY, Blaisdell Publishing Company, 1963

[4] T. Homma, M. Ochiai, and M. Takahashi, An algorithm for recognizing S^3 in 3-manifolds with Heegaard splittings of genus two, Osaka J. Math., 17 (1980), 625-648.

[5] T. Kaneto, On presentations of the fundamental group of the 3-sphere associated with Heegaard diagrams, J. Math. Soc. Japan Vol. 33, No. 1 (1981), 147-158.

[6] O. Morikawa, A counterexample to a conjecture of Whitehead, MATH. SEM. NOTES, Vol. 8 (1980), 295-298.

[7] M. Ochiai, A counterexample to a conjecture of Whitehead and Volodin-Kuznetzov-Fomenko, J. Math. Soc. Japan 31 (1979), 687-691.

[8] , Heegaard-Diagrams and Whitehead-Graphs, MATH. SEM. NOTES OF KOBE UNIV., Vol 7 (1979), 573-590.

[9] O. Ja. Viro, The Volodin-Kuznetzov-Fomenko conjecture on Heegaard diagrams is false, Uspehi Math. Nauk, 32 (1977),175-176.

[10] I.A.Volodin,V.E.Kuznetzov, and A.T.Fomenko, The problem of discriminating algorithmically the standard three-dimensional sphere, Russian Math. Surveys, 29:5 (1974), 71-172.

[11] J.H.C.Whitehead, On certain sets of elements in a free group, Proc. London Math. Soc., (2), 41 (1936), 48-56.

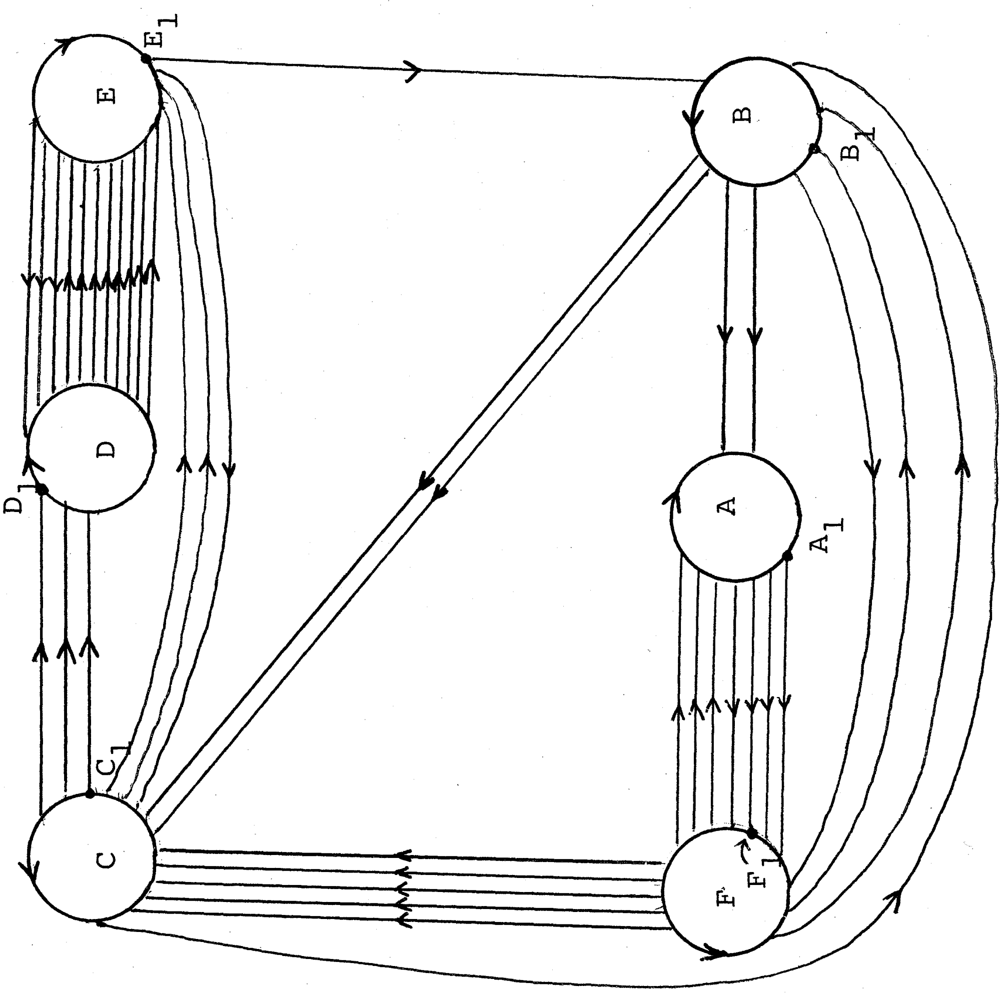

Figure 1.